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ON THE CONVERGENCE OF LAPLACE INTEGRAL

N. K. CHAKRAVARTY

1. Introduction.

Consider Q: the n dimensional Euclidean space R_n of elements $\mathbf{x} = (x_1, \ldots, x_n)$, $x_k \geqslant 0$. Then $\Gamma = C \cup \{\phi\}$, C set of cells $C: \{\mathbf{x} : a_i < x_i < b_i, i=1, 2, \ldots, n\}$ and $\{\phi\}$, the null set, is a semi-ring. If for a cell $U = (a_1, b_1; \ldots; a_n, b_n)$, $\mu(\phi)=0$ and $\mu(U) = \prod_{j=1}^{n} (b_j - a_j)$, then μ is a measure on Γ. By means of the extension procedure we then obtain the σ-ring $\triangle$ of all μ-measurable sets A, where A is a Lebesgue measurable kernel on Q. The n-dimensional Lebesgue measure $\mu = \mu(A)$ (countably additive), the measure of A, is totally σ-finite and (Q, A, μ) constitutes a totally σ-finite measure space.

Let F and F_0 be two A-measurable functions

$$\{x \in A : a < F, F_0 \leqslant \infty, a, \text{ any real number } \geqslant 0\}$$

defined on Q such that

$$F_0 = F \text{ on } A \quad \text{and} \quad F_0 = 0 \text{ on } Q \setminus A.$$

Let F_0 be Lebesgue summable on $\Omega : a \leqslant x_j \leqslant X_j$, $j=1, 2, \ldots, n$, a subspace of A and $\mathbf{p} = (p_1, \ldots, p_n)$, where $p_j = \sigma_j + i\,\lambda_j$, $i = (-1)^{\frac{1}{2}}$, $j=1, 2, \ldots, n$, are independent complex parameters. Let $\mathbf{p}.\mathbf{x}$ denote the inner product of $\mathbf{p}$ and $\mathbf{x}$ and

$$(1.1) \quad f(X;\ \mathbf{p}) = \int_{\Omega} e^{-\mathbf{p}.\mathbf{x}} F d\mu, \quad \mathbf{x} \in \Omega \subset Q.$$

If $\lim\limits_{X \to \infty} f(X; \mathbf{p})$ exists, we say that the n-dimensional Laplace integral

$$(1.2) \qquad \int_0^{\infty} e^{-\mathbf{p}.\mathbf{x}} F d\mu$$

converges at $\mathbf{p}$ to the value

$$(1.3) \qquad f(\mathbf{p}) = \lim_{X \to \infty} f(X; \mathbf{p}).$$

The Laplace integral (1.2) is said to converge boundedly at $\mathbf{p}$, if (1.2) exists and

$$(1.4) \qquad |f(X; \mathbf{p})| \leqslant M,$$

where $X \in A$ and M is independent of X. (1.1) may be termed a section of the Laplace integral (1.2).

It follows from the example given by Ditkin and Prudnikov ([4], P.7) as also from examples by Voelker and Doetsch [6] and Bernstein and Coon [2] that the convergence of a Laplace integral on a Euclidean plane at a point (p, q) does not imply convergence of all its sections at the same point (p, q). The same result obviously holds for the n-dimensional Laplace integral and the mere convergence of (1.2) at a point $\mathbf{p}_0$ does not imply convergence of (1.1) for all X at the same point $\mathbf{p}_0$. Hence for the existence of the n-dimensional Laplace integral we must ensure the simultaneous existence of (1.1) and (1.2) at the same point $\mathbf{p}$. As such we are lead to consider the bounded convergence of (1.2).

2. Bounded convergence of (1.2).

Let $\phi(\mathbf{x})$ represent the indefinite integral

$$\phi(\mathbf{x}) = \int_E e^{-\mathbf{p}_0.\mathbf{x}} F d\mu \tag{2.1}$$

where $E \subset \Omega$ and $\mathbf{p}_0 = (p_{10}, \ldots, p_{n0})$, $p_{j0} = \sigma_{j0} + i\lambda_{j0}$.

Since $e^{-\mathbf{p}_0.\mathbf{x}}F$ is summable on Ω and therefore on E, it follows that $\phi(x)$ is finite and is absolutely continuous with respect to

μ: $\phi \ll \mu$ and $d\phi = e^{-\mathbf{p}_0 \mathbf{x}} F d\mu$.

Let $h = (h_1, h_2, \ldots, h_n) = \mathbf{p} - \mathbf{p}_0$, so that $h_j = p_j - p_{j0} = \sigma_j - \sigma_{j0} + i(\lambda_j - \lambda_{j0}) = a_j + ib_j$, say. Thus $a_j > 0$, when re $\mathbf{h} = \text{re}(\mathbf{p} - \mathbf{p}_0) > 0$.

Then (1.1) can be written in the form

$$f(X \mid \mathbf{p}) = \int_\Omega e^{-\mathbf{h}.\mathbf{x}} d\phi \tag{2.2}$$

where re $\mathbf{h} > 0$.

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_{i,j \ldots, l} = (\alpha_i, \alpha_j, \ldots, \alpha_l)$ with similar notations for β.

$\triangle F$ represents the n-dimensional increment of $F(\mathbf{x})$ from $F(\alpha)$ to $F(\beta)$, which means a summation of which the first term is $F(\beta)$ and the subsequent terms are obtained from this by putting one or more of the β_j equal to α_j and then effecting a change of sign for each such substitution.

By $\triangle_{i;j,\cdots l}(F)$ we mean $\triangle F$ in which the i th, the j th, ..., the l th terms have been omitted from α and β.

Then Young's formula for integration by parts [7] reads as follows :

$$(2.3)\quad \int_{\alpha}^{\beta} g.\,df = \Delta(f.g) - \sum_{i} \int_{\alpha_i}^{\beta_i} \Delta_i(f.dg) + \sum_{i,j} \int_{\alpha_{i,j}}^{\beta_{i,j}} \Delta_{i,j}\,(f.dg)$$

$$- \sum_{i,j,k} + \ .\ + (-1)^n \int_{\alpha}^{\beta} g.df$$

We apply (2.3) to (2.2) and make use of the property that ϕ vanishes when $x_j = 0,\ 0 \leqslant x_j \leqslant X_j$. We then obtain

$$f(X\,;\,\mathbf{p}) = \exp\,(-\mathbf{h}.X)\ \phi(X) + \sum_{i} h_i\ \exp(-\mathbf{h}.X + h_i X_i) \int_0^{x_i} \phi(X\,;\ x_i) \exp\,(-h_i x_i) dx_i$$

$$+ \sum_{i,j} h_i h_j \exp\,(-\mathbf{h}.X + h_{i,j} X_{i,j}) \int_0^{X_{i,j}} \phi(X\,;\ x_i, x_j)\ \exp\,(-h_{i,j} x_{i,j}) dx_{i,j}$$

$$(2.4)\quad + \sum_{i,j,k} + \ldots + H \int_0^{X} \phi(X)\ \exp\,(-\mathbf{h}.\mathbf{x}) d\mathbf{x}$$

where $H = h_1 \ldots h_n$, $d\mathbf{x} = dx_1 \ldots dx_n$, $dx_{i,j \ldots, l} = dx_i\ dx_j\ \ldots\ dx_l$, and $\phi(X_j, x_i, x_j, \ldots, x_l)$ is the expression $\phi(X)$ in which the elements $X_i, X_j, \ldots, X_l$ of X are replaced by $x_i, x_j, \ldots, x_l$.

If the sections (1.1) of the Laplace integral (1.2) satisfy (1.4) at $\mathbf{p} = \mathbf{p}_0$, then ϕ is uniformly bounded : $|\,\phi\,| \leqslant M$, independent of the x_j. Hence from (2.4) after some easy steps

$$(2.5)\quad |\,f(X\,;\,p)\,| \leqslant M\left(1 + \sum_{i} \frac{|\,h_i\,|}{a_i} + \sum_{i,j} \frac{|h_i||h_j|}{a_i a_j} + \ldots + \frac{|h_1| \ldots |h_n|}{a_1 \ldots a_n}\right)$$

It follows, therefore, from (2.5) that if the Laplace integral (1.2) converges boundedly at $\mathbf{p}_0$, the Laplace integral converges boundedly in the region for which $\operatorname{re}\,(\mathbf{p} - \mathbf{p}_0) \geqslant 0$.

It may be noted that for the validity of the above result it is necessary that both the conditions (1.3) and (1.4) must have to be satisfied simultaneously at $\mathbf{p}_0$. (See the example on P.7, Ditkin and Prudnikov [4] for the corresponding result in connection with the two dimensional Laplace integral.)

3. Absolute Convergence of (1.2).

The Laplace integral (1.2) is absolutely convergent, if $\int_0^{\infty} |\,e^{-\mathbf{p}.\mathbf{x}}\,F\,|\,d\mu$ exists finitely.

Since $\left|\int\limits_{\Omega} e^{-\mathbf{p}.\mathbf{x}} F d\mu\right| \leqslant \int\limits_{\Omega}\left| e^{-\mathbf{p}.\mathbf{x}} F\right| d\mu \leqslant \int\limits_{0}^{\infty}\left| e^{-\mathbf{p}.\mathbf{x}} F\right| d\mu,$

it follows that the absolute convergence of the Laplace integral (1.2) at $\mathbf{p}_0$ implies the bounded convergence of the integral at the same point.

Let $\mathbf{p} \equiv (p_1, .., p_n)$ and $\mathbf{p}_0 \equiv (p_{10}, \ldots, p_{n0})$ be real and let the Laplace integral (1.2) be absolutely convergent when $\mathbf{p} = \mathbf{p}_0$ (fixed). Then, evidently, the Laplace integral is absolutely convergent for real p_i, p_j, where $i \neq j = 1, 2, \ldots, n$ for fixed values $p_i = p_{i0}$, $p_j = p_{j0}$. If p_i, p_j are complex and $\alpha_i(p_{j0})$ and $\alpha_j(p_{i0})$ are the convergence abscissae (see Voelker and Doetsch [6] or the characteristics of convergence—See Ditkin and Prudnikov [4], p.9) with respect to p_i, p_j, respectively, and if C_i and C_j represent the domains

$$C_i : \ \text{re}\, p_i \geqslant p_{i0} > A_i \ ; \ \text{re}\, p_j > \alpha_j \ (p_{i0})$$

and $$C_j : \ \text{re}\, p_i > \alpha_i (p_{j0}) \ ; \ \text{re}\, p_j \geqslant p_{j0} > A_j$$

(A_i, A_j are fixed real numbers).

Then as in Voelker and Doetsch [6], the domain of absolute convergence of the Laplace integral (1.2) with respect to p_i, p_j is determined by

$$D_{i,j} = C_i \cup C_j$$

We now give to i, j all possible values $i \neq j = 1, 2, \ldots, n$. Then when the Laplace integral (1.2) is absolutely convergent at the real point $\mathbf{p}_0$, the domain determined by the complex values $\mathbf{p}$ for which (1.2) is absolutely convergent, is the set

$$\{ \cup D_{i,j} : \ i \neq j = 1, 2, \ldots, n \}.$$

Let $D(A)$ represent the set of points $\mathbf{p}$ for which $\text{re}\, \mathbf{p} \geqslant A$, $A = (A_1, \ldots, A_n)$, with $D[A]$ as the closure. Then $D[A]$ is uniquely determined, when one assumes that the curves $p_i = \alpha_i(p_j)$ and $p_j = \alpha_j(p_i)$ in the real p_i-p_j plane are, respectively, parallel to the p_j and p_i axes. For simplicity of our discussion we can choose $D[A]$ as the domain of absolute convergence of the Laplace integral (1.2). $D[A]$ then also represents the domain of bounded convergence of (1.2).

4. Uniform Convergence of (1.2).

Let $L(X,Y)$ be the line segment

$$L(X,Y) = \{ z \mid z = \theta X + (1-\theta) Y, \ 0 < \theta < 1 \}$$

joining two points X, Y of A.

Put $D_r \, \psi(x \, ; z_r) = \dfrac{\partial}{\partial z_r} \psi \, (x \, ; z_r)$

and $\text{grad}\, \psi(x \, ; z) = (D_1 \psi(x \, ; z_1), \ldots, D_n \psi \, (x \, ; z_n)).$

Then if $\psi(X)$ have a differential everywhere in a neighbourhood $N(X)$ of $X \in A$ and Y be a point which $\in N'(x)$, we have the mean value theorem

(4.1) $\psi(Y) - \psi(X) = (Y - X).\ \text{grad}\ \psi(X\,;z)$

(See Apostol [1] ; p. 135)

where $z \equiv (z_1, \dots, z_n)$, $z_r \in L(x_r, y_r)$, $r=1,2,\dots,n$.

If all the $D_r\psi(x\,;z_r)$ be uniformly bounded, such that

$$\max |D_r\psi| = \frac{M_1}{n},\ r=1,\dots,n \text{ and } \max|Y-X| = \delta, \text{ we have}$$

$$|\psi(Y)-\psi(X)| \leqslant M_1\delta. \tag{4.2}$$

Let E_{rs} be the domain

$$E_{rs}: \begin{cases} 0 \leqslant x_s \leqslant X_s, & \text{for } 1 \leqslant s \leqslant r-1 \\ Y_r \leqslant x_r \leqslant X_r, & \text{for } s=r \\ 0 \leqslant x_s \leqslant Y_s, & \text{for } r < s \leqslant n \end{cases} \tag{4.3}$$

Also let $\Omega_X : 0 \leqslant x_i \leqslant X_i$, $\Omega_Y : 0 \leqslant y_i \leqslant Y_i$, $i=1, 2, \dots, n$, be two hyper-rectangles having one common corner at the origin and the common adjacent sides through the origin along the co-ordinate axes, the end point of the diagonal through the origin of one being at $X=(X_1, \dots, X_n)$, while that of the other at $Y = (Y_1, \dots, Y_n)$. Then by considering the space included within Ω_X, Ω_Y, we obtain, if $X > Y$, the following decomposition

$$\Omega_X - \Omega_Y = \sum_{r=1}^{n} E_{rs}.$$

If now ψ is integrable on Ω_X, ψ is integrable on Ω_Y and also on each E_{rs} and we have

$$\left(\int_{\Omega_X} - \int_{\Omega_Y}\right)\psi(x)d\mu = \sum_{r=1}^{n}\int_{E_{rs}}\psi d\mu \tag{4.4}$$

(See Kolmogorov and Fomin [5], P. 298)

Since $\phi \ll \mu$, $\exp.(-\mathbf{h}\mathbf{x}.)\,\phi(\mathbf{x})$ is also so and

(4.5) $|D_r(\exp(-\mathbf{h}.\mathbf{x})\phi(\mathbf{x})| \leqslant M_2$,

$r=1,2,\dots,n$ in $\Omega :\ 0 \leqslant x_j \leqslant X_j$, $j=1,2,\dots,n$.

Also when the sections of the Laplace integral (1.2) are bounded at $\mathbf{p}_0$,

$$|\phi(\mathbf{x})| \leqslant M_3 \tag{4.6}$$

for $x_j \geqslant 0$, $j = 1, 2, \dots, n$.

We now apply the formula (2.4) to the functions $f(X\,;\mathbf{p})$ and $f(Y\,;\mathbf{p})$, $X > Y$, simplify each term in $f(X\,;\mathbf{p}) - f(Y\,;\mathbf{p})$ by the mean value theorem (4.1), make use of the formula of type (4.4) holding when $n=2, 3, \dots$ and utilize

relations of type (4.2), (4.5) and (4.6) as and when necessary. Then after some easy reductions, it follows, when Y is large enough, that

$$|f(X;\mathbf{p})-f(Y;\mathbf{p})|$$

$$\leqslant M\Delta\Big[e^{-aY}+\sum_i \frac{|h_i|}{a_i}e^{-a_iY_i}+\sum_{i,j}\frac{|h_i|\,|h_j|}{a_i\,a_j}\left(e^{-a_iY_i}+e^{-a_jY_j}\right)$$

$$+\ldots+\frac{|H|}{\alpha}\left(\sum_{i=1}^{n}e^{-a_iY_i}\right)\Big], \tag{4.7}$$

where $\Delta=\max_j|X_j-Y_j|$, $M=$max. of all the constants involved in the process, $\alpha=a_1\ldots a_n$ and $H=h_1\ldots h_n$.

We can assume that the hyper-rectangles constructed above with corners at X, Y and common corner at $O,(X_i>Y_i)$ are such that X is a linear function of Y. Then although Δ may tend to infinity with Y_j, $\Delta\exp(-a_jY_j)$ must tend to zero with Y_j.

Let $|h_i|\leqslant K_ia_i$, where K_i are finite constants and $a_i\geqslant\delta_i>0$. Then (4.7) reduces to

$$|f(X;p)-f(Y;p)|\leqslant M\Delta\left[e^{-\delta_iY_i}+\sum_i K_ie^{-\delta_iY_i}+\ldots+\mathbf{k}\sum_{i=1}^{n}e^{-\delta_iY_i}\right], \tag{4.8}$$

$\mathbf{k}=k_1\ldots k_n$.

The right hand side of (4.8) is independent of $\mathbf{p}$ and can be made less than ϵ by taking Y large enough.

The Laplace integral is therefore uniformly convergent for

$|h_i|\leqslant k_ia_i$, $a_i\geqslant\delta_i>0$ i.e. for $|p_i-p_{i0}|\leqslant k_i(\sigma_i-\sigma_{i0})$;

$\sigma_i\geqslant\sigma_{i0}+\delta_i$, $i=1,2,\ldots,n$, k_i are positive constants which may be as large as we please.

We thus have the following theorem.

Theorem : Given that the Laplace integral (1.2) is absolutely convergent at $p_i=p_{i0}$. Then the Laplace integral is (absolutely and also) boundedly convergent for re $p_i\geqslant$re p_{i0} and is uniformly convergent for re $p_i\geqslant$re $p_{i0}+\delta_i$, $|p_i-p_{i0}|\leqslant k_i$ re (p_i-p_{i0}), where $i=1,2,\ldots,n$ and the k_i are real constants which may be as large as possible.

Thus the Laplace integral (1.2) which is absolutely convergent in $D[A]$, is boundedly convergent in $D[A]$ and in any compact region $\subset D[A]$, not containing A, the integral is uniformly convergent.

REFERENCES

[1] Apostol, T. M.—*Mathematical Analysis* (1968) (Addison-Wesly Publishing Co. Inc.)

[2] Bernstein, D. L. and Coon, G. A.—Some properties of the double Laplace Transformation—*Trans. Amer. Math. Soc.* **74** (1953).

[3] Chakravarty, N. K.– *Calcutta D. Phil Thesis* (1961) (unpublished).

[4] Ditkin, V. A. and Prudnikov, A. P.—*Operational Calculus in two variables and its applications* (1962) (Pergamon Press).

[5] Kolmogorov, A. N. and Fomin, S. V.—*Introductory Real Analysis* (1970) (Prentice Hall Inc.)

[6] Voelker, D and Doetsch, G.—*Die Zweidimensionale Laplace transformation* (1950) (Verlag Birkhaüser, Basle)

[7] Young, W. H.—On multiple integration by parts and the second theorem of the mean—*Proc. Lond. Math. Soc.* (2), **16** (1917).

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Dept. of Pure Math.
Calcutta University

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SOME MULTILATERAL GENERATING RELATIONS INVOLVING HERMITE AND TCHEBICHEF POLYNOMIALS

S. K. CHATTERJEA

1. In a recent paper [4], P. A. Lee obtained the following trilateral generating relation involving Charlier and Tchebichef polynomials :

$$(1.1)\quad \sum_{n=0}^{\infty} z^n \mathrm{T}_n(x) \mathrm{C}_n(k\,;\,\alpha) \mathrm{C}_n(l\,;\,\beta)/n\,!$$

$$=\frac{1}{2}\Big[e^{\rho}\Big(1-\frac{\rho}{\alpha}\Big)^k\Big(1-\frac{\rho}{\beta}\Big)^l \mathrm{C}_k\Big\{l\,;\,-\frac{(\alpha-\rho)(\beta-\rho)}{\rho}\Big\}$$

$$+e^{\rho'}\Big(1-\frac{\rho'}{\alpha}\Big)^k\Big(1-\frac{\rho'}{\beta}\Big)^l \mathrm{C}_k\Big\{l\,;\,-\frac{(\alpha-\rho')(\beta-\rho')}{\rho'}\Big\}\Big],$$

where $\rho=(x+\sqrt{x^2-1})z,\quad \rho'=(x-\sqrt{x^2-1})z,$

and $\mathrm{C}_n(x\,;\,a)={}_2\mathrm{F}_0\Big(-n,\,-x\,;\,-\,;\,-\frac{1}{a}\Big)$; $a>0$ and $x=0, 1, 2, \ldots\ldots$

The object of the present paper is to derive some multilateral generating relations involving Hermite and Tchebichef polynomials by means of a method which is much shorter than that adopted by Lee. It is interesting to remark that our method of obtaining multilateral generating relations is perfectly general and straightforward in the sense that one can easily apply our method to any m-lateral generating relation in order to obtain the corresponding $(m+1)$-lateral generating relation involving Tchebichef polynomial as an extra polynomial. The main results are contained in (2.3), (2.5), (3.5), (3.6) and (3.8).

2. First we consider the following generating relation of W. A. Al-Salam [1] :

$$(2.1)\quad \sum_{n=0}^{\infty} \Delta_{n,\,1,\,2\,;\,x}\,(\mathrm{H})t^n/n\,!=\frac{2x}{(1-t^2)^{3/2}}\,\exp\Big(\frac{x^2t}{1+t}\Big)$$

where $\Delta_{n,\,1,\,2\,,\,x}\,(\mathrm{H})=\mathrm{H}_{n+1}(x)\mathrm{H}_{n+2}(x)-\mathrm{H}_n(x)\mathrm{H}_{n+3}(x)$

and $\exp\,(xt-\frac{1}{2}t^2)=\sum_{n=0}^{\infty}\mathrm{H}_n(x)t^n/n\,!$.

The following relation of Tchebichef polynomials will be utilized throughout the paper :

$$(2.2)\quad \mathrm{T}_n(x)=\tfrac{1}{2}[(x+\sqrt{x^2-1})^n+(x-\sqrt{x^2-1})^n].$$

2

Instead of using (2.2), Lee used a number of relations (viz. generating relation of $T_n(x)$, successive differentiation, multiplication and Rodrigues' formulas for Laguerre polynomials) in order to prove (1.1).

Now it follows from (2.1) and (2.2) that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \Delta_{n,1,2;x}(H)\, T_n(y)$$

$$= \frac{1}{2}\Big[\sum_{n=0}^{\infty} \frac{\Delta_{n,1,2;x}(H)}{n!}\{t(y+\sqrt{y^2-1})\}^n + \sum_{n=0}^{\infty} \frac{\Delta_{n,1,2;x}(H)}{n!}\{t(y-\sqrt{y^2-1})\}^n\Big]$$

(2.3)

$$= x\Big[(1-\rho_1^2)^{-3/2} \exp\left(\frac{x^2\rho_1}{1+\rho_1}\right) + (1-\rho_2^2)^{-3/2} \exp\left(\frac{x^2\rho_2}{1+\rho_2}\right)\Big],$$

where $\rho_1 = t(y+\sqrt{y^2-1})$ and $\rho_2 = t(y-\sqrt{y^2-1})$.

Next we notice that

$$(2.4)\quad \sum_{n=0}^{\infty} T_{n+k}(x) t^n/n!$$

$$= \tfrac{1}{2}[(x+\sqrt{x^2-1})^k \exp\{t(x+\sqrt{x^2-1})\} + (x-\sqrt{x^2-1})^k \exp\{t(x-\sqrt{x^2-1})\}].$$

Thus we have

$$\sum_{n=0}^{\infty} T_n(y)\, t^n/n! \sum_{k=0}^{n} \binom{n}{k} \Delta_{k,1,2,x}(H)$$

$$= \sum_{k=0}^{\infty} \frac{\Delta_{k,1,2;x}(H)}{k!}\, t^k \sum_{n=0}^{\infty} T_{n+k}(y)\, t^n/n!$$

(2.5)

$$= x\Big[(1-\rho_1^2)^{-3/2} \exp\left(\rho_1 + \frac{x^2\rho_1}{1+\rho_1}\right) + (1-\rho_2^2)^{-3/2} \exp\left(\rho_2 + \frac{x^2\rho_2}{1+\rho_2}\right)\Big],$$

where $\rho_1 = t(y+\sqrt{y^2-1})$ and $\rho_2 = t(y-\sqrt{y^2-1})$.

3. For the sequence of Hermite polynomials $\{H_n(x)\}$, let us suppose

$$(3.1)\quad \sum_{n=0}^{\infty} A_n\, H_{m+n}(x)\, H_n(y)\, t^n = \frac{f(x,y,t)}{[g(x,y,t)]^m} H_m\{h(x,y,t)\},$$

where the sequence of coefficients A_n is selected in such a way that the series on the left of (3.1) gives rise to a generating function separated like the right member of (3.1), f, g, h being functions of x, y, t and let

$$\text{(3.2)} \qquad F(x, t) = \sum_{n=0}^{\infty} a_n H_n(x) t^n$$

be a unilateral generating relation, where a_n's ($\neq 0$) are arbitrary constants.

Then

$$\sum_{n=0}^{\infty} H_n(x) t^n \sum_{k=0}^{n} a_{n-k} A_k H_k(y) z^k$$

$$= \sum_{n=0}^{\infty} \frac{f(x, y, zt)}{[g(x, y, zt)]^n} H_n(h(x, y, zt)) a_n t^n$$

$$= f(x, y, zt) \, F(h(x, y, zt), t/g(x, y, zt))$$

Thus we obtain

$$\text{(3.3)} \qquad \sum_{n=0}^{\infty} H_n(x) \sigma_n(y, z) t^n = f(x, y, zt) \, F(h(x, y, zt), t/g(x, y, zt))$$

where $\sigma_n(y, z) = \sum_{k=0}^{n} a_{n-k} A_k H_k(y) z^k$.

Now for the existence of a relation like (3.1) we notice that [3] :

$$\text{(3.4)} \qquad \sum_{n=0}^{\infty} \frac{t^n}{2^n n!} H_{n+m}(x) H_n(y)$$

$$= (1-t^2)^{-\frac{1}{2}(m+1)} \exp\left[\frac{2xyt-(x^2+y^2)t^2}{1-t^2}\right] H_m\left(\frac{x-yt}{(1-t^2)^{1/2}}\right),$$

where $\exp(2xt-t^2) = \sum_{n=0}^{\infty} H_n(x) t^n/n!$, which is slightly different from the symbol $H_n(x)$ used in section 2.

Thus using $A_n = (2^n \, n!)^{-1}$ in (3.1) we observe that

$$f(x, y, t) = (1-t^2)^{-1/2} \exp\left[\frac{2xyt-(x^2+y^2)t^2}{1-t^2}\right]$$

$$g(x, y, t) = (1-t^2)^{1/2}$$

$$h(x, y, t) = (x-yt)/(1-t^2)^{1/2},$$

so that we derive the trilateral generating relation in the form of the following theorem :

Theorem 1. If $F(x, t) = \sum_{n=0}^{\infty} a_n H_n(x) t^n/n!$,

where $$\exp(2xt - t^2) = \sum_{n=0}^{\infty} H_n(x) t^n/n!,$$

then

$$(3.5)\quad \sum_{n=0}^{\infty} H_n(x)\sigma_n(y, z) t^n/n!$$
$$= (1 - z^2t^2)^{-1/2} \exp\left[\frac{2xyzt - (x^2+y^2)z^2t^2}{1 - z^2t^2}\right].F\left(\frac{x - yzt}{(1-z^2t^2)^{1/2}}, \frac{t}{(1-z^2t^2)^{1/2}}\right)$$

where $\sigma_n(y, z) = \sum_{k=0}^{n} \binom{n}{k} a_{n-k}(z/2)^k H_k(y)$, $|zt| < 1$.

Now we are in a position to adjoin Tchebichef polynomial to (3.5) in order to derive a quadrilateral generating relation. Indeed we have the following theorem :

Theorem 2. If $F(x, t) = \sum_{n=0}^{\infty} a_n H_n(x) t^n/n!$,

where $$\exp(2xt - t^2) = \sum_{n=0}^{\infty} H_n(x) t^n/n!,$$

then

$$(3.6)\quad \sum_{n=0}^{\infty} H_n(x)\sigma_n(y, z) T_n(u) t^n/n!$$
$$= \frac{1}{2}\sum_{i=1}^{2}\left[(1 - z^2\rho_i^2)^{-1/2} \exp\left\{\frac{2xyz\rho_i - (x^2+y^2)z^2\rho_i^2}{1 - z^2\rho_i^2}\right\}.\right.$$
$$\left. .F\left(\frac{x - yz\rho_i}{(1-z^2\rho_i^2)^{1/2}}, \frac{\rho_i}{(1-z^2\rho_i^2)^{1/2}}\right)\right],$$

where $\rho_1 = t(u + \sqrt{u^2-1})$, $\rho_2 = t(u - \sqrt{u^2-1})$,

$$\sigma_n(y, z) = \sum_{k=0}^{n} \binom{n}{k} a_{n-k}(z/2)^k H_k(y), \quad |z\rho_i| < 1 \ (i = 1, 2).$$

As a nice application of theorem 2, we take $a_n = (c)_n$ and use the following divergent generating function due to F. Brafman [2] :

$$(3.7)\quad (1-2xt)^{-c}\ {}_2F_0\left(\tfrac{1}{2}c,\ \tfrac{1}{2}c+\tfrac{1}{2}\ ;\ -\ ;\ \frac{-4t^2}{(1-2xt)^2}\right) \cong \sum_{n=0}^{\infty}(c)_n H_n(x)t^n/n\,!$$

in order to derive at once from Theorem 2 the following relation :

$$(3.8)\quad \sum_{n=0}^{\infty} H_n(x)\sigma_n(y,z)T_n(u)t^n/n\,!$$

$$\cong \tfrac{1}{2}\sum_{i=1}^{2}\Big[(1-z^2\rho_i^{\,2})^{c-\frac{1}{2}}(1-z^2\rho_i^{\,2}-2\rho_i x+2yz\rho_i^{\,2})^{-c}.$$

$$.\exp\left(\frac{2xyz\rho_i-(x^2+y^2)z^2\rho_i^{\,2}}{1-z^2\rho_i^{\,2}}\right){}_2F_0\left(\tfrac{1}{2}c,\ \tfrac{1}{2}c+\tfrac{1}{2}\ ;\ -\ ;\ \left(\frac{2\rho_i}{1-z^2\rho_i^{\,2}-2\rho_i x+2yz\rho_i}\right)^2\right)\Big]$$

where $\rho_1=t(u+\sqrt{u^2-1})$, $\rho_2=t(u-\sqrt{u^2-1})$

and $\sigma_n(y,z)=\displaystyle\sum_{k=0}^{n}\binom{n}{k}(c)_{n-k}(z/2)^k H_k(y)$.

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REFERENCES

[1] Al-Salam, W. A.—*Rev. Mat. Fis. Teo. Univ. Nac. Tucuman, Argentina* **13** (1960), 85-93.
[2] Brafman, F.—*Proc. Amer. Math. Soc.* **2** (1951), 942-949.
[3] Carlitz, L.—*Boll. Un. Mat. Ital.* **1** (1970), 43-46.
[4] Lee, P. A.—*Nanta Math.* **8** (1975), 83-87.

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Dept. of Pure Math.
Calcutta University

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ON (p+3)-DIMENSIONAL COMPLEX LIE GROUP AND SPECIAL FUNCTIONS

S. K. CHATTERJEA

1. Introduction : In a recent paper [4], C. F. Wong and R. N. Kesarwani have considered the (p+3)-dimensional complex Lie group K_{p+3} with elements

(1.1) $g=g(\tau ; c ; a_0, a_1, \ldots a_p) ; \tau, c, a_i \in C,$

where the multiplication law is given by

(1.2) $g(\tau ; c ; a_0, a_1, \ldots a_p)\, g(\tau' ; c' ; a'_0, a'_1 \ldots a'_p)$
$= g(\tau+\tau' ; c+e^{-\tau}c' ; \phi_0, \phi_1, \ldots \phi_p)$

and

$$(1.3) \quad \varphi_l = a_l + \sum_{k=l}^{p} \binom{k}{l} c^{k-l} e^{k\tau} a'_k \; ; \; o \leqslant l \leqslant p.$$

Indeed, K_{p+3} is a generalization of 5-dimensional complex Lie group K_5 studied by W. Miller, Jr. [2].

Now the operators $A(g)$, $g \in K_{p+3}$, defining the multiplier representation of K_{p+3} on the complex Vector Space V of all analytic functions $f(z)$ which are defined for all non-zero values of z, are defined by

$$(1.4) \quad [A(g)f](z) = \exp\left(m_0\tau + \mu \sum_{l=0}^{p} a_l z^l\right)(1+c/z)^{w+m_0} f(e^\tau z + e^\tau c),$$

$w+m_0$ being not an integer.

The matrix elements $A_{lk}(g)$ of $A(g)$ with respect to the basis $h_k(z)=z^k$; $k=0, \pm 1, \pm 2, \ldots$, of V, as defined in [4, p 120] are as follows :

$$(1.5) \quad [A(g)h_k](z) = \sum_{l=-\infty}^{\infty} A_{lk}(g) h_l(z) \; ; \; k=0, \pm 1, \pm 2, \ldots$$

$$\text{i.e.} \quad \exp\left[(m_0+k)\tau + \mu \sum_{l=0}^{p} a_l z^l\right](1+c/z)^{m_0+w+k} z^k$$

$$= \sum_{l=-\infty}^{\infty} A_{lk}(g) z^l, \; | c/z | < 1.$$

Wong and Kesarwani have examined one special case of $A_{lk}(g)$, viz. the function ${}_pF_p$, from which ${}_1F_1$ or usual Laguerre function $L_w^{(\alpha)}(z)$ can be easily studied.

The object of the present paper is to examine the general matrix element $A_{lk}(g)$ and then to consider some interesting special cases.

2. General Matrix Element : Let $g=g(\tau\,;\,c\,;\,a_0, a_1,\ldots,a_p)$. Then (1.5) reduces to

(2.1) $$\exp((m_0+k)\tau+\mu a_0)\exp(\mu(a_1z+a_2z^2+\ldots+a_pz^p))(1+c/z)^{m_0+w+k}z^k$$

$$=\sum_{l=-\infty}^{\infty} A_{lk}(g)z^l.$$

Expanding the left member of (2.1) with the help of binomial expansion and the following expansion [3, p. 719]

(2.2) $\exp\,(b_1x+b_2x^2+\ldots)=1+a_1x+a_2x^2+\ldots$

where

$$a_n=\frac{1}{n\,!}\begin{vmatrix} b_1 & -1 & 0 & \ldots & \ldots & 0 \\ 2b_2 & b_1 & -2 & 0 & \ldots & 0 \\ 3b_3 & 2b_2 & b_1 & -3 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}_n$$

$$=\sum\frac{b_1^{e_1}\;b_2^{e_2}\;\ldots b_n^{e_n}}{e_1\,!\,e_2\,!\,..\,e_n\,!}$$

and e_i's are to have all positive integral values inclusive of zero and $e_1+2e_2+3e_3+\ldots+ne_n=n$,

in a power series and computing the coefficient of z^l, we obtain the explicit representation for the matrix element $A_{lk}(g)$ as follows

(2.3) $$A_{lk}(g)=\exp\,((m_0+k)\tau+\mu a_0)\frac{(-c)^{k-l}}{\Gamma(-m_0-w-k)}\sum_{m=0}^{\infty}\frac{\Gamma(m-l-m_0-w)}{\Gamma(k+m-l+1)}\,b_mc^m$$

where

(2.4) $b_0=1$

(2.5) $$b_m=\frac{1}{m!}\begin{vmatrix} \mu a_1 & -1 & 0 & 0 & \ldots & 0 \\ 2\mu a_2 & \mu a_1 & -2 & 0 & \ldots & 0 \\ 3\mu a_3 & 2\mu a_2 & \mu a_1 & -3 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}_m$$

$$=\sum\frac{(\mu a_1)^{e_1}(\mu a_2)^{e_2}\ldots(\mu a_m)^{e_m}}{e_1\,!\,e_2\,!\ldots e_m\,!}$$

and $e_1+2e_2+3e_3+\ldots+me_m=m$.

If may be noted that there is no simple expression for $A_{lk}(g)$, where $g=g(\tau\,;\,c\,;\,a_0, a_1,\ldots\,a_p)$, in terms of functions of hypergeometric type.

However, if we put $\beta_m=(-1)^m b_m$, then we obtain the following system of equations :

$$
(2.6)\quad \begin{cases}
\beta_1-\mu a_1=0\\
2\beta_1-2\mu a_2-\beta_1\mu a_1=0\\
3\beta_3-3\mu a_3-\beta_1.2\mu a_2-\beta_2\mu a_1=0\\
\dots \quad \dots \quad \dots\\
p\beta_p-p\mu a_p-\beta_1(p-1)\mu a_{p-1}-\beta_2(p-2)\mu a_{p-2}-\dots-\beta_{p-1}\mu a_1=0\\
(p+1)\beta_{p+1}-\beta_1 p\mu a_p-\beta_2(p-1)\mu a_{p-1}-\dots-\beta_p\mu a_1=0\\
(p+2)\beta_{p+2}-\beta_2 p\mu a_p-\dots-\beta_p .2\mu a_2=0\\
\dots \quad \dots \quad \dots \quad \dots
\end{cases}
$$

By Laurent expansion the coefficients $A_{lk}(g)$ in (2.1) are given by

(2.7) $A_{lk}(g)$

$$=\frac{1}{2\pi i}\int^{(0+)}\exp[(m_0+k)\tau+\mu(a_0+a_1z+\dots+a_pz^p)](1+c/z)^{m_0+w+k}z^{k-l-1}\,dz,$$

where $g=g(\tau\,;\,c\,;\,a_0,a_1,\dots a_p)$.

Comparing (2.3) with (2.7) we obtain

$$(2.8)\quad \frac{1}{2\pi i}\int^{(0+)}\exp[\mu(a_1z+\dots+a_pz^p)](1+c/z)^{m_0+w+k}z^{k-l-1}dz$$

$$=\frac{(-c)^{k-l}}{\Gamma(-m_0-w-k)}\sum_{m=0}^{\infty}\frac{\Gamma(m-l-m_0-w)}{\Gamma(k+m-l+1)}\,c^m\,\sum\frac{(\mu a_1)^{e_1}(\mu a_2)^{e_2}\dots(\mu a_m)^{e_m}}{e_1\,!\,e_2\,!\dots e_m\,!}$$

where $e_1+2e_2+3e_3+\dots+me_m=m$ and $|c/z|<1$.

3. Special Cases : If in particular, $g=g(\tau\,;\,c\,;\,a,0,\dots 0,b)$, then from (2.3) we have

$$(3.1)\quad A_{lk}(g)=\exp(m_0+k)\tau+\mu a)\frac{(-c)^{k-l}}{\Gamma(-m_0-w-k)}\sum_{m=0}^{\infty}\frac{\Gamma(m-l-m_0-w)}{\Gamma(k+m-l+1)}\beta_m\,(-c)^m$$

where

$$\beta_0=1,\ \beta_1=\beta_2=\dots=\beta_{p-1}=0,\ \beta_p=\mu b,$$

$$\beta_{p+1}=\beta_{p+2}=\dots=\beta_{2p-1}=0,\ \beta_{2p}=\frac{(\mu b)^2}{2\,!},$$

$$\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots$$

$$\beta_{kp}=\frac{(\mu b)^k}{k\,!}.$$

Hence we deduce from (3.1)

$$(3.2)\quad A_{lk}(g)=\exp((m_0+k)\tau+\mu a)\frac{(-c)^{k-l}}{\Gamma(-m_0-w-k)}\sum_{m=0}^{\infty}\frac{\Gamma(mp-l-m_0-w)}{\Gamma(mp+k-l+1)}\,.\,[\mu b(-c)^p]^m/m\,!$$

which has been obtained by Wong and Kesarwani for ${}_pF_p$. This ${}_pF_p$ is a particular case of our matrix element $A_{lk}(g)$ given in (2.3).

Again in particular, if $g=g(0\,;0\,;0,a,b,0,\ldots 0)$, then the generating function (2.1) becomes

$$(3.3)\quad \exp(\mu(az+bz^2))z^k=\sum_{l=-\infty}^{\infty}A_{lk}(g)z^l.$$

Comparing (3.3) with the generating function of the Hermite polynomials, viz.

$$(3.4)\quad \exp(2xy-y^2)=\sum_{l=0}^{\infty}\frac{y^l}{l\,!}H_l(x),$$

we obtain

$$(3.5)\quad A_{lk}(g)=\begin{cases}0 \text{ if } l<k\\ \dfrac{(-\mu b)^{(l-k)/2}}{(l-k)\,!}H_{l-k}\left(\dfrac{\mu a}{2(-\mu b)^{1/2}}\right), \text{ if } l\geqslant k.\end{cases}$$

Furthermore, in particular, if $g=g(0\,;0\,;0,a,0,\ldots 0,b)$, then the generating function (2.1) becomes

$$(3.6)\quad \exp(\mu(az+bz^p))z^k=\sum_{l=-\infty}^{\infty}A_{lk}(g)z^l.$$

Comparing (3.6) with the generating function of the polynomials $g_l^p(x)$ considered by L. R. Bragg [1], viz.

$$(3.7)\quad \exp(pzx-z^p)=\sum_{l=0}^{\infty}\frac{z^l}{l\,!}g_l{}^p(x),$$

we obtain

$$(3.8)\quad A_{lk}(g)=\begin{cases}0 \text{ if } l<k\\ \dfrac{(-\mu b)^{(l-k)/p}}{(l-k)\,!}g_{l-k}^p\left(\dfrac{\mu a}{p(-\mu b)^{1/p}}\right), \text{ if } l\geqslant k.\end{cases}$$

When $p=2$, $A_{lk}(g)$ in (3.8) reduces to that in (3.5).

REFERENCES

[1] Bragg, L. R.—*Boll. Un. Mat. Ital.* 1, Ser IV (1968), 347-354.
[2] Miller, W. Jr.—Lie theory and special functions, Academic Press, New York (1968) p. 300.
[3] Muir, T.—A treatise on the theory of determinants, Dover Publ. Inc., New York (1928), p. 719.
[4] Wong, C. F. and Kesarwani, R. N.—*Portugal Math.* **34** (1975), 119-125.

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Dept. of Pure Math.
Calcutta University

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ON A FIXED POINT THEOREM OF KIYOSHI

KANAN MAJUMDAR

1. Introduction : In a recent paper [6], I. Kiyoshi generalized a theorem of K. Goebel and E. Zlotkiewicz [3] in the following form :

Theorem 1. Let F be a mapping of a Banach space X into itself. If F satisfies the conditions

(1) $F^2=I$, where I is the identity mapping

(2) $\| F(x)-F(y) \| \leqslant \alpha \| x-y \| + \beta(\| x-F(x) \| + \| y-F(y) \|)$ for every $x, y \in X$, where $0 \leqslant \alpha, \beta$ and $0<\alpha+4\beta<2$,

then F has at least one fixed point.

It may be of interest to remark that the first part in the right member of (2) is involved in the contraction mapping of S. Banach [1] and the second part of (2) is contained in the contraction type mapping of R. Kannan [5]. Also S. K. Chatterjea [2] introduced another contraction type mapping by means of the metric relation

$$d(F(x), F(y)) \leqslant \alpha[d(x, F(y))+d(y, F(x))].$$

Finally G. E. Hardy and T. D. Rogers [4] introduced the concept of generalized contraction type mapping by means of the metric relation

$$d(F(x), F(y)) \leqslant \alpha_1 d(x, y)+\alpha_2 d(x, F(x))+\alpha_3 d(y, F(y))+\alpha_4 d(x, F(y))+\alpha_5 d(y, F(x)).$$

Noticing these various types of contraction type relations, we are led to consider an extension of the theorem of Kiyoshi in the following form :

Theorem 2. Let F be a mapping of a Banach space X into itself. If F satisfies the conditions

(i) $F^3=I$ where I is the identity mapping

(ii) $\| F(x)-F(y) \| \leqslant \alpha \| x-y \| + \beta \| x-F(x) \| + \gamma \| y-F(y) \| + \delta \| x-F(y) \| + \epsilon \| y-F(x) \|$

for every $x, y \in X$ where $0 \leqslant \alpha, \beta, \gamma, \delta, \epsilon$ and $\alpha^2+2\alpha(\beta+\gamma+\delta+\epsilon+1)+3(\beta+\gamma+\delta+\epsilon)<1$,

then F has at least one fixed point.

2. Proof of theorem 2. Let x be a point of X and we put $y=\frac{1}{2}(F+I)(x)$, $z=F(y)$, $u=2y-3$. Now by conditions (i) and (ii) we have

$$\| z-x \| = \| F(y)-F^3(x) \| \leqslant \alpha \| y-F^2(x) \| + \beta \| y-F(y) \| + \gamma \| F^2(x)-F^3(x) \| + \delta \| y-F^3(x) \| + \epsilon \| F^2(x)-F(y) \|$$

Again, by symmetry

$$\| z-x \| = \| F^2(x)-F(y) \| \leqslant \alpha \| F^2(x)-y \| +\beta \| F^2(x)-F^3(x) \| \\ +\gamma \| y-F(y) \| +\delta \| F^2(x)-F(y) \| +\epsilon \| y-F^3(x) \|$$

Thus it follows from the above two relations that

$$\| z-x \| \leqslant \alpha \| y-F^2(x) \| +\frac{\beta+\gamma}{2} \| y-F(y) \| +\frac{\beta+\gamma}{2} \| F^2(x)-F^3(x) \| \\ +\frac{\delta+\epsilon}{2} \| F^2(x)-F(y) \| +\frac{\delta+\epsilon}{2} \| y-F^3(x) \|$$

Now $\| y-F^2(x) \| \leqslant \| y-F(y) \| + \| F(y)-F^2(x) \|$

$$\leqslant \| y-F(y) \| +\alpha \| y-F(x) \| +\beta \| y-F(y) \| +\gamma \| F(x)-F^2(x) \| \\ +\delta \| y-F^2(x) \| +\epsilon \| F(x)-F(y) \|$$

$$\leqslant \| y-F(y) \| +\frac{\alpha}{2} \| x-F(x) \| +\beta \| y-F(y) \| +\gamma \| F(x)-y \| +\gamma \| y-F^2(x) \| \\ +\delta \| y-F^2(x) \| +\epsilon \| F(x)-y \| +\epsilon \| y-F(y) \|$$

$$\leqslant (1+\beta+\epsilon) \| y-F(y) \| +\tfrac{1}{2}(\alpha+\gamma+\epsilon) \| x-F(x) \| +(\gamma+\delta) \| y-F^2(x) \|$$

Also by symmetry,

$$\| y-F^2(x) \| = \| F^2(x)-y \| \leqslant \| F^2(x)-F(y) \| + \| F(y)-y \| \\ = \| y-F(y) \| + \| F^2(x)-F(y) \| \\ \leqslant \| y-F(y) \| +\alpha \| F(x)-y \| + \\ \beta \| F(x)-F^2(x) \| +\gamma \| y-F(y) \| +\delta \| F(x)-F(y) \| +\epsilon \| y-F^2(x) \| \\ \leqslant \| y-F(y) \| +\frac{\alpha}{2} \| x-F(x) \| +\beta \| F(x)-y \| + \\ \beta \| y-F^2(x) +\gamma \| y-F(y) \| +\delta \| F(x)-y \| +\delta \| y-F(y) \| +\epsilon \| y-F^2(x) \|$$

Thus we have,

$$\| y-F^2(x) \| \leqslant \frac{2+\beta+\gamma+\delta+\epsilon}{2-(\beta+\gamma+\delta+\epsilon)} \| y-F(y) \| +\frac{2\alpha+\beta+\gamma+\delta+\epsilon}{2\{2-(\beta+\gamma+\delta+\epsilon)\}} \| x-F(x) \|$$

and $\| F^3(x)-F^2(x) \| \leqslant \| y-F^3(x) \| + \| y-F^2(x) \|$

$$\therefore \quad \| z-x \| \leqslant \alpha \| y-F^2(x) \| +\frac{\beta+\gamma}{2} \| F^2(x)-y \| +\frac{\beta+\gamma}{2} \| y-F^3(x) \| + \\ \frac{\beta+\gamma}{2} \| y-F(y) \| +\frac{\delta+\epsilon}{2} \| F^2(x)-y \| +\frac{\delta+\epsilon}{2} \| y-F(y) \| +\frac{\delta+\epsilon}{2} \| y-F^3(x) \|$$

Also, $\| u-x \| = \| F(x)-F(y) \| \leqslant \| F(x)-y \| + \| y-F(y) \|$

$$= \tfrac{1}{2} \| x-F(x) \| + \| y-F(y) \|$$

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Now, $\| z-u \| \leqslant \| z-x \| + \| u-x \|$

and $\| z-u \| = 2 \| y-F(y) \|$

$$\therefore \quad 2\|y-F(y)\| \leqslant \frac{2\alpha+\beta+\gamma+\delta+\epsilon}{2}\left[\frac{2+\beta+\gamma+\delta+\epsilon}{2-(\beta+\gamma+\delta+\epsilon)}\|y-F(y)\| + \frac{2\alpha+\beta+\gamma+\delta+\epsilon}{2\{2-(\beta+\gamma+\delta+\epsilon)\}}\|x-F(x)\|\right]+\frac{\beta+\gamma+\delta+\epsilon}{2}\|y-F(y)\| + \frac{\beta+\gamma+\delta+\epsilon}{2}\|x-F(x)\| + \tfrac{1}{2}\|x-F(x)\| + \|y-F(y)\|$$

In other words, we have

$$\|y-F(y)\| \leqslant \frac{4\alpha^2+4\alpha(\beta+\gamma+\delta+\epsilon)+4}{2\{4-6(\beta+\gamma+\delta+\epsilon)-2\alpha(\beta+\gamma+\delta+\epsilon)-4\alpha\}}\|x-F(x)\|$$

Let $G=\frac{1}{2}(F+I)$. Now for any $x \in X$, we have

$$\begin{aligned}\|G^2(x)-G(x)\| &= \|G(y)-y\| = \|\tfrac{1}{2}(F+I)(y)-y\| = \tfrac{1}{2}\|y-F(y)\| \\ &\leqslant \frac{1}{2}\,\frac{4\alpha^2+4\alpha(\beta+\gamma+\delta+\epsilon)+4}{2\{4-6(\beta+\gamma+\delta+\epsilon)-2\alpha(\beta+\gamma+\delta+\epsilon)-4\alpha\}}\|x-F(x)\| \\ &= \frac{1}{2}\,\frac{\alpha^2+\alpha(\beta+\gamma+\delta+\epsilon)+1}{2-3(\beta+\gamma+\delta+\epsilon)-\alpha(\beta+\gamma+\delta+\epsilon)-2\alpha}\|x-F(x)\| \\ &= \frac{\alpha^2+\alpha(\beta+\gamma+\delta+\epsilon)+1}{2-3(\beta+\gamma+\delta+\epsilon)-\alpha(\beta+\gamma+\delta+\epsilon)-2\alpha}\|G(x)-x\|.\end{aligned}$$

By the hypothesis, we have

$$0 \leqslant \left[\frac{\alpha^2+\alpha(\beta+\gamma+\delta+\epsilon)+1}{2-3(\beta+\gamma+\delta+\epsilon)-\alpha(\beta+\gamma+\delta+\epsilon)-2\alpha}\right] < 1.$$

Thus $\{G^n(x)\}$ is a Cauchy sequence in X. By the completeness $\{G^n(x)\}$ converges to some element x_0 in X, i.e. $\lim_{n\to\infty} G^n(x)=x_0$. Thus $G(x_0)=x_0$. Hence $F(x_0)=x_0$ is a fixed point of F. This completes the proof of our proposed theorem.

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REFERENCES

[1] Banach, S.—*Fund. Math.* **3** (1922), 133-181.

[2] Chatterjea, S. K.—*C. R. Acad, Bulgare Sci.* **25** (1972) 727-730 (abstract published in Research Report No. 2 (1971) Centre of Adv. Studies in Appl. Math. Cal. Univ. p. 12).

[3] Goebel, K. and Zlotkiewicz, E.—*Colloquium Math.* **23** (1971) 103-106.

[4] Hardy, G. E. and Rogers, T. D.—*Canad. Math. Bull.* **16** (1973), 201-206.

[5] Kannan, R.—*Bull. Cal. Math. Soc.* **60** (1968) 71-76.

[6] Kiyoshi, I.—*Math. Balkanica* **5** (1975) 143-144.

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Canning Dwarikanath Balika Vidyalaya
24 Parganas

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ON STARLIKE FUNCTIONS

S. K. CHATTERJEA

1. Introduction : In a recent paper [1], Ming-Po Chen has considered the class of analytic functions

(1.1) $$f(z)=z+\sum_{k=n+1}^{\infty} a_k z^k, \quad n\geqslant 1$$

satisfying the condition

(1.2) $$| zf'(z)/f(z)-1 | <\alpha$$

for a given α, $0<\alpha\leqslant 1$, for $|z|<1$.

It is well known that if S denote the class of functions of the form $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ that are analytic and univalent in the unit disk $|z|<1$, then f is said to be starlike of order α, denoted by $f\in S_\alpha$, if $Re\{zf'(z)/f(z)\}>\alpha$ $(|z|<1)$. A subclass $S_{\{\alpha\}}$ of S_α consisting of those $f(z)$ for which $|zf'(z)/f(z)-1|<1-\alpha$ for $|z|<1$ was considered by C. P. Mc Carty [2]. Also it is known that if S denote the class of functions $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ that are analytic and univalent in the unit disk $|z|<1$, then f is said to be convex of order α $(0\leqslant\alpha<1)$, denoted by $f\in K_\alpha$, if $Re\{1+zf''(z)/f'(z)\}>\alpha$ $(|z|<1)$.

Recently H. Silverman [3] has considered the subclass T (of the class of functions of the form $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ that are analytic and univalent in the unit disk $|z|<1$) consisting of functions expressible in the form

(1.3) $$f(z)=z-\sum_{n=2}^{\infty} |a_n| z^n.$$

In this paper we like to consider the class of functions(1.1) of Ming-Po Chen from the view-point of coefficient inequalities. For this purpose we make the following definitions :

Definition 1. If S(n) denote the class of functions (1.1) that are analytic and univalent in the unit disk $|z|<1$, then f is said to be starlike of order α $(0\leqslant\alpha<1)$, denoted by $f\in S_\alpha(n)$, if $Re\,\{zf'(z)/f(z)\}>\alpha$ $(|z|<1)$.

A subclass $S_{\{\alpha\}}(n)$ of $S_\alpha(n)$ consists of those functions $f(z)$ for which $|zf'(z)/f(z)-1|<1-\alpha$ for $|z|<1$.

Definition 2. If S(n) denote the class of functions (1.1) that are analytic and univalent in the unit disk $|z|<1$, then f is said to be convex of order α $(0\leqslant\alpha<1)$, denoted by $f\in K_\alpha(n)$, if $Re\,\{1+zf''(z)/f'(z)\}>\alpha$ $(|z|<1)$.

Definition 3. Let T be the subclass of functions (1.1) consisting of functions expressible in the form

(1.4) $$f(z)=z-\sum_{k=n+1}^{\infty}|a_k|z^k,\ n\geqslant 1.$$

Then $T_\alpha(n)$ and $C_\alpha(n)$ are defined respectively as the subclasses of T that are starlike of order α and convex of order α.

2. Sufficient condition for $f(z)\in S_{\{\alpha\}}(n)$

Theorem 1. Let $f(z)=z+\sum_{k=n+1}^{\infty}a_k z^k,\ n\geqslant 1.$

If $\sum_{k=n+1}^{\infty}[(k-\alpha)/(1-\alpha)].\,|a_k|\leqslant 1,$

then $f(z)\in S_{\{\alpha\}}(n)$ for $\alpha\in[0,1)$.

Proof. On $|z|=1$, we have

$$(1-\alpha)|f(z)|-|zf'(z)-f(z)|$$

$$=(1-\alpha)\left|z+\sum_{k=n+1}^{\infty}a_k z^k\right|-\left|\sum_{k=n+1}^{\infty}(k-1)a_k z^k\right|$$

$$\geqslant(1-\alpha)-\sum_{k=n+1}^{\infty}(k-\alpha)|a_k|\geqslant 0, \text{ by hypothesis.}$$

In other words,

$$|zf'(z)/f(z)-1|\leqslant 1-\alpha.$$

Hence by the maximum modulus theorem we have

(2.1) $$|zf'(z)/f(z)-1|<1-\alpha \text{ for } |z|<1.$$

So $f(z)\in S_{\{\alpha\}}(n)$ for $\alpha\in[0,1)$.

Special case of theorem 1 was proved by Mc Carty [2] for $n=1$. It may be noted that theorem 1 relates the modulus of coefficients to the order of starlikeness. Further we remark that $f(z)=z-[(1-\alpha)/(k-\alpha)]z^k$ is an extremal function with respect to the above theorem since $|zf'(z)/f(z)-1|=1-\alpha$ for $z=1$, $\alpha\in[0,1)$, $k=n+1, n+2, \ldots$ and $n\geqslant 1$.

The condition of theorem 1 is not necessary owing to the fact that

$$f(z)=ze^{(1-\alpha)z^n/n}\in S_{\{\alpha\}}(n),$$

whereas

$$\sum_{k=n+1}^{\infty}[(k-\alpha)/(1-\alpha)]\,|a_k|=\sum_{m=1}^{\infty}\frac{mn+1-\alpha}{1-\alpha}\cdot\frac{(1-\alpha)^m}{m!\,n^m}$$

$$>2e^{(1-\alpha)/n}-1>1,$$

for all $\alpha\in[0,1)$ $n\geqslant 1$.

3. Sufficient condition for $f(z) \in S_\alpha(n)$

Theorem 2. Let $f(z)=z+\sum_{k=n+1}^{\infty} a_k z^k$, $n \geqslant 1$.

If $\sum_{k=n+1}^{\infty} [(k-\alpha)/(1-\alpha)].\ |a_k| \leqslant 1$,

then $f(z) \in S_\alpha(n)$ for $\alpha \in [0, 1)$.

Proof. It is sufficient to show that the values for zf'/f lie in a circle centered at $w=1$ whose radius is $1-\alpha$. In other words, we are to show that $|zf'(z)/f(z)-1| < 1-\alpha$ for $|z|<1$, which is already proved in theorem 1 under the same hypothesis. Hence the theorem is proved.

Special case of theorem 2 has been proved by Silverman [3] for $n=1$. Also particular cases of theorem 2 were proved by Goodman [4] for $n=1$, $\alpha=0$, and by Schild [5] for $n=1$, $\alpha=\frac{1}{2}$.

Corollary. Let $f(z)=z+\sum_{k=n+1}^{\infty} a_k z^k$, $n \geqslant 1$,

If $\sum_{k=n+1}^{\infty} [k(k-\alpha)/(1-\alpha)]\ |a_k| \leqslant 1$, then $f(z) \in K_\alpha(n)$.

Proof. $f(z) \in K_\alpha(n)$ if and only if $z\,f'(z) \in S_\alpha(n)$.

Now since $zf'=z+\sum_{k=n+1}^{\infty} k a_k z^k$, one may replace a_k with ka_k in theorem 2.

4. Necessary and sufficient condition for $f(z) \in T_\alpha(n)$

Theorem 3. A function $f(z)=z-\sum_{k=n+1}^{\infty} |a_k|\, z^k$, $n \geqslant 1$, is in $T_\alpha(n)$ if and only if

$$\sum_{k=n+1}^{\infty} [(k-\alpha)/(1-\alpha)]\ |a_k| \leqslant 1.$$

Proof. The sufficiency part follows from theorem 2. Now to prove the necessary part we assume that

$$(4.1) \quad Re\,\{zf'(z)/f(z)\} = Re \left\{ \frac{z-\sum_{k=n+1}^{\infty} k\,|a_k|\,z^k}{z-\sum_{k=n+1}^{\infty} |a_k|\,z^k} \right\} > \alpha, \ (\,|z|<1).$$

Choosing values of z on the real axis so that zf'/f is real and letting $z \to 1$ through real values, we obtain from (4·1)

$$1-\sum_{k=n+1}^{\infty} k\,|a_k| \geqslant \alpha\,(1-\sum_{k=n+1}^{\infty} |a_k|\,),$$

which is equivalent to

$$\sum_{k=n+1}^{\infty} [(k-\alpha)/(1-\alpha)]\ |a_k| \leqslant 1.$$

This completes the proof.

Corollary 1. If $f(z) \in T_\alpha(n)$ then $|a_k| \leqslant (1-\alpha)/(k-\alpha)$, with equality only for functions of the form $f_k(z)=z-[(1-\alpha)/(k-\alpha)]z^k$.

Corollary 2. A function $f(z)=z-\sum_{k=n+1}^{\infty}|a_k|z^k$, $n\geqslant 1$, is in $C_\alpha(n)$ if and only if $\sum_{k=n+1}^{\infty}[k(k-\alpha)/(1-\alpha)]|a_k| \leqslant 1$.

Proof. The proof follows as that of corollary to theorem 2.

5. Remarks on Starlike Functions

We know that [6] an analytic function which is normalized by the condition $f(0)=f'(0)-1=0$, is said to be in the class of functions known as prestarlike of order α, $0\leqslant\alpha<1$, if $f*g_\alpha \in S_\alpha$ where $g_\alpha(z)=z/(1-z)^{2(1-\alpha)}$ and $f*g_\alpha$ is the Hadamard product of $f(z)$ and $g_\alpha(z)$. Moreover a necessary and sufficient condition for f to be prestarlike of order α is that the functional

$$G(\alpha,z)=\left\{f(z)*\frac{g_\alpha(z)}{1-z}\right\}\Big/\left\{f(z)*g_\alpha(z)\right\}$$

satisfies $Re\, G(\alpha, z)>1/2$ $(|z|<1)$.

Since the Hadamard product of two starlike functions of the same order is a starlike function of the same order, it follows that all starlike functions of order α are obviously prestarlike functions of order α. Hence the necessary condition in order that a function $f(z)$ is starlike of order α is that

$$Re\, G(\alpha, z)>1/2 \quad (|z|<1).$$

REFERENCES

[1] CHEN, MING-FO, On a class of starlike functions, *Nanta Math.* **8** (1975), 79-82.

[2] Goodman, A. W.,—Univalent functions and non-analytic curves, *Proc. Amer. Math. Soc.* **8** (1957), 598-601.

[3] Mc Carty, Carl P.,—Starlike functions, *Proc. Amer. Math. Soc.* **43** (1974), 361-366.

[4] Schild, A.,—On a class of univalent star shaped mappings, *Proc. Amer. Math. Soc.* **9** (1958), 751-757.

[5] Silverman, Herb,—Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.* **51** (1975), 109-116.

[6] Silverman, H. and Silvia, E. M.,—Prestarlike functions with negative coefficients, *Internat. J. Math. & Math. Sci* **2** (1979), 427-439.

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Dept. of Pure Math.
Calcutta University

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A NEW CLASS OF GENERATING RELATIONS INVOLVING LAGUERRE AND GEGENBAUER POLYNOMIALS

A. K. CHONGDAR

1. Introduction : Recently unified theory for obtaining generating functions of special functions has received much attention owing to the fact that a number of particular bilateral (or bilinear) generating functions for some special functions, for example the well known Hille-Hardy formula for Laguerre polynomials, Mehler's formula for Hermite polynomials and others, are available in the literature. Such unified theory consists in deriving a class of bilateral or trilateral generating functions for special functions whereby the particular bilateral or trilateral generating functions will follow easily as consequences of the theory. A question in this direction was raised by C. Truesdell [7] and some answers are given by W. A. Al-Salam [1] and S. K. Chatterjea [3,4,] in connection with bilateral generating functions involving Laguerre, Hermite and Gegenbauer polynomials in recent years. It is interesting to note that the particular bilateral generating relation of L. Weisner [8], viz.

$$(1.1)\quad \rho^{-2\lambda}\ \exp\left[\frac{-yt(x-t)}{\rho^2}\right]\ {}_0F_1\left[-\ ;\ \lambda+\tfrac{1}{2},\frac{y^2t^2(x^2-1)}{4\rho^4}\right]$$

$$=\sum_{r=0}^{\infty} r\,!\ L_r^{(2\lambda-1)}(y)\ C_r^{\lambda}(x)\frac{t^r}{(2\lambda)_r}\ ;\quad \rho=(1-2xt+t^2)^{1/2}$$

follows at once from the following theorem of Chatterjea on a class of bilateral generating functions for Gegenbauer polynominals :

If there exists a unilateral generating relation of the form

$$(1.2)\quad F(x,t)=\sum_{m=0}^{\infty} a_m t^m C_m^{\lambda}(x)$$

then there will exist a bilateral generating relation of the form

$$(1.3)\quad \rho^{-2\lambda}\ F\left(\frac{x-t}{\rho},\ \frac{ty}{\rho}\right)=\sum_{r=0}^{\infty} t^r b_r(y) C_r^{\lambda}(x)$$

where

$$b_r(y)=\sum_{m=0}^{r}\binom{r}{m}a_m y^m.$$

The importance of this class of bilateral generating relations lies in the fact that one can at once derive a large number of bilateral generating relations for Gegenbauer polynomials by attributing different values to a_m.

Now in the investigation of such class of generating relations, group theoretic-method seems to be a potent one in comparison with analytic method, because the unknown generating function can only be obtained by group theoretic-method, whereas the known generating function can be verified and then extended by analytic method. As an illustration of this statement we may cite the following extension of Mehler's formula by Chatterjea [5] :

$$(1.4)\sum_{k=0}^{\infty}\frac{t^k}{2^k k\,!}\sum_{r=0}^{n}2^r r\,!\binom{n}{r}\binom{n}{r}\left(\frac{-t}{1-t^2}\right)^r \mathrm{H}_{k+n-r}(x)\,\mathrm{H}_{k+n-r}(y)$$

$$=\left(1-t^2\right)^{-(n+\frac{1}{2})}\exp\left[\frac{2xyt-(x^2+y^2)t^2}{1-t^2}\right]\mathrm{H}_n\left(\frac{x-yt}{\sqrt{1-t^2}}\right)\mathrm{H}_n\left(\frac{y-xt}{\sqrt{1-t^2}}\right),$$

which could not be derived by analytic method prior to its existence. In fact, the very nature of Chatterjea's formula helps L. Carlitz [2] to extend further.

So in the present paper we shall adopt group-theoretic method to obtain a new class of mixed trilateral generating relation involving Laguerre and Gegenbauer polynomials. We shall use the raising operators of the Lie algebras in connection with Laguerre and Gengenbauer polynomials [6]. Our main theorem can be stated in the following form :

Theorem : If there exists a bilateral generating relation of the form

$$(1.5)\qquad \mathrm{G}(x,z,w)=\sum_{n=0}^{\infty}a_n\mathrm{L}_n^{(\alpha)}(x)\mathrm{C}_n^{\lambda}(z)w^n$$

then there exists a mixed trilateral generating relation

$$(1-w)^{-\alpha-1}\,(1-2wz+w^2)^{-\lambda}\,\exp\left(-\frac{wx}{1-w}\right).$$

$$\cdot\,\mathrm{G}\left(\frac{x}{1-w},\ \frac{z-w}{\sqrt{w^2-2wz+1}},\ \frac{wv}{(1-w)\sqrt{w^2-2wz+1}}\right)$$

$$(1.6)\qquad =\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}f_{m\,n}\,(w,v,x)\,\mathrm{C}_n^{\lambda}(z)$$

where

$$f_{m,n}(w,v,x)=\sum_{p=0}^{\min(m,n)}\binom{m+n-2p}{n-p}\binom{n}{p}a_{n-p}\,w^{m+n-p}\,v^{n-p}\,\mathrm{L}_{n+m-2p}^{(\alpha)}(x)$$

and $\binom{n}{p}=\dfrac{n\,!}{p\,!\,(n-p)\,!}$.

The above theorem is illustrated by means of a well known bilateral generating relation (1.1) due to L. Weisner.

2. Group-theoretic method : For the Laguerre polynomials $L_n^{(\alpha)}(x)$ defined by

$$(2.1) \quad (1-t)^{-\alpha-1} \exp\left(\frac{-xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\, t^n,$$

we consider the operator R_1, where

$$(2.2) \quad R_1 = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + (-x+\alpha+1)y$$

such that

$$(2.3) \quad R_1[F_n^{(\alpha)}(x, y)] = (n+1)\, F_{n+1}^{(\alpha)}(x, y), \text{ where } F_n^{(\alpha)}(x, y) = L_n^{(\alpha)}(x)\, y^n.$$

The corresponding extended form of the group generated by R_1 is given by

$$(2.4) \quad (\exp wR_1)f(x, y) = (1-w)^{-\alpha-1} \exp\left(\frac{-wxy}{1-wy}\right) f\left(\frac{x}{1-wy}, \frac{y}{1-wy}\right).$$

Also for the Gegenbauer polynomials $C_n^{\lambda}(x)$ defined by

$$(2.5) \quad (1-2zt+t^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^{\lambda}(z)\, t^n,$$

we consider the operator R_2, where

$$(2.6) \qquad R_2 = (z^2-1)t\frac{\partial}{\partial z} + zt^2\frac{\partial}{\partial t} + 2\lambda zt$$

such that

$$(2.7) \qquad R_2[F_n^{\lambda}(z, t)] = (n+1)F_{n+1}^{\lambda}(z, t), \text{ where } F_n^{\lambda}(z, t) = C_n^{\lambda}(z)t^n$$

The corresponding extended form of the group generated by R_2 is given by

$$(2.8) \qquad (\exp wR_2)\, f(z, t)$$

$$= (w^2t^2-2wzt+1)^{-\lambda} f\left(\frac{z-wt}{\sqrt{w^2t^2-2wzt+1}}, \frac{t}{\sqrt{w^2t^2-2wzt+1}}\right)$$

Let us consider the bilateral generating function

$$(2.9) \qquad G(x, z, w) = \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x) C_n^{\lambda}(z) w^n$$

Replacing w by $wytv$, we get

$$(2.10) \qquad G(x,\, z,\, wytv)=\sum_{n=0}^{\infty} a_n\{L_n^{(\alpha)}(x)y^n\}\{C_n^{\lambda}(z)t^n\}(wv)^n$$

$$=\sum_{n=0}^{\infty} a_n F_n^{(\alpha)}(x,\, y)F_n^{\lambda}(z,\, t)(wv)^n$$

Applying the operator $(\exp wR_1)(\exp wR_2)$ on both sides of (2.10) we get

$$(2.11) \qquad (\exp wR_1)(\exp wR_2),\ G(x,\, z,\, wytv)$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty} a_n \frac{w^m}{m\,!} R_1^m F_n^{(\alpha)}(x,\ y)\ \frac{w^p}{p\,!} R_2^p F_n^{\lambda}(z,\, t)(wv)^n$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\min(m,n)} a_{n-p}(n-p+1)_{m-p}(n-p+1)_p\, v^{n-p}\ \frac{w^{n+m-p}}{(m-p)\,!\ p\,!}$$

$$F_{n+m-2p}^{(\alpha)}(x,\, y)\ F_{n}^{\lambda}(z,\, t)$$

where $(a)_n=a(a+1)\ldots\ldots\ (a+n-1)$.

But

$$(\exp wR_1)(\exp wR_2)\ G(x,\, z,\, wytv)$$

$$(2.12) \qquad =(1-w)^{-\alpha-1}(w^2t^2-2wzt+1)^{-\lambda} \exp\left(\frac{-wxy}{1-wy}\right).$$

$$.G\left(\frac{x}{1-wy},\ \frac{z-wt}{\sqrt{w^2t^2-2wzt+1}},\ \frac{wytv}{(1-wy)\sqrt{w^2t^2-2wzt+1}}\right).$$

So we get from (2.11) and (2.12)

$$(1-w)^{-\alpha-1}(w^2-2wz+1)^{-\lambda} \exp\left(-\frac{wx}{1-w}\right).$$

$$.G\left(\frac{x}{1-w},\ \frac{z-w}{\sqrt{w^2-2wz+1}},\ \frac{wv}{(1-w)\sqrt{w^2-2wz+1}}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f_{m,n}(w,\, v,\, x)\ C_n^{\lambda}(z)$$

where $f_{m,n}(w,v,x)=\sum_{p=0}^{\min(m,n)}\binom{m+n-2p}{n-p}\binom{n}{p}a_{n-p}w^{m+n-p}\,v^{n-p}\,L_{n+m-2p}^{(\alpha)}(x)$,

on putting $y=t=1$, which is (1.6).

3. Application : As an application of the above theorem we consider the following generating function due to Weisner :

$$(1-2wz+w^2)^{-\lambda}\exp\left\{-\frac{wx(z-w)}{1-2wz+w^2}\right\}\ {}_0F_1\left[-\ ;\ \lambda+\tfrac{1}{2}\ ;\ \frac{x^2w^2(z^2-1)}{4(1-2wz+w^2)^2}\right]$$

$$(3.1)\qquad =\sum_{n=0}^{\infty}\frac{n!}{(2\lambda)_n}\,L_n^{(2\lambda-1)}(x)\ C_n^{\lambda}(z)w^n.$$

Putting $a_n=\dfrac{n!}{(2\lambda)_n}$, $\alpha=2\lambda-1$, $\nu=1$ in our theorem, we obtain

$$\rho^{-2\lambda}\exp\left[-\frac{wx}{1-w}\left\{1+\frac{(1-w)(z-w)-w}{\rho^2}\right\}\right]{}_0F_1\left[-\ ;\ \lambda+\tfrac{1}{2}\ :\ \frac{x^2w^2(z^2-1)}{4\rho^4}\right]$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}f_{m,n}(w,\,x)\ C_n^{\lambda}(z),$$

where $f_{m,n}(w,\,x)=\displaystyle\sum_{p=0}^{\min(m,n)}\binom{n+m-2p}{n-p}\binom{n}{p}\frac{(n-p)!}{(2\lambda)_{n-p}}\,w^{m+n-p}L_{n+m-2p}^{(2\lambda-1)}(x),$

and $\rho^2=(1-w)^2(w^2-2wz+1)-2w(z-w)(1-w)+w^2,$

which (viz, this particular case) also does not seem to appear earlier.

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REFERENCES

[1] Al-Salam, W. A.—Operational representations for the Laguerre and other polynomials, *Duke Math. Jour.* **31** (1964), 127-142.

[2] Carlitz, L.—An extension of Mehler's formula, *Boll. Un. Mat. Ital.* (1970), 43-46.

[3] Chatterjea, S. K.—Une nouvelle fonction génératrice bilatéral pour les polynomes d' Hermite, *C. R. Acad. Sc. Paris Série A*, **266** (1968), 979-981.

[4] Chatterjea, S. K.—A bilateral generating function for the ultraspherical polynomials, *Pacific Jour. Math.* **29**, No. 1 (1969), 73-76.

[5] Chatterjea, S. K.—Quelques fonction génératrice des polynomes d' Hermite, du point de vue de l'algebra de Lie, *C. R. Acad. Sc. Paris. Série A* **268** (1969) 600-602.

[6] McBride, E. B.—Obtaining generating functions, Springer-Verlag, New York Heidelberg, Berlin (1971).

[7] Truesdell, C—Special function, Princeton University Press, Annals Math. Studies No-18 (1948).

[8] Weisner, L.—Group-theoretic origin of certain generating functions, *Pacific Jour. Math.* **5** (1955) 1033-1039.

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Bangabasi Evening College
Calcutta-700 009

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ON AN ASYMPTOTIC FORMULA FOR A SERIES INVOLVING THE EIGENVALUES OF A DIFFERENTIAL OPERATOR

N. K. CHAKRAVARTY and S. K. ACHARYYA

1. The Problem: Let $C^k(R)$, $R: 0 \leqslant x < \infty$, be the set of all real-valued functions, having k continuous derivatives on R.

Consider the differential equations

$$\begin{aligned} \frac{d^2u}{dx^2} - pu &= -\lambda u \\ \frac{d^2v}{dx^2} - qv &= -\lambda v \end{aligned} \tag{1.1}$$

where $p, q \in C^1(R)$ or p, q are absolutely continuous over any compact sub-interval of R, and $\lambda \in C$, the set of all complex numbers. Let u, v be linearly independent, $\phi = \binom{u}{v}$, and $\phi \in D$, the basic Hilbert space $L_2\,[0, \infty)$. Let us assume further that $p\phi, q\phi \in D$.

The boundary conditions considered are

$$u(0) = v(0) = 0 \text{ and } u, v \in L_2 \text{ at } \infty \tag{1.2}$$

$$\text{or } u'(0) = v'(0) = 0 \text{ and } u, v \in L_2 \text{ at } \infty. \tag{1.3}$$

As usual we call (1.2) the Dirichlet and (1.3) the Neumann boundary conditions.

The differential equations (1.1) with either the Dirichlet or the Neumann boundary conditions, give rise to an eigenvalue problem.

Let $p > 0, q > 0$ be steadily increasing in x, for $x \in [0, \infty)$. Then it follows by the analysis of Chakravarty and Sengupta [1], that the sequence of eigenvalues $\{\lambda_n\}$ for the problem (1.1) and (1.2) or the problem (1.1) and (1.3) is positive, and is a discrete collection with $\lim\limits_{n\to\infty} \lambda_n = \infty$.

Let $\psi_n \equiv \binom{\psi_{1n}}{\psi_{2n}}$ be the eigenvector corresponding to the eigenvalue λ_n.

Then $\psi_n \in D$.

The object of the present note is to prove the following theorem.

Theorem.

(i) Let λ_n and ψ_n be the eigenvalue and the eigenvector for the system (1.1), with either the Dirichlet or the Neumann boundary conditions, and let

(ii) $|p(\xi)-p(x)|, |q(\xi)-q(x)| \leqslant C|\xi-x|(p\wedge q)^{\frac{3}{2}}(x)$ for $0<|\xi-x|\leqslant 1$, C a positive constant and $(p\wedge q)(x) = \text{Min}\,(p(x), q(x))$;

(iii) $p(\xi), q(\xi) \leqslant K_0 \exp\,[\frac{1}{3}|\xi-x|\;(p\wedge q)^{\frac{1}{2}}(x)]$ for $|\xi-x|>1$, K_0, a positive constant;

(iv) $\int_0^\infty p^{-\frac{1}{2}}\,dx$ and $\int_0^\infty q^{-\frac{1}{2}}\,dx$, are convergent.

(v) $(p\wedge q)(x) \geqslant \frac{1}{4}x^2\,(p\wedge q)(a)$ for all sufficiently large x; $0<a<\infty$;

(vi) $|p(x)-q(x)| \leqslant A\,e^{-\bar{a}_0 x}$, $A, \bar{a}_0$ positive constants.

Then (i) $\sum_{n=0}^{\infty} \frac{1}{\lambda_n^2}$ is convergent;

(ii) as $\mu\to\infty$, $\sum_{n=0}^{\infty} \frac{1}{(\lambda_n+\mu)^2} \sim \frac{1}{4}\,\text{I}$, where

$$\text{I}=\int_0^\infty \left[\frac{1}{(\mu+p)^{\frac{3}{2}}}+\frac{1}{(\mu+q)^{\frac{3}{2}}}\right] dx.$$

2. Proof of the theorem.

If $g(\xi, x, K) \equiv \begin{pmatrix} g_{11} & g_{21} \\ g_{12} & g_{22} \end{pmatrix}$ be the Green's matrix for the "Fourier system", i.e. the system

$$\text{(2.1)}\qquad \begin{aligned} \frac{d^2u}{d\xi^2} &= K^2\,u(\xi) \\ \frac{d^2v}{d\xi^2} &= K^2\,v(\xi) \end{aligned}$$

with either the Dirichlet or the Neumann boundary conditions, then it is easy to derive, that for any $x \in [0, \infty)$, and with $K^2 \doteq \mu + p(x)$, where $\mu > 0$,

$$\frac{\psi_{1n}(x)}{\lambda_n+\mu} = -\int_0^\infty \big(g_{11}(\xi, x, \mathrm{K}),\ g_{12}(\xi, x, \mathrm{K})\big)\begin{pmatrix}\psi_{1n}(\xi)\\ \psi_{2n}(\xi)\end{pmatrix}d\xi$$

$$+\frac{1}{\lambda_n+\mu}\int_0^\infty \big(g_{11}(\xi, x, \mathrm{K})\ \ g_{12}(\xi, x, \mathrm{K})\big)\begin{pmatrix}p(\xi)-p(x) & 0\\ 0 & q(\xi)-q(x)\end{pmatrix}\psi_n(\xi)d\xi$$

(2.2) $$= a_n(x) + b_n(x), \quad \text{say.}$$

And

$$\frac{\psi_{2n}(x)}{\lambda_n+\mu} = -\int_0^\infty \big(g_{21}(\xi, x, \mathrm{K})\ \ g_{22}(\xi, x, \mathrm{K})\big)\begin{pmatrix}\psi_{1n}(\xi)\\ \psi_{2n}(\xi)\end{pmatrix}d\xi$$

$$+\frac{1}{\lambda_n+\mu}\int_0^\infty \big(g_{21}(\xi, x, \mathrm{K})\ \ g_{22}(\xi, x, \mathrm{K})\big)\begin{pmatrix}p(\xi)-p(x) & 0\\ 0 & q(\xi)-q(x)\end{pmatrix}\psi_n(\xi)d\xi$$

$$+\frac{1}{\lambda_n+\mu}\int_0^\infty \big(g_{21}(\xi, x, \mathrm{K})\ \ g_{22}(\xi, x, \mathrm{K})\big)\begin{pmatrix}0 & 0\\ 0 & q(x)-p(x)\end{pmatrix}\psi_n(\xi)d\xi$$

(2.2)$^{\mathrm{A}}$ $$= \alpha_n(x) + \beta_n(x) + \gamma_n(x), \quad \text{say}$$

It can be easily verified, by the application of the Schwarz inequality, that the infinite integrals involved are convergent.

It is easy to see that the explicit form of the Green's matrix for (2.1), with Dirichlet's and Neumann's boundary conditions are, respectively,

(i) with Dirichlet's boundary conditions,

$$(g_{ij}(\xi, z, \mathrm{K})) = \mathrm{M}(\xi, z)\,\mathrm{E}_2 \quad \text{for } z \leqslant \xi$$
$$= \mathrm{M}(z, \xi)\,\mathrm{E}_2 \quad \text{for } z > \xi$$

where $\mathrm{M}(\xi, z) = \dfrac{e^{-\mathrm{K}(z+\xi)} - e^{-\mathrm{K}(\xi-z)}}{2\mathrm{K}}$, and E_2 is the unit matrix $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.

(ii) with Neumann's boundary conditions,

$$(g_{ij}(\xi, z, \mathrm{K})) = \mathrm{L}(\xi, z)\,\mathrm{E}_2 \quad \text{for } z \leqslant \xi$$
$$= \mathrm{L}(z, \xi)\,\mathrm{E}_2 \quad \text{for } z > \xi,$$

where $\mathrm{L}(\xi, z) = \dfrac{e^{-\mathrm{K}(z+\xi)} + e^{-\mathrm{K}(\xi-z)}}{-2\mathrm{K}}$.

Therefore

(2.3) $$\int_0^\infty g_{11}^2(\xi, x, \mathrm{K})\,d\xi = \begin{cases} \dfrac{1}{4\mathrm{K}^3} - \dfrac{e^{-2\mathrm{K}x}}{4\mathrm{K}^3} - \dfrac{xe^{-2\mathrm{K}x}}{2\mathrm{K}^2}, & \text{with Dirichlet's boundary conditions;} \\[2ex] \dfrac{1}{4\mathrm{K}^3} + \dfrac{e^{-2\mathrm{K}x}}{4\mathrm{K}^3} + \dfrac{xe^{-2\mathrm{K}x}}{2\mathrm{K}^2}, & \text{with the Neumann boundary conditions.} \end{cases}$$

In the following we prove the theorem for the system (1.1), with Dirichlet's boundary conditions. The same analysis holds, when we take the system with Neumann's boundary conditions.

It follows from (2.2) and (2.3), with $K^2=\mu+p$, and the Parseval relation, that

$$\int_0^\infty \sum_{n=0}^\infty a_n^2(x)\,dx = \frac{1}{4}\int_0^\infty \frac{dx}{(\mu+p)^{\frac{3}{2}}} + O\left(\mu^{-1}\int_0^\infty \frac{dx}{p^{\frac{1}{2}}}\right) \tag{2.4}$$

as μ tends to infinity.

A similar result holds for $\alpha_n(x)$ in (2.2)A.

From (2.2)

$$b_n(x) = \frac{1}{\lambda_n+\mu}\left[\int_{R_1}(g_{11},\ g_{12})\begin{pmatrix} p(\xi)-p(x) & 0 \\ 0 & q(\xi)-q(x)\end{pmatrix}\psi_n(\xi)d\xi\right.$$

$$\left. + \int_{R_2}(g_{11},\ g_{12})\begin{pmatrix} p(\xi) & 0 \\ 0 & q(\xi)\end{pmatrix}\psi_n(\xi)d\xi - \int_{R_2}(g_{11},\ g_{12})\begin{pmatrix} p(x) & 0 \\ 0 & q(x)\end{pmatrix}\psi_n(\xi)d\xi\right]$$

$$= \frac{1}{\lambda_n+\mu}\left[b_{1n}(x) + b_{2n}(x) - b_{3n}(x)\right], \text{ say,} \tag{2.5}$$

where $R_1 \equiv |\xi-x| \leqslant 1 \subset R$ and $R_2 = R \setminus R_1$. Using the conditions (ii), satisfied by $|p(\xi)-p(x)|$, $|q(\xi)-q(x)|$, holding for R_1, it follows that

$$b_{1n}^2 \leqslant \frac{4c^2}{\mu p^{\frac{1}{2}}(x)}, \text{ leading to}$$

$$\int_0^\infty b_{1n}^2(x)\,dx = O\left(\mu^{-1}\int_0^\infty \frac{dx}{p^{\frac{1}{2}}}\right), \text{ as } \mu\to\infty. \tag{2.6}$$

Similarly, using the conditions (iii) satisfied by $p(\xi)$, $q(\xi)$ holding for R_2,

$$\int_0^\infty b_{2n}^2(x)\,dx = O(\mu^{-1}), \text{ as } \mu\to\infty. \tag{2.7}$$

Again $b_{3n}^2 \leqslant p^2(x)\int_{R_2} g_{11}^2\,d\xi \int_{R_2}\psi_m^2\,d\xi \leqslant \dfrac{p^2\exp(-2(\mu+p)^{\frac{1}{2}})}{(\mu+p)^{\frac{3}{2}}}$,

which leads to

$$\int_0^\infty b_{3n}^2(x)\,dx = O(\mu^{-1}) \text{ as } \mu\to\infty. \tag{2.8}$$

Hence, altogether, we have from (2.5)

$$\int_0^\infty b_n^2(x)dx = O\left(\frac{\mu^{-1}}{(\lambda_n+\mu)^2}\right), \text{ as } \mu\to\infty. \tag{2.9}$$

A similar result holds also for $\beta_n(x)$.

To find $\int_0^\infty \gamma_n^2(x)\, dx$, we have

$$\gamma_n^2(x) = \frac{(q(x)-p(x))^2}{(\lambda_n+\mu)^2}\left(\int_0^\infty g_{22}(\xi, x, K)\,\psi_{2n}(\xi)\, d\xi\right)^2$$

$$\leqslant \frac{(q(x)-p(x))^2}{(\lambda_n+\mu)^2}\int_0^\infty g_{22}^2(\xi, x, K)\, d\xi.$$

We have therefore,

$$\int_0^\infty \gamma_n^2(x)\, dx = O\left(\mu^{-1}\frac{1}{(\lambda_n+\mu)^2}\right), \text{ by condition (vi).}$$

Therefore from (2.2) and (2.2)^A, for any positive integer m, we obtain by using the condition (v)

$$S_m \leqslant \tfrac{1}{2}\int_0^\infty \frac{dx}{(\mu+p)^{3/2}} + O(\mu^{-1} S_m) + O(\mu^{-1}), \text{ as } \mu\to\infty, \text{ where} \tag{2.10}$$

$$S_m = \sum_{n=0}^{m} \frac{1}{(\lambda_n+\mu)^2}.$$

Hence $\sum_{n=0}^{\infty} \frac{1}{\lambda_n^2}$ is convergent, when $\int_0^\infty p^{-\frac{1}{2}}dx$ is so.

Putting $K^2=\mu+q$, and proceeding as before, we also obtain that

$$\sum_{n=0}^{\infty}\frac{1}{\lambda_n^2} \text{ is convergent, when } \int_0^\infty q^{-\frac{1}{2}}(x)\, dx \text{ is so.}$$

Since $\int_0^\infty \sum_{n=0}^{m} \left[a_n(x)\, b_n(x)\right] dx = O\left[\, I_1^{\frac{1}{2}}\, S^{\frac{1}{2}}\, \mu^{-\frac{1}{2}}\,\right]$ where

$I_1 = \int_0^\infty \frac{dx}{(\mu+p)^{3/2}}$ and $S = \sum_{n=0}^{\infty} \frac{1}{(\lambda_n+\mu)^2}$, it follows by squaring (2.2),

proceeding as before and making m tend to infinity, that

$$S\int_0^\infty \psi_{1n}^2\, dx = \tfrac{1}{4} I_1 + O(\mu^{-1}S) + O(\mu^{-1}) + O(\mu^{-\frac{1}{2}} S^{\frac{1}{2}} I_1^{\frac{1}{2}}),$$

as $\mu \to \infty$, with a similar relation for $S\int_0^\infty \psi_{2n}^2\, dx.$

Therefore, by addition

$$(2.11)\quad S = \tfrac{1}{2} I_1 + O(\mu^{-1}S) + O(\mu^{-1}) + O(\mu^{-\frac{1}{2}} S^{\frac{1}{2}} I_1^{\frac{1}{2}}), \text{ as } \mu \to \infty.$$

Again, proceeding similarly with $K^2 = \mu + q$, we obtain

$$(2.12)\quad S = \tfrac{1}{2} I_2 + O(\mu^{-1}S) + O(\mu^{-1}) + O(\mu^{-\frac{1}{2}} S^{\frac{1}{2}} I_2^{\frac{1}{2}}) \text{ as } \mu \to \infty,$$

where $I_2 = \int_0^\infty \frac{dx}{(\mu+q)^{3/2}}$.

Combining (2.11) and (2.12), we obtain

$S \sim \frac{1}{4} I$, as $\mu \to \infty$.

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REFERENCES

[1] Chakrabarty, N. K. and Sengupta P. K.—On the distribution of the eigenvalues of a matrix differential operator—*Jour. Ind. Inst. Sc.*, **61(B)** (1979) 19-42.

[2] Titchmarsh, E. C.—Eigenfunction expansions associated with a second order differential equation, Part II (Clarendon Press) (1958).

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Dept. of Pure Math.
Calcutta University

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GENERAL REAL INVERSION OPERATOR

D. K. BHATTACHARYA

Abstract : The paper deals with the inversion formula of Laplace-Stieltjes integrals in one and more variables by using a general real inversion operator from which similar operators used by A. Erdelyi [2] and P. G. Rooney [3] follow as particular cases.

1.1. Introduction :

$$\text{Let } f(s)=\int_0^\infty e^{-st}\, d\,\{\alpha(t)\}$$

then the real inversion operator is taken as

$$\int_0^t L_{k,t}^{\lambda,\mu,\nu,}\,[f(s)]\quad dt, \text{ where}$$

$$L_{k,t}^{\lambda,\mu,\nu,}[f(s)]=(A_1)\int_0^\infty x^\nu\,[J_{kx}]_\nu^\mu\; f\left[\frac{k(x+1)}{t}\right]dx,\ \lambda\geqslant 0,\ \nu>-1,\ (\mu,k)>0.$$

$$\text{and}\quad A_1=k^{\lambda+\nu+\frac{3}{2}}\,2^{2\lambda-\frac{1}{2}}\,\{\mu(\mu+1)^{\frac{1}{2}}\}\,(t\ \sqrt{\pi})^{-1}\,[W_{\lambda,\nu}\,(4k)^{-1}]$$

$$[J_{kx}]_\nu^\mu=J_\nu^\mu\,(k^{\mu+1}\,x^\mu),$$

$$J_\nu^\mu\,(x)=\sum_{k=0}^\infty\frac{(-x)^k}{k\,!\,\Gamma(1+\nu+\mu k)}$$

$$\text{Since}\quad J_\nu^1\,(x)=x^{-\frac{\nu}{2}}\,J_\nu\,(2x^{\frac{1}{2}})$$

$$\text{and}\quad W_{o,\nu}\,(x)=\left(\frac{x}{\pi}\right)^{\frac{1}{2}}K_\nu\left(\frac{x}{2}\right),$$

we have,

$$L_{k,t}^{0,1,\nu}[f(s)]=[2t\,K_\nu\,(2k)]^{-1}\,k\int_0^\infty x^{\frac{\nu}{2}}\,J_\nu\ (2K\,x^{\frac{1}{2}})\,f\left[\frac{k(x+1)}{t}\right]dx$$

—the operator $L_{k,t}\ [f(s)]$ discussed by A. Erdelyi [2].

Again $L_{k,t}^{0,1,\frac{1}{2}}\ [f(s)]$ are the cases discussed by P. G. Rooney [3].

Next Let $f(s.t) = \int_0^\infty \int_0^\infty e^{-sx-ty}\, d\{\alpha(x, y)\}$, then the real inversion operator is of the form

$$\int_0^x \int_0^y L^{\lambda\,;\,\mu_1,\mu_2\,;\,\nu_1,\nu_2}_{k_1,k_2\,;\,x,y}\,[f(s, t)]\, dx\, dy$$

$$=(A_2)\int_0^\infty \int_0^\infty \prod_1^2 \left[z_i^{\nu_i} \left\{J_{kz}\right\}_{\nu_i}^{\mu_i} \right] f\left[\frac{k_1(z_1+1)}{x}, \frac{k_2(z_2+1)}{y}\right] dz_1\, dz_2,$$

$\lambda \geqslant 0,\ \nu_i > -1,\ (\mu_i, k_i) > 0,\ i=1, 2$

where $A_2 = \prod_1^2 \left[k_i^{\lambda+\nu_i+\frac{3}{2}}\, 2^{4\lambda-1} \{\mu_i(\mu_i+1)\}^{\frac{1}{2}}\ (\pi xy)^{-1}\, \{W_{\lambda_i, \nu_i}(4K_i)\}^{-1} \right]$

1.2. Notations.

In what follows, we use the following notations.

(a) A function $\phi(x, y)$ is said to be quasi-normalized on $S : [o, o\,;\, R_1, R_2]$ if $\phi(x, y) = \frac{1}{4}[\phi]_{x\pm,\, y\pm}$,

where, $[\phi]_{x\pm\ y\pm} = \phi(x+, y+) + \phi(x+, y-) + \phi(x-, y+) + \phi(x-, y-)$

and $\phi(x\pm, y\pm)$ means the limit of $\phi(x, y)$ as $(x, y) \to (x\pm, y\pm)$.

along the line joining the points (x, y) and $(x\pm, y\pm)$.

(b) A function $\phi(x, y)$ belongs to the class H on S, if

(i) for all possible rectangles consisting of lines of the form

$x=x_i, y=y_j\ (i=1, 2, \ldots, m\,;\, j=1, 2, \ldots, n)$

the set of all summations

$\sum_{i,j} |\{\phi\}_{D_{ij}}|,\ D_{i,j} = [x_{i-1}, y_{j-1}\,;\, x_i, y_j]$

is bounded on S.

(ii) $\phi(x_0, y)$ and $\phi(x, y_0)$ for some fixed values x_0, y_0 are also functions of bounded variation on S,

(c) A function $\phi(x, y)$ belongs to the class H_0 on S. if

(i) $\phi \in H$ on S

(ii) $\phi(x, 0)=0=\phi(0, y)$.

1.3 Theorem 1.3.1.

Let (i) $\alpha(t)$ be a normalized function of bounded variation in $(0, R)$, $R > 0$,

(ii) $\int_0^\infty e^{-st}\, d\{\alpha(t)\}$

converges with $s=\gamma > 0$, then

$$\lim_{k\to\infty} \int_0^t L_{k,t}^{\lambda,\mu,\nu}\,[f(s)]\, dt = \tfrac{1}{2}[\alpha(t+)+\alpha(t-)]$$

almost everywhere for $t > 0$, where $\alpha(t\pm)$ exist.

Proof: By hypothesis,

$$f(s)=s.\int_0^\infty e^{-st}\,\alpha(t)\,dt.$$

Now $$L_{k,t}^{\lambda,\mu,\nu}\Big[f(s)\Big]=(A_1).\int_0^\infty x^{\nu}\Big[J_{kx}\Big]_{\nu}^{\mu}\left\{\int_0^\infty \frac{k(x+1)}{t}\, e^{\frac{-k(x+1)u}{t}}\alpha(u)\,du\right\}$$

$$=\left(\frac{k}{t}\right)(A_1).\int_0^\infty e^{\frac{-ku}{t}}\alpha(u)\left\{\int_0^\infty e^{\frac{-kxu}{t}}\left(x^{\nu+1}+x^{\nu}\right)\Big[J_{kx}\Big]_{\nu}^{\mu}dx\right\}du$$

[by change of order of integration, which is easily justifiable]

$$=\left(\frac{k}{t}\right).(A_1).\,k^{-(\nu+2)}\,t^{\nu+2}\int_0^\infty u^{-(\nu+2)}\,e^{-k\left\{\frac{u}{t}+\left(\frac{t}{u}\right)^{\mu}\right\}}.\,\alpha(u)$$

$$.\left\{(\nu+1)+\left(k\frac{u}{t}\right)-\mu k\left(\frac{t}{u}\right)^{\mu}\right\}du.$$

$$=t.k^{-(\nu+1)}(A_1)\,.\int_0^\infty u^{-(\nu+2)}\,.\,\alpha(u)\left[t^{\nu}\left\{(\nu+1)+\left(k\frac{u}{t}\right)-\mu k\left(\frac{t}{u}\right)^{\mu}\right\}.\; e^{-k\left\{\frac{u}{t}+\left(\frac{t}{u}\right)^{\mu}\right\}}\right]du$$

$$=t\,.\,k^{-(\nu+1)}\,(A_1)\int_0^\infty u^{-(\nu+2)}.\alpha(u)\,\frac{d}{dt}\left[e^{-k\left\{\frac{u}{t}+\left(\frac{t}{u}\right)^{\mu}\right\}}.\,t^{(\nu+1)}\right]du$$

$$\therefore\;\int_0^t L_{k,t}^{\lambda,\mu,\nu}\Big[f(s)\Big]\,dt$$

$$=t\,.\,k^{-(\nu+1)}\,(A_1)\,.\,t^{\nu+1}\int_0^\infty e^{-k\left\{\frac{u}{t}+\left(\frac{t}{u}\right)^{\mu}\right\}}\,u^{-(\nu+2)}\,.\,\alpha(u)du$$

(differentiation under the sign of integration being permissible as by hypothesis

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$\int_0^\infty e^{-st} \,.\, t^{-(\nu+2)}\, \alpha(t)dt$ converves uniformly and absolutely for $s > \gamma$).

$$\sim \frac{k^{\lambda+\frac{1}{2}} \,.\, 2^{2\lambda-\frac{1}{2}}\{\mu(\mu+1)\}^{\frac{1}{2}}}{(\sqrt{\pi}) \,.\, e^{-2k} \,.\, (4k)^{\lambda}} \,.\, t^{\nu+1} \,.\, t^{-(\nu+2)} \,.\, \{\alpha(t+)+\alpha(t-)\} e^{-2k}$$

$$\cdot \frac{t\sqrt{\pi}}{\{2k\mu(\mu+1)\}^{\frac{1}{2}}}, \quad \text{as } k \to \infty.$$

(assuming that the asymptotic evaluation is justified).

$$= \frac{\alpha(t+)+\alpha(t-)}{2}, \quad t > 0.$$

Let us now justify the asymptotic evaluation.

Let I_1 and I_2 be respectively the integrals

$$k^{\lambda+\frac{1}{2}} \int_0^{t-\delta} e^{-k\left\{\frac{u}{t}+\left(\frac{t}{u}\right)^{\mu}\right\}} u^{-(\nu+2)} \alpha(u) du$$

and $$k^{\lambda+\frac{1}{2}} \int_{t+\delta}^{\infty} e^{-k\left\{\frac{u}{t}+\left(\frac{t}{u}\right)^{\mu}\right\}} u^{-(\nu+2)} \alpha(u) du$$

We choose $k_0 > \gamma t$, then for $k > k_0$,

$$| I_2 | \leqslant k^{\lambda+\frac{1}{2}} \Big[W_{\lambda,\mu}(4k)\Big]^{-1} \int_{t+\delta}^{\infty} e^{-k_0\left\{\frac{u}{t}+\left(\frac{t}{u}\right)^{\mu}\right\}} e^{-(k-k_0)\left\{\frac{u}{t}+\left(\frac{t}{u}\right)^{\mu}\right\}} .$$

$$| u^{-(\nu+2)} \alpha(u) | \; du$$

$$< k^{\lambda+\frac{1}{2}} \Big[W_{\lambda,\nu}(4k)\Big]^{-1} e^{(k-k_0)\left[\frac{(t+\delta)}{t}+\left\{\frac{(t+\delta)}{t}\right\}^{-\mu}\right]} .$$

$$\int_{t+}^{\infty} e^{-k_0\left\{\frac{u}{t}+\left(\frac{t}{u}\right)^{\mu}\right\}} | u^{-(\nu+2)} \alpha(u) | \; du$$

$$A(t) \,.\, k^{\lambda+\frac{1}{2}} \Big[W_{\lambda,\nu}(4k)\Big]^{-1} e^{-k\alpha}, \quad [\alpha > 2, \; \mu \leqslant 1]$$

$$\sim A(t) \,.\, k^{\lambda+\frac{1}{2}} e^{2k} e^{-k\alpha}$$

$$\to 0 \text{ as } k \to \infty.$$

Similarly it may be shown that $| I_1 | \to 0$ as $k \to \infty$.

Thus the theorem is completely proved.

Theorem. 1.3.2

Let (i) $\alpha(x, y)$ be a normalized function belonging to H_0 on S

(ii) $\int_0^\infty \int_0^\infty e^{-(sx-ty)} d\{\alpha(x, y)\}$

converges with $s = \gamma_1 > 0, \quad t = \gamma_2 > 0,$

then, $\lim\limits_{(k_1, k_2) \to (\infty, \infty)} \int_0^x \int_0^y L^{\lambda\,;\,\mu_1, \mu_2\,;\,\nu_1, \nu_2}_{k_1, k_2\,;\,x, y} [f(s, t)]\, dx\, dy = \frac{1}{4}[\alpha]_{x\pm,\, y\pm}$

almost everywhere, where $\alpha(x\pm, y\pm)$ exist.

Proof : By hypothesis, we have

$$f(s, t) = st \int_0^\infty \int_0^\infty e^{-sx-ty} \alpha(x, y)\, dx\, dy$$

Therefore,

$$L^{\lambda\,.\,\mu_1, \mu_2\,,\,\nu_1, \nu_2}_{k_1, k_2\,;\,x, y} [f(s, t)] = (k_1 k_2/xy) \int_0^\infty \int_0^\infty e^{-k_1 \frac{u}{x}} e^{-k_2 \frac{v}{y}} \alpha(u, v).$$

$$\left[e^{-k_1 z_1 \frac{u}{x}}\, e^{-k_2 z_2 \frac{v}{y}} \prod_1^2 \left\{z_i^{\nu_i+1} + z_i^{\nu_i}\right\} . \left[J_{k_i z_i}\right]_{\nu_i}^{\mu_i} dz_i\right\}\Big] du\, dv$$

$$= (xy) \prod_1^2 \left\{K_i^{-(\nu_i+1)}\right\} (A_2) \,.\, \int_0^\infty \int_0^\infty u^{-(\nu_1+2)}\, v^{-(\nu_1+2)}\, \alpha(u, v).$$

$$\frac{d}{dx}\left[e^{-k_1 \left\{\frac{u}{x} + \left(\frac{x}{u}\right)^{\mu_1}\right\}} x^{\nu_1+1}\right] . \frac{d}{dy}\left[e^{-k_2 \left\{\frac{v}{y} + \left(\frac{y}{v}\right)^{\mu_2}\right\}} y^{\nu_2+1}\right] du\, dv$$

$$\therefore \int_0^x \int_0^y L^{\lambda\,;\,\mu_1, \mu_2\,;\,\nu_1, \nu_2}_{k_1, k_2\,;\,x, y} [f(s, t)]\, dx\, dy$$

$$= (xy)\, k_1^{-(\nu_1+1)}\, k_2^{-(\nu_2+1)} (A_2) \,.\, x^{\nu_1+1}\, y^{\nu_2+1}.$$

$$\left[\int_0^\infty e^{-k_1 \left\{\frac{u}{x} + \left(\frac{x}{u}\right)^{\mu_1}\right\}} u^{-(\nu_1+2)} du \,.\, \int_0^\infty e^{-k_2 \left\{\frac{v}{y} + \left(\frac{y}{v}\right)^{\mu_2}\right\}} v^{-(\nu_2+2)} \alpha(u, v) dv\right]$$

$$\sim k_1^{\lambda+\frac{1}{2}}\, k_2^{\lambda+\frac{1}{2}}\, 2^{4\lambda-1} \{\mu_1 \mu_2 (\mu_1+1)(\mu_2+1)\}^{\frac{1}{2}}\, \pi^{-1} \left[W_{\lambda, \nu_1}(4k_1)\right]^{-1}$$

$$e^{2k_2} (4k_2)^{-\lambda}$$

$$x^{\nu_1+1} y^{\nu_2+1} y^{-(\nu_2+2)} \cdot \int_0^\infty e^{-k_1\left\{\frac{u}{x}+\left(\frac{x}{u}\right)^{\mu_1}\right\}} u^{-(\nu_1+2)}.$$

$$\{\alpha(u, y+)+\alpha(u, y-)\}\, du \,.\, e^{-2k_2} \, (y\sqrt{\pi})\{2k_2\mu_2(\mu_2+1)\}^{-\frac{1}{2}} \text{ as } k_2 \to \infty$$

$$\sim \prod_1^2 \left[k_i^{\lambda+\frac{1}{2}} \{\mu_i(\mu_i+1)\}^{\frac{1}{2}} \,.\, e^{2k_i} \,.\, (4k_i)^{\lambda} \right] . \, 2^{4\lambda-1} \,.\, \pi^{-1} \,.\, x^{\nu_1+1} \,.\, y^{\nu_2+1}$$

$$x^{-(\nu_1+2)} y^{-(\nu_2+2)} e^{-2k_2} e^{-2k_1} [\alpha]_{x\pm,\, y\pm} \cdot (x\sqrt{\pi})(y\sqrt{\pi}) \,.\, 2^{-1}$$

$$\prod_1^2 \left\{ k_i \mu_i (\mu_{i+1}) \right\}^{-\frac{1}{2}},$$

as $k_1 \to \infty$, $k_2 \to \infty$.

$$= \tfrac{1}{4}[\alpha]_{x\pm,\, y\pm}.$$

The same result holds when $k_2 \to \infty$, $k_1 \to \infty$.

This completes the theorem.

Remark : Similarity of the results of theorem 1.3.1. & 1.3.2 obviously reveals that such real inversion formulae may be generalised to functions of n variables.

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REFERENCES

[1] Bernstein, D. L.—The double Laplace Integral—*Duke Math. J.*, 8 (1941).

[2] Erdelyi, A.—The inversion of Laplace transformation—*Math. Mag.* 29 (1950-51).

[3] Rooney, P. G.—A new representation and inversion theory for the Laplace transformation—*Canad. J. Math.*, Vol. (IV) (1952), 433-444.

[4] Widder, D. V.—The Laplace Transform—Princeton, 1946.

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Dept. of Pure Math.
Calcutta University

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FIXED POINT THEOREMS

K. M. Ghosh

In this paper we shall prove two fixed point theorems which are extensions of a theorem of A. A. Ivanov [1] and a theorem of S. Reich [2].

Theorem 1. (Extension of Ivanov's theorem)

Let X be a non-empty metric space and $T : X \to X$, be a self-mapping of X. If X is T-orbitally complete, T is orbitally continuous and for every distinct x, y in X there exist real numbers $a_i (i=1, 2 \ldots 7)$ such that

$$(1) \quad a_1 d(x, y)+a_2 d(x, Tx)+a_3 d(y, Ty)+a_4 d(y, Tx)+a_5 d(x, Ty)$$
$$+a_6 d(Tx, Ty)+a_7 \frac{d(x, Tx)\, d(y, Ty)}{d(x, y)} \geqslant 0,$$

where,

$$(2) \quad a_1+a_2+a_3+a_6+a_7 < \min \{0, -(a_4+a_5)\},$$

$$(3) \quad a_7+a_6+\frac{a_2+a_3}{2}+\frac{a_4+a_5}{2} < 0,$$

then T has a fixed point in X.

Proof: By the symmetric property of metric, we can easily obtain

$$(4) \quad a_1 d(x, y)+\frac{a_2+a_3}{2}\Big[d(x, Tx)+d(y, Ty) \Big]+\frac{a_4+a_5}{2}\Big[d(y, Tx)+d(x, Ty) \Big]$$
$$+a_6 d(Tx, Ty)+a_7 \frac{d(y, Tx)\, d(y, T_y)}{d(x, y)} \geqslant 0.$$

Since x and y are arbitrary, let $y=Tx$. Then from (4) we have

$$(5) \quad a_1 d(x, Tx)+\frac{a_2+a_3}{2}\Big[d(x, Tx)+d(Tx, T^2x) \Big]+\frac{a_4+a_5}{2} d(x, T^2x)$$
$$+a_6 d(Tx, T^2x)+a_7 d(Tx, T^2x) \geqslant 0.$$

Now we consider the following two cases:

Case (i): When $a_4+a_5 \geqslant 0$, then $d(x, T^2x) \leqslant d(x, Tx)+d(Tx, T^2x)$ and we obtain by virtue of (5)

$$(6) \quad \left(a_1+\frac{a_2+a_3}{2}+\frac{a_4+a_5}{2}\right) d(x, Tx)+\left(\frac{a_2+a_3}{2}+\frac{a_4+a_5}{2}+a_6+a_7\right)$$
$$.d(Tx, T^2x) \geqslant 0.$$

Since $a_4+a_5 \geqslant 0$, it follows therefore from (2) and (3)

$$\frac{2a_1+a_2+a_3}{2}+\frac{a_4+a_5}{2}+\frac{a_2+a_3}{2}+\frac{a_4+a_5}{2}+a_6+a_7 < 0,$$

or, $$\left(a_1+\frac{a_2+a_3}{2}+\frac{a_4+a_5}{2}\right)\left(\frac{a_2+a_3}{2}+\frac{a_4+a_5}{2}+a_6+a_7\right)^{-1}+1>0$$

or, $$-(2a_1+a_2+a_3+a_4+a_5)(a_2+a_3+a_4+a_5+2a_6+2a_7)^{-1} < 1.$$

Now from (6), we have $(2a_1+a_2+a_3+a_4+a_5) \geqslant 0$

Thus we get

(7) $0 \leqslant -(2a_1+a_2+a_3+a_4+a_5)(a_2+a_3+a_4+a_5+2a_6+2a_7)^{-1} < 1.$

Combining (6) and (7) we have,

$$d(Tx, T^2x) \leqslant -(2a_1+a_2+a_3+a_4+a_5)$$
$$.(a_2+a_3+a_4+a_5+2a_6+2a_7)^{-1}\, d(x, Tx).$$

By induction it may be shown that $\{T^n x\}_{n=0}^{\infty}$ is a Cauchy sequence. Since the metric space X is T-orbitally complete, $\lim\limits_{n\to\infty} T^n x = u \in X$.

Next we shall show that u is a fixed point.

Since T is orbitally continuous, we have

$$Tu = T\lim_{n\to\infty} T^n x = \lim_{n\to\infty} T^{n+1}x = u,$$

which implies that u is a fixed point of T.

Case (ii) : When $a_4+a_5 < 0$, then

$d(x, T^2x) \geqslant d(Tx, T^2x)-d(x, Tx)$ and then we obtain by virtue of (5)

(8) $$\left(a_1+\frac{a_2+a_3}{2}-\frac{a_4+a_5}{2}\right) d(x, Tx)+\left(\frac{a_2+a_3}{2}+\frac{a_4+a_5}{2}+a_6+a_7\right)$$
$$.d(Tx, T^2x) \geqslant 0$$

By similar argument of case (i), it may be easily shown that

$$-(2a_1+a_2+a_3-a_4-a_5)(a_2+a_3+a_4+a_5+2a_6+2a_7)^{-1} < 1$$

Thus from (8) we have

$$2a_1+a_2+a_3-a_4-a_5 \geqslant 0,$$

so that

$$0 \leqslant -(2a_1+a_2+a_3-a_4-a_5)(a_2+a_3+a_4+a_5+2a_6+2a_7)^{-1} < 1.$$

It follows therefore from (8) that

$$d(Tx, T^2x) \leqslant -(2a_1+a_2+a_3-a_4-a_5)(a_2+a_3+a_4+a_5+2a_6+2_7)^{-1}\, d(x, Tx).$$

The remaining part of the proof is similar to that of case (i).

Remark : Putting $a_7=0$ in theorem 1 we get the theorem (1) of Ivanov [1] as a particular case of our theorem.

Theorem 2. (Extension of Reich's theorem)

Let (X, d) be a complete metric space and $T : X \to X$ and let $t : X \to$ set of real numbers be defined by $t(x)=d(x, Tx)$. If for any $x, y \in X$,

(9) $d(Tx, Ty) < a_1\, t(x)+a_2\, t(y)+a_3\, d(x, y)+a_4\, d(x, Ty)+a_5\, d(y, Tx)$

where a_i's are non-negative real numbers and $a_3+a_4+a_5 < 1$,

(10) t is lower semi-continuous

(11) there exists a sequence $\{x_n\} \subset X$ such that

$t(x_n) \to 0$ as $n \to \infty$,

then T has a unique fixed point in X.

Proof : Let $\{x_n\}$ be any sequence with $t(x_n) \to 0$.

Now for $m > n$,

$$\begin{aligned} d(x_n, x_m) &\leqslant d(x_n, Tx_n)+d(Tx_n, Tx_m)+d(x_m, Tx_m) \\ &\leqslant d(x_n, Tx_n)+d(x_m, Tx_m)+a_1\, t(x_n)+a_2\, t(x_m)+a_3\, d(x_n, x_m) \\ &\qquad +a_4\, d(x_n, Tx_m)+a_5\, d(x_m, Tx_n) \\ &\leqslant (1+a_1+a_5)\, t(x_n)+(1+a_2+a_4)\, t(x_m)+(a_3+a_4+a_5)\, d(x_n, x_m). \end{aligned}$$

Thus

$$d(x_n, x_m) \leqslant \frac{1+a_1+a_5}{1-a_3-a_4-a_5}\, t(x_n)+\frac{1+a_3+a_4}{1-a_3-a_4-a_5}\, t(x_m).$$

Since $t(x_n) \to 0$ as $n \to \infty$, there exists an integer N such that

$$t(x_n) \leqslant \frac{(1-a_3-a_4-a_5)\,\epsilon}{2(1+a_2+a_5)} \text{ and } t(x_m) \leqslant \frac{(1-a_3-a_4-a_5)}{2(1+a_3+a_4)}\,\epsilon, \quad n > N,$$

where $\epsilon > 0$. Thus $\{x_n\}$ is a Cauchy sequence.

Hence $x_n \to x \in X$. Since t is lower semi-continuous, so, $t(x)=0$ and then $Tx=x$. Uniqueness of the fixed point follows easily.

Remark : Putting $a_4 = a_5 = 0$, in Theorem 2 we get the theorem of S. Reich [2] as a special case of our theorem.

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REFERENCES

[1] Ivanov, A. A.—*Math. Balkanica* **4** (1974), 283-287.
[2] Reich, S.—*Amer. Math. Monthly* 79 (1972) 94-95.

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Dept. of Pure Math.
Calcutta University

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BIANCHI AND VEBLEN IDENTITIES FOR THE PROJECTIVE CURVATURE TENSOR OF A SEMI-SYMMETRIC AFFINE CONNECTION IN AN AFFINELY CONNECTED SPACE

B. BARUA (nee GUPTA) and ASOKE RAY

0. Introduction.

Let V_N be an N-dimensional affinely connected space with a symmetric affine connection Γ^i_{jk}. An affine connection L^i_{jk} given by

$$(0.1) \quad L^i_{jk} = \Gamma^i_{jk} + \delta^i_j \phi_k - \delta^i_k \phi_j$$

where ϕ_j is a covariant vector, is called a semi-symmetric affine connection [1, *p*.36] in V_N. If B^i_{jkl} and L^i_{jkl} are the curvature tensors with respect to Γ^i_{jk} and L^i_{jk} respectively, then

$$(0.2) \quad L^i_{jkl} = B^i_{jkl} + \delta^i_k \phi_{jl} - \delta^i_l \phi_{jk} + \delta^i_j (\phi_{lk} - \phi_{kl})$$

where

$$(0.3) \quad \varphi_{jk} = \nabla_k \phi_j + \phi_j \phi_k = \overline{\nabla}_k \phi_j + \phi_j \phi_k,$$

∇ and $\overline{\nabla}$ being the operators of covariant differentiation with respect to the connections Γ^i_{jk} and L^i_{jk} respectively.

The projective curvature tensors for the connection Γ^i_{jk} and L^i_{jk} are given by

$$(0.4) \quad W^i_{jkl} = B^i_{jkl} + \frac{2}{N+1} \delta^i_j \beta_{kl} + \frac{1}{N-1}\left(\delta^i_k B_{jl} - \delta^i_l B_{jk}\right) + \frac{2}{N^2-1}\left(\delta^i_l \beta_{jk} - \delta^i_k \beta_{jl}\right)$$

and

$$(0.5) \quad P^i_{jkl} = L^i_{jkl} + \frac{2}{N+1} \delta^i_j \lambda_{kl} + \frac{1}{N-1}\left(\delta^i_k L_{jl} - \delta^i_l L_{jk}\right) + \frac{2}{N^2-1}\left(\delta^i_l \lambda_{jl} - \delta^i_k \lambda_{jl}\right)$$

where $B_{jk} = B^t_{jkt}$, $L_{jk} = L^t_{jkt}$

$$2\beta_{jk} = B_{jk} - B_{kj}, \quad 2\lambda_{jk} = L_{jk} - L_{kj}$$

It is known that the curvature tensor of any symmetric connection satisfies the Bianchi identity and the Veblen identity [1, p.56] which are

$$(0.6)\quad \nabla_m B^i_{jkl} + \nabla_k B^i_{jlm} + \nabla_l B^i_{jmk} = 0$$

and

$$(0.7)\quad \nabla_m B^i_{jkl} + \nabla_j B^i_{mlk} + \nabla_k B^i_{ljm} + \nabla_l B^i_{kmj} = 0$$

The Bianchi identity [2] and Veblen identity [3] for the projective curvature tensor in a Riemannian space are

$$(0.8)\quad \nabla_m W^i_{jkl} + \nabla_k W^i_{jlm} + \nabla_l W^i_{jmk} - \frac{1}{N-2}\nabla_t\left(\delta^i_m W^t_{jkl} + \delta^i_k W^t_{jlm} + \delta^i_l W^t_{jmk}\right) = 0$$

and

$$(0.9)\quad \nabla_m W^i_{jkl} + \nabla_j W^i_{mlk} + \nabla_k W^i_{ljm} + \nabla_l W^i_{kmj} - \frac{1}{N-2}\nabla_t\left(\delta^i_m W^t_{jkl} + \delta^i_j W^t_{mlk} + \delta^i_k W^t_{ljm} + \delta^i_j W^t_{kmj}\right) = 0$$

In this paper analogous identities for the projective curvature tensor P^i_{jkl} of a semi-symmetric affine connection in V_N have been derived.

1. Bianchi and Veblen identities for the curvature tensor L^i_{jkl}.

From (0, 2), we get

$$(1.1)\quad L^i_{jkl} + L^i_{klj} + L^i_{ljk} = 2\left[\delta^i_j(\phi_{lk} - \phi_{kl}) + \delta^i_k(\phi_{jl} - \phi_{lj}) + \delta^i_l(\phi_{kj} - \phi_{jk})\right].$$

Therefore, $L^i_{jkl} + L^i_{klj} + L^i_{ljk} = 0$ iff $\phi_{jk} = \phi_{kj}$ for $N \geqslant 3$. But from (0.3), this condition is equivalent to $\nabla_j \phi_k = \nabla_k \phi_j$ which implies that ϕ_i is a gradient vector. Thus

Theorem 1. The curvature tensor of a semi-symmetric affine connection satisfies

$L^i_{jkl} + L^i_{klj} + L^i_{ljk} = 0$ iff ϕ_j is a gradient vector.

In the remaining part of this paper ϕ_j will be considered as a gradient vector.

Let w_i be an arbitrary non-null covariant vector in V_N. The generalized Ricci identity gives

$$\overline{\nabla}_k\overline{\nabla}_j w_i - \overline{\nabla}_j\overline{\nabla}_k w_i = w_t L^t_{ijk} - 2\{(\overline{\nabla}_j w_i)\phi_k - (\overline{\nabla}_k w_i)\phi_j\} \tag{1.2}$$

Operating both sides by $\overline{\nabla}_l$ we get

$$\overline{\nabla}_l\overline{\nabla}_k\overline{\nabla}_j w_i - \overline{\nabla}_l\overline{\nabla}_j\overline{\nabla}_k w_i = (\overline{\nabla}_l w_t) L^t_{ijk} + w_t\overline{\nabla}_l L^t_{ijk}$$
$$- 2\{(\overline{\nabla}_l\overline{\nabla}_j w_i)\phi_k - (\overline{\nabla}_l\overline{\nabla}_k w_i)\phi_j + (\overline{\nabla}_j w_i)(\overline{\nabla}_l\phi_k) - (\overline{\nabla}_k w_i)(\overline{\nabla}_l\phi_j)\} \tag{1.3}$$

Permuting j, k, l cyclically and then adding all possible expressions obtained from (1.3), we get, by virtue of (1.2)

$$\overline{\nabla}_l(\overline{\nabla}_k\overline{\nabla}_j w_i - \overline{\nabla}_j\overline{\nabla}_k w_i) + \overline{\nabla}_k(\overline{\nabla}_j\overline{\nabla}_l w_i - \overline{\nabla}_l\overline{\nabla}_j w_i)$$
$$+ \overline{\nabla}_j(\overline{\nabla}_l\overline{\nabla}_k w_i - \overline{\nabla}_k\overline{\nabla}_l w_i)$$
$$= \{(\overline{\nabla}_l w_t) L^t_{ijk} + (\overline{\nabla}_k w_t) L^t_{ilj} + (\overline{\nabla}_j w_t) L^t_{ikl}\}$$
$$+ w_t(\overline{\nabla}_l L^t_{ijk} + \overline{\nabla}_j L^t_{ikl} + \overline{\nabla}_k L^t_{ilj}) + 2w_t(\phi_l L^t_{ijk} + \phi_j L^t_{ikl} + \phi_k L^t_{ilj}) \tag{1.4}$$

Applying generalized Ricci identity to $\overline{\nabla}_j w_i$ we get

$$\overline{\nabla}_l\overline{\nabla}_k(\overline{\nabla}_j w_i) - \overline{\nabla}_k\overline{\nabla}_l(\overline{\nabla}_j w_i) = (\overline{\nabla}_t w_i) L^t_{jkl} + (\overline{\nabla}_j w_t) L^t_{ikl}$$
$$- 2\{(\overline{\nabla}_k\overline{\nabla}_j w_i)\phi_l - (\overline{\nabla}_l\overline{\nabla}_j w_i)\phi_k\} \tag{1.5}$$

Adding the expressions obtained from (1.5) by all possible cyclic permutations of j, k, l and using the resulting equation in (1.4), we get, by virtue of Theorem 1,

$$2\{(\overline{\nabla}_l w_t) L^t_{ijk} + (\overline{\nabla}_k w_t) L^t_{ilj} + (\overline{\nabla}_j w_t) L^t_{ikl}\}$$
$$+ 2w_t(\overline{\nabla}_l L^t_{ijk} + \overline{\nabla}_j L^t_{ikl} + \overline{\nabla}_k L^t_{ilj})$$
$$+ 4w_t(\phi_l L^t_{ijk} + \phi_j L^t_{ikl} + \phi_k L^t_{ilj})$$
$$= 2\{(\overline{\nabla}_l w_t) L^t_{ijk} + (\overline{\nabla}_j w_t) L^t_{ikl} + (\overline{\nabla}_k w_t) L^t_{ilj}\}$$
$$+ w_t(\overline{\nabla}_l L^t_{ijk} + \overline{\nabla}_j L^t_{ikl} + \overline{\nabla}_k L^t_{ilj})$$

From the above equation we get

$$(\overline{\nabla}_l L^t_{ijk} + 4\phi_l L^t_{ijk}) + (\overline{\nabla}_j L^t_{ikl} + 4\phi_j L^t_{ikl})$$
$$+ (\overline{\nabla}_k L^t_{ilj} + 4\phi_k L^t_{ilj}) = 0 \tag{1.6}$$

Put

(1.7) $\bar{L}^t_{ijk} = e^{4\phi} L^t_{ijk}\,, \quad \bar{L}_{ij} = e^{4\phi} L_{ij}\,, \quad \bar{\lambda}_{ij} = e^{4\phi}\lambda_{ij}$

Then

(1.8) $\bar{\nabla}_l \bar{L}^t_{ijk} = e^{4\phi}(\bar{\nabla}_l L^t_{ijk} + 4\phi_l L^t_{ijk}), \; \bar{\nabla}_k \bar{L}_{ij} = e^{4\phi}(\bar{\nabla}_k L_{ij} + 4\phi_k L_{ij})$

Substituting from (1.8) in (1.6), we get

(1.9) $\bar{\nabla}_l \bar{L}^t_{ijk} + \bar{\nabla}_j \bar{L}^t_{ikl} + \bar{\nabla}_k \bar{L}^t_{ilj} = 0$

This is the Bianchi identity for the curvature tensor L^t_{ijk} of the semi-symmetric affine connection (0.1), for which ϕ_j is a gradient vector.

From Theorem 1, and (1.7) we get

$$\bar{L}^t_{ijk} + \bar{L}^t_{jki} + \bar{L}^t_{kij} = 0.$$

Applying $\bar{\nabla}_l$ we get

(1.10) $\bar{\nabla}_l \bar{L}^t_{ijk} = \bar{\nabla}_l \bar{L}^t_{jik} + \bar{\nabla}_l \bar{L}^t_{kji}.$

Similarly,

(1.11) $\bar{\nabla}_j \bar{L}^t_{ikl} = \bar{\nabla}_j \bar{L}^t_{kil} + \bar{\nabla}_j \bar{L}^t_{lki}$

Applying (1.10) and (1.11) in (1.9) and using (1.9) again in the resulting equation, we get

(1.12) $\bar{\nabla}_l \bar{L}^t_{jik} + \bar{\nabla}_j \bar{L}^t_{lki} + \bar{\nabla}_i \bar{L}^t_{kjl} + \bar{\nabla}_k \bar{L}^t_{ilj} = 0.$

This is the Veblen identity for the curvature tensor L^t_{jkl} of the semi-symmetric affine connection (0.1) for which ϕ_j is a gradient vector.

2. Bianchi and Veblen identities for the projective curvature tensor P^i_{jkl} :

From (0.5) and (1.7) we can write

(2.1) $$\bar{P}^t_{ijk} = e^{4\phi}\, P^t_{ijk}$$

By virtue of (2.1) and (1.9) we have

(2.2) $$\bar{\nabla}_l \bar{P}^t_{ijk} + \bar{\nabla}_j \bar{P}^t_{ikl} + \bar{\nabla}_k \bar{P}^t_{ilj} = \frac{2}{N+1}\delta^t_i(\bar{\nabla}_j \bar{\lambda}_{kl} + \bar{\nabla}_k \bar{\lambda}_{lj} + \bar{\nabla}_l \bar{\lambda}_{jk})$$
$$+ \frac{1}{N^2-1}[\delta^t_l \{\bar{\nabla}_k(N\bar{L}_{ij} + \bar{L}_{ji}) - \bar{\nabla}_j(N\bar{L}_{ik} + \bar{L}_{ki})\}$$
$$+ \delta^t_k \{\bar{\nabla}_j(N\bar{L}_{il} + \bar{L}_{li}) - \bar{\nabla}_l(N\bar{L}_{ij} + \bar{L}_{ji})\}$$
$$+ \delta^t_j \{\bar{\nabla}_l(N\bar{L}_{ik} + \bar{L}_{ki}) - \bar{\nabla}_k(N\bar{L}_{il} + \bar{L}_{li})\}]$$

Contracting t and l in (1.9) and applying the result to

$\bar{L}^t_{ijk} + \bar{L}^t_{jki} + \bar{L}^t_{kij} = 0$, we get

$$\overline{\nabla}_j\bar{\lambda}_{kl} + \overline{\nabla}_k\bar{\lambda}_{ij} + \overline{\nabla}_l\bar{\lambda}_{jk} = 0. \tag{2.3}$$

Again, contracting t and l in (2.2), we get, by virtue of (2.3),

$$\overline{\nabla}_h\bar{P}^h_{ijk} = \frac{N-2}{N-1}[\overline{\nabla}_k(N\bar{L}_{ij} + \bar{L}_{ji}) - \overline{\nabla}_j(N\bar{L}_{ik} + \bar{L}_{ki})]. \tag{2.4}$$

Applping (2.4) to (2.2), we get

$$\overline{\nabla}_l\bar{P}^t_{ijk} + \overline{\nabla}_j\bar{P}^t_{ikl} + \overline{\nabla}_k\bar{P}^t_{ilj} - \frac{1}{N-2}\overline{\nabla}_h(\delta^t_l\ \bar{P}^h_{ijk} + \delta^t_j\ \bar{P}^h_{ikl} + \delta^t_k\ \bar{P}^h_{ilj}) = 0. \tag{2.5}$$

This is the Bianchi identity for the projective curvature tensor of a semi-symmetric affine connection for which ϕ_j is a gradient vector.

The Veblen identity for the projective curvature tensor of a semi-symmetric affine connection for which ϕ_j is a gradient vector, is given by

$$\overline{\nabla}_l\bar{P}^t_{jik} + \overline{\nabla}_j\bar{P}^t_{lki} + \overline{\nabla}_i\bar{P}^t_{kjl} + \overline{\nabla}_k\bar{P}^t_{ilj} - \frac{1}{N-2}\overline{\nabla}_h(\delta^t_l\ \bar{P}^h_{jik} + \delta^t_j\ \bar{P}^h_{lki} + \delta^t_i\ \bar{P}^h_{kjl} + \delta^t_k\ \bar{P}^h_{ilj}) = 0. \tag{2.6}$$

The calculation is same as for the Veblen identity (1.12) for the tensor $\bar{L}^t_{ijk}$

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REFERENCES

[1] Eisenhart, L. P.—Non-Riemannian Geometry—Princeton Univ. Press, (1927).

[2] Gupta, Bandana—Number of Bianchi Identities for the Projective curvature tensor in a Riemannian space—*Bull. de l'acad., Polon. des Scie.*, Vol. XIII, No. 2, (1965), pp. 151-154.

[3] Roy Chowdhury, A. N.—The Bianchi and Veblen Identities for the Projective curvature tensor in a Riemannian space—*Bul. Math. de la soc. Sci.*, Math dela R. S. de Roumanie, Tome 9 (57) (1965), pp. 177-181.

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Dept. of Pure Math.
Calcutta University

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ON THE PRODUCT OF FUZZY SUBSETS

M. K. CHAKRABORTY and MILI DAS

1. Introduction.

Since the introduction of the concept of fuzzy subsets by Zadeh [8] in 1965, many researchers have contributed to the development of the theory in various directions and the theory has been found to be extremely fruitful in the application field ([1], [2], [3], [4], [6], [7], [9]). We know that if U be the reference set, any ordinary subset A of U can be represented by the corresponding characteristic function f_A mapping the elements of U to the set {0, 1}. If instead, a function $f_{\underset{\sim}{A}}$ maps U to the closed interval [0, 1], the parallel notion is called a fuzzy subset of U and will be denoted by $\underset{\sim}{A}$. Equality, inclusion, union, intersection and complementation are defined as below

Def. (i) $\underset{\sim}{A}=\underset{\sim}{B} \rightleftharpoons f_{\underset{\sim}{A}}(x)=f_{\underset{\sim}{B}}(x)$ for all $x \in U$

(ii) $\underset{\sim}{A}\subseteq\underset{\sim}{B} \rightleftharpoons f_{\underset{\sim}{A}}(x)\leqslant f_{\underset{\sim}{B}}(x)$ for all $x \in U$

(iii) $\underset{\sim}{A}\cup\underset{\sim}{B} \rightleftharpoons \max\,[f_{\underset{\sim}{A}}(x), f_{\underset{\sim}{B}}(x)]$ for all $x \in U$

(iv) $\underset{\sim}{A}\cap\underset{\sim}{B} \rightleftharpoons \min\,[f_{\underset{\sim}{A}}(x), f_{\underset{\sim}{B}}(x)]$ for all $x \in U$

(v) $\bar{\underset{\sim}{A}} \rightleftharpoons 1-f_{\underset{\sim}{A}}(x)$ for all $x \in U$.

The notion of fuzzy subset and the above definitions obviously generalise the theory of ordinary subsets of a set.

The notion of fuzzy relation among members of an ordinary set and some immediate consequences have been brilliantly studied by Kaufmann [5] and some further developments in this respect have been made in [2]. A fuzzy relation in an ordinary set S is a fuzzy subset of the product $S\times S$. The product of fuzzy subsets has, however, not been defined so far. In this paper we shall introduce this notion and shall derive some theorems showing thereby analogies and departures from ordinary set theory.

2. Product of fuzzy subsets.

Definition. Let U be the reference set and [0, 1] the membership set. Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be fuzzy subsets of U defined by the membership functions $f_{\underset{\sim}{A}}$ and $f_{\underset{\sim}{B}}$ respectively. The product $\underset{\sim}{A}\times\underset{\sim}{B}$ is the fuzzy subset of $U\times U$ defined by the membership function

$$f_{\underset{\sim}{A}\times\underset{\sim}{B}} : f_{\underset{\sim}{A}\times\underset{\sim}{B}}(x,y)=\min\,[f_{\underset{\sim}{A}}(x), f_{\underset{\sim}{B}}(y)] \text{ for all } x,y \in U.$$

Example : Let $U = \{a, b, c, d\}$,

A : $f_A(a) = \cdot 1$, $f_A(b) = \cdot 2$, $f_A(c) = 0$, $f_A(d) = 1$,

B : $f_B(a) = 0$, $f_B(b) = 1$, $f_B(c) = 1$, $f_B(d) = \cdot 5$.

Then $A \times B$ is the fuzzy subset of $U \times U$ defined by the membership function $f_{A \times B}$ which maps

$$
\begin{array}{llll}
(a,a) \to 0 & (b,a) \to 0 & (c,a) \to 0 & (d,a) \to 0 \\
(a,b) \to \cdot 1 & (b,b) \to \cdot 2 & (c,b) \to 0 & (d,b) \to 1 \\
(a,c) \to \cdot 1 & (b,c) \to \cdot 2 & (c,c) \to 0 & (d,c) \to 1 \\
(a,d) \to \cdot 1 & (b,d) \to \cdot 2 & (c,d \to 0 & (d,d) \to \cdot 5.
\end{array}
$$

This is a straightway generalisation of the cartesian product of two ordinary subsets of U.

The following results follow directly from the definition.

Theorem 2.1. (i) $A \times B \neq B \times A$

(ii) $(A \times B) \times C = A \times (B \times C)$

(iii) $A \times \phi = \phi \times A = \phi$, where ϕ is the null fuzzy set.

(iv) $f_{A \times U}(x,y) = f_A(x)$ for all pairs x,y of U.

We give below some more results that hold also in case of ordinary set theory.

Theorem 2.2. (i) $A \subseteq B \Rightarrow A \times C \subseteq B \times C$

and $C \times A \subseteq C \times B$

(ii) $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$

(iii) $(A \cup B) \times (C \cup D) \neq (A \times C) \cup (B \times D)$

(iv) $(A \cap B) \times C = (A \times C) \cap (B \times C)$

$C \times (A \cap B) = (C \times A) \cap (C \times B)$

(v) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

$(B \cup C) \times A = (B \times A) \cup (C \times A)$.

Proof. (i) $A \subseteq B$ implies $f_A(x) \leqslant f_B(x)$ for all x in U.

Now, $f_{A \times C}(x,y) = \min [f_A(x), f_C(y)]$
and $f_{B \times C}(x,y) = \min [f_B(x), f_C(y)]$ $\Big\}$ for all $x,y \in U$.

Case I. $f_A(x) \leqslant f_C(y)$.

Then $f_A(x) \leqslant f_C(y)$ and $f_B(x)$.

Hence $f_A(x) \leqslant \min[f_B(x), f_C(y)]$.

Hence $\min[f_A(x), f_C(y)] = f_A(x) \leqslant \min[f_B(x), f_C(y)]$.

Case II. $f_C(y) < f_A(x) \leqslant f_B$.

Obviously, $\min[f_A(x), f_C(y)] = \min[f_B(x), f_C(y)]$.

So, $A \times C \subseteq B \times C$.

Similarly, $C \times A \subseteq C \times B$.

(ii) $$f_{(A\cap B)\times(C\cap D)}(x, y) = \min[\min(f_A(x), f_B(x)), \min(f_C(y), f_D(y))]$$
$$= \min[f_A(x), f_B(x), f_C(y), f_D(y)]$$
$$= \min[\min(f_A(x), f_C(y)), \min(f_B(x), f_C(y))]$$
$$= f_{(A\times C)\cap(B\times D)}(x, y).$$

(iii) $$f_{(A\cup B)\times(C\cup D)}(x, y) = \min[\max(f_A(x), f_B(x)), \max(f_C(y), f_D(y))].$$
$$f_{(A\times C)\cup(B\times D)}(x, y) = \max[\min(f_A(x), f_C(y)), \min(f_A(x), f_D(y))].$$

Taking $f_A(x) = 1$, $f_B(x) = 0$, $f_C(y) = 0$, $f_D(y) = 1$, it is clear that the two membership functions are not same.

(iv) $(A \cap B) \times C = (A \cap B) \times (C \cap C) = (A \times C) \cap (B \times C)$ by (ii).

Similarly, $C \times (A \cap B) = (C \times A) \cap (C \times B)$.

(v) $$f_{A\times(B\cup C)}(x, y) = \min[f_A(x), \max(f_B(y), f_C(y))].$$
$$f_{(A\times B)\cup(A\times C)}(x, y) = \max[\min(f_A(x), f_B(y), \min(f_A(x), f_C(y))].$$

There are six possibilies

$$f_A(x) \geqslant f_B(y) \geqslant f_C(y)$$
$$f_A(x) \geqslant f_C(y) \geqslant f_B(y)$$
$$f_B(y) \geqslant f_A(x) \geqslant f_C(y)$$
$$f_B(y) \geqslant f_C(y) \geqslant f_A(x)$$
$$f_C(y) \geqslant f_A(x) \geqslant f_B(y)$$
$$f_C(y) \geqslant f_B(y) \geqslant f_A(x).$$

In all the cases the above membership functions are identical.

Definition. The difference, disjunctive sum, algebraic product and algebraic sum of two fuzzy sets have been defined respectively as follows [5].

$$\underset{\sim}{A} - \underset{\sim}{B} = \underset{\sim}{A} \cap \overline{\underset{\sim}{B}}.$$

$$\underset{\sim}{A} \oplus \underset{\sim}{B} = (\underset{\sim}{A} \cap \overline{\underset{\sim}{B}}) \cup (\overline{\underset{\sim}{A}} \cap \underset{\sim}{B}).$$

$\underset{\sim}{A} \,.\, \underset{\sim}{B}$ by the membership function $f_{\underset{\sim}{A}}(x).\, f_{\underset{\sim}{B}}(x)$.

$\underset{\sim}{A} \hat{+} \underset{\sim}{B}$ by $f_{\underset{\sim}{A}}(x) + f_{\underset{\sim}{B}}(x) - f_{\underset{\sim}{A}}(x).\, f_{\underset{\sim}{B}}(x)$.

The following theorem contains the departure from the theory of ordinary sets.

Theorem 2.2. (i) $(\underset{\sim}{A} - \underset{\sim}{B}) \times \underset{\sim}{C} \neq (\underset{\sim}{A} \times \underset{\sim}{C}) - (\underset{\sim}{B} \times \underset{\sim}{C})$.

(ii) $(\underset{\sim}{A} \oplus \underset{\sim}{B}) \times \underset{\sim}{C} \neq (\underset{\sim}{A} \times \underset{\sim}{C}) \oplus (\underset{\sim}{B} \times \underset{\sim}{C})$.

(iii) $(\underset{\sim}{A} \,.\, \underset{\sim}{B}) \times \underset{\sim}{C} \neq (\underset{\sim}{A} \,.\, \underset{\sim}{C}) \times (\underset{\sim}{B} \,.\, \underset{\sim}{C})$.

(iv) $(\underset{\sim}{A} \hat{+} \underset{\sim}{B}) \times \underset{\sim}{C} \neq (\underset{\sim}{A} \times \underset{\sim}{C}) \hat{+} (\underset{\sim}{B} \times \underset{\sim}{C})$.

Proof: (1)
$$f_{(\underset{\sim}{A} - \underset{\sim}{B}) \times \underset{\sim}{C}}(x, y) = \min [f_{\underset{\sim}{A} \cap \overline{\underset{\sim}{B}}}(x), f_{\underset{\sim}{C}}(y)]$$
$$= \min [f_{\underset{\sim}{A}}(x), 1 - f_{\underset{\sim}{B}}(x), f_{\underset{\sim}{C}}(y)].$$

and
$$f_{(\underset{\sim}{A} \times \underset{\sim}{C}) - (\underset{\sim}{B} \times \underset{\sim}{C})}(x,y) = \min [\min (f_{\underset{\sim}{A}}(x), f_{\underset{\sim}{C}}(y)), 1 - \min (f_{\underset{\sim}{B}}(x), f_{\underset{\sim}{C}}(y))].$$

Taking $f_{\underset{\sim}{A}}(x) = {\cdot}5$, $f_{\underset{\sim}{B}}(x) = 1$, $f_{\underset{\sim}{C}}(y) = {\cdot}5$ the non-identity of the above membership functions are established.

(ii)
$$(\underset{\sim}{A} \oplus \underset{\sim}{B}) \times \underset{\sim}{C} = [(\underset{\sim}{A} \cap \overline{\underset{\sim}{B}}) \cup (\overline{\underset{\sim}{A}} \cap \underset{\sim}{B})] \times \underset{\sim}{C}$$
$$= [(\underset{\sim}{A} \cap \overline{\underset{\sim}{B}}) \times \underset{\sim}{C}] \cup [(\overline{\underset{\sim}{A}} \cap \underset{\sim}{B}) \times \underset{\sim}{C}] \quad \text{by th. 2·2}$$
$$= [(\underset{\sim}{A} \times \underset{\sim}{C}) \cap (\overline{\underset{\sim}{B}} \times \underset{\sim}{C})] \cup [(\overline{\underset{\sim}{A}} \times \underset{\sim}{C}) \cap (\underset{\sim}{B} \times \underset{\sim}{C})] \quad \text{by th. 2·2}$$

So
$$f_{(\underset{\sim}{A} \oplus \underset{\sim}{B}) \times \underset{\sim}{C}}(x, y) =$$
$$\max [\min (f_{\underset{\sim}{A}}(x), 1 - f_{\underset{\sim}{B}}(x), f_{\underset{\sim}{C}}(y)), \min (1 - f_{\underset{\sim}{A}}(x), f_{\underset{\sim}{B}}(x), f_{\underset{\sim}{C}}(y))],$$

and
$$(\underset{\sim}{A} \times \underset{\sim}{C}) \oplus (\underset{\sim}{B} \times \underset{\sim}{C}) =$$
$$[(\underset{\sim}{A} \times \underset{\sim}{C}) \cap \overline{(\underset{\sim}{B} \times \underset{\sim}{C})}] \cup [\overline{(\underset{\sim}{A} \times \underset{\sim}{C})} \cap (\underset{\sim}{B} \times \underset{\sim}{C})].$$

So
$$f_{(\underset{\sim}{A} \times \underset{\sim}{C}) \oplus (\underset{\sim}{B} \times \underset{\sim}{C})}(x, y) =$$
$$\max [\min \{\min (f_{\underset{\sim}{A}}(x), f_{\underset{\sim}{C}}(y)), 1 - \min (f_{\underset{\sim}{B}}(x), f_{\underset{\sim}{C}}(y))\},$$
$$\min \{1 - \min (f_{\underset{\sim}{A}}(x), f_{\underset{\sim}{C}}(y)), \min (f_{\underset{\sim}{B}}(x), f_{\underset{\sim}{C}}(y))\}].$$

The non-identity of the two functions is proved by taking

$$f_{\underset{\sim}{A}}(x) = \cdot 9,\ f_{\underset{\sim}{B}}(x) = \cdot 8,\ f_{\underset{\sim}{C}}(y) = \cdot 3.$$

(iii) $f_{(\underset{\sim}{A}.\underset{\sim}{B})\times\underset{\sim}{C}}(x, y) = \min\,[f_{\underset{\sim}{A}}(x).f_{\underset{\sim}{B}}(x),\ f_{\underset{\sim}{C}}(y)].$

$f_{(\underset{\sim}{A}\times\underset{\sim}{C}).(\underset{\sim}{B}\times\underset{\sim}{C})}(x, y) = \min\,[f_{\underset{\sim}{A}}(x),\ f_{\underset{\sim}{C}}(y)].\min\,[f_{\underset{\sim}{B}}(x),\ f_{\underset{\sim}{C}}(y)].$

Take $f_{\underset{\sim}{A}}(x) = \cdot 5,\ f_{\underset{\sim}{B}}(x) = \cdot 5,\ f_{\underset{\sim}{C}}(y) = \cdot 1.$

(iv) $f_{(\underset{\sim}{A}\hat{+}\underset{\sim}{B})\times\underset{\sim}{C}}(x, y) = \min\,[f_{\underset{\sim}{A}}(x) + f_{\underset{\sim}{B}}(x) - f_{\underset{\sim}{A}}(x).f_{\underset{\sim}{B}}(x),\ f_{\underset{\sim}{C}}(y)].$

$f_{(\underset{\sim}{A}\times\underset{\sim}{C})\hat{+}(\underset{\sim}{B}\times\underset{\sim}{C})}(x, y) = \min\,[f_{\underset{\sim}{A}}(x),\ f_{\underset{\sim}{C}}(y)] + \min\,[f_{\underset{\sim}{B}}(x),\ f_{\underset{\sim}{C}}(y)]$

$- \min\,[f_{\underset{\sim}{A}}(x),\ f_{\underset{\sim}{C}}(y)].\min\,[f_{\underset{\sim}{B}}(x),\ f_{\underset{\sim}{C}}(y)].$

Take $f_{\underset{\sim}{A}}(x) = 1,\ f_{\underset{\sim}{B}}(x) = 1,\ f_{\underset{\sim}{C}}(y) = \cdot 5.$

Remarks. In ordinary set theory equality holds in all the cases stated in theorem 2.3. Algebraic product and sum reduce to intersection and union when the sets are ordinary.

3. Conclusion.

Fuzzy relation between two ordinary sets has been extensively studied ([5], [2]). With the introduction of the product of two fuzzy subsets, a natural possibility has developed to think about the concept of relation between two fuzzy subsets that can be defined as a fuzzy subset of the product. Reflexivity, symmetry etc. may then be defined and corresponding results obtained in this direction will be presented in a future publication.

REFERENCES

[1] Bellman, R. E. & Zadeh, L. A.—Decision making in a fuzzy environment, Management Science, **17** (4) Dec. 1970, 141-164.

[2] Chakraborty, M. K. & Sen. D.—An introduction to the theory of Fuzzy sets and its Applications, Proceedings of the Second Mathematics Conference, Bangladesh Math. Society, Dacca, October 1980.

[3] Chang, C. L.—Fuzzy Topological spaces, *Jour. Math.* Anal. and appl. ; **24** (1968), 182-190.

[4] Goguen, J. A.—L-Fuzzy sets, *Jour. Math. Anal. and Appl.* **18** (1967), 145-174.

[5] Kaufmann, A.—Introduction to the theory of Fuzzy subsets, Vol. I, Acad. Press Inc, 1975.

[6] Tanaka K & Mijumato M.—Fuzzy programs and their executions. Fuzzy Sets and their applications to Cognitive and Decision Process, Acad. Press Inc. 1975.

[7] Wong C. K.—Covering properties of fuzzy topological spaces, *Jour. Math. Anal. & Appl.* **43** (1973), 697-704.

[8] Zadeh, L. A.—Fuzzy sets ; *Information and control*, 8 (1965) 338-353.

[9] ——————— ; A Fuzzy-algorithmic approach to the definition of complex and imprecise concepts ; *Int. Jour. Man-Machine studies*, 8 (1976), 249-291.

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Calcutta University

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KINNEY'S FUNCTIONS IN STUDY OF SOME PROPERTIES OF THE CANTOR SET

D. K. GANGULY

1. Introduction and Notations :

For any $x \in [0, 1]$, let $x = \sum_{i=1}^{\infty} \frac{x_i}{3^i}$, $x_i = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ for every i.

Kinney [3] defined two functions $f(x)$ and $\nu(x)$ by the relations :

$$f(x) = \sum_{i=1}^{\infty} \frac{f_i(x)}{3^i} \text{ and } \nu(x) = \sum_{i=1}^{\infty} \frac{\nu_i(x)}{3^i} \text{ where } f_i(x) = 2\delta(x_i, 2) \text{ and } \nu_i(x) = 2\delta(x_i, 1)$$

with the properties that $\delta(a, b) = 1$ if $a = b$ and $\delta(a, b) = 0$ if $a \neq b$. It follows that $x = f(x) + \frac{1}{2}\nu(x)$ and $f(x) \in c$ and $\nu(x) \in$ C where C is the Cantor set for any $x \in [0, 1]$.

To avoid the ambiguity, we make the convention that any ternary representation of x in [0. 1] should not end with a chain of 2's but we take $f(1) = 1$ and $\nu(1) = 0$, by definitions.

It is known that the Cantor set C possesses the Steinhaus property as well as the Randolph property i.e. for any $d \in [0, 1]$ we can find a pair x_1, x_2 of Cantor points such that $x_2 - x_1 = d$ and also a pair y_1, y_2 of Cantor points such that $\frac{y_1 + y_2}{2} = d$, ([1], [4], [5], [7]).

A set E is said to be an (SD)-set if its distance set fills an interval about the origin, whose length is equal to the diameter of the set E [2].

A set E defined in [0, 1] is said to be symmetrical if x and $(1-x)$ both belong to E.

Cantor set C is an (SD)-set and is also symmetrical. That the Cantor set C possesses both the Steinhaus and the Randolph properties has been shown by the above mathematicians.

I propose to give here another method of the proof of the same properties, which seems to me to be shorter and direct, compared to the proofs given earlier.

Theorem (Randolph). Every point between 0 and 1 is the arithmatical mean of at least one pair of Cantor points.

Proof: Let $f(x)$ and $\nu(x)$ be the Kinney's functions defined in [0, 1]. If x is any point in [0, 1], then $x=f(x)+\frac{1}{2}\nu(x)$ . . .. (1)

and $x=\sum_{i=1}^{\infty}\frac{x_i}{3^i}$, $x_i=\begin{pmatrix}0\\1\\2\end{pmatrix}$ for every $i=1, 2, 3,\ldots\ldots\ldots$

It follows that, $f(x)=\sum_{i=1}^{\infty}\frac{f_i(x)}{3^i}$ and $\nu(x)=\sum_{i=1}^{\infty}\frac{\nu_i(x)}{3^i}$

where $f_i(x)=2\delta(0, 2)=0$ and $\nu_i(x)=2\delta(0, 1)=0$,

when $x_i=0$.

Thus $f_i(x)+\nu_i(x)=0$, when $x_i=0$,

Again $f_i(x)=2\delta(1, 2)=0$ and $\nu_i(x)=2\delta(1, 1)=2$,

when $x_i=1$.

Thus $f(x)+\nu_i(x)=2$, when $x_i=1$.

Finally $f_i(x)=2\delta(2, 2)=2$ and $\nu_i(x)=2\delta(2, 1)=0$,

when $x_i=2$.

Thus $f_i(x)+\nu_i(x)=2$, when $x_i=2$.

Therefore, $f(x)+\nu(x)=\sum_{i=1}^{\infty}\frac{f_i(x)+\nu_i(x)}{3^i}=\sum_{i=1}^{\infty}\frac{a_i}{3^i}$

where $a_i=f_i(x)+\nu_i(x)=0$ or 2 for every $i=1, 2, 3\ldots\ldots\ldots$

It follows that $\psi(x)=f(x)+\nu(x)\in C$ for every $x\in[0, 1]$.

Now from (1), we have $2x=2f(x)+\nu(x)=f(x)+\{f(x)+\nu(x)\}=f(x)+\psi(x)$.

Thus $x=\frac{f(x)+\psi(x)}{2}$ where $f(x)\in C$, $\psi(x)\in C$ and x is any point in [0, 1]. Hence the theorem.

Corollary (Steinhau's Theorem): The distance set of the Cantor middle third set C fills the unit interval $0\leqslant d\leqslant 1$ [i.e. C is an (SD)-set].

Proof: The proof of this theorem follows from Randolph's theorem and a theorem given by Bose Majumdar [1] stated below.

"A necessary and sufficient condition that a linear symmetrical set E defined in [0, 1] may be an (SD)-set is that it satisfies Randolph's property".

[In fact, if $d \in [0, 1]$, then taking $1-2x=d$, we get $0 \leqslant x \leqslant \frac{1}{2}$ and hence by Randolph's theorem $x=\frac{f(x)+\varphi(x)}{2}$ where $f(x) \in C$ and $\psi(x) \in C$ and thus $d=f_1(x)-\psi(x)$ where $f_1(x)=1-f(x) \in C$].

2. In elementary books on algebra and trigonometry methods are given for solving equations and also constructing equations when roots are given.

In the following particular case, we propose to show that we can construct equations whose roots are precisely all the numbers of the perfect set C and no number outside it.

Theorem : Each of the equations (1) $f(x)=x$ and (2) $\nu(x)=0$ are satisfied if and only if $x \in C$, where $f(x)$ and $\nu(x)$ are Kinney's functions defined in $[0, 1]$, and C is the Cantor set.

Proof : The condition is sufficient.

We are given any $x \in C$. It is to be shown that x is a solution of each of the equation (1) and (2).

Let $x=\sum_{i=1}^{\infty} \frac{a_i}{3^i}$ where $a_i=\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ for each $i=1, 2, 3\ldots\ldots$

Then $f(x)=\sum_{i=1}^{\infty} \frac{f_i(x)}{3^i}$ and $\nu(x)=\sum_{i=1}^{\infty} \frac{\nu_i(x)}{3^i}$

Now, $f_i(x)=0$ and $\nu_i(x)=0$, when $a_i=0$

Again $f_i(x)=2$ and $\nu_i(x)=0$, when $a_i=2$.

Hence $f_i(x)=a_i$ and $\nu_i(x)=0$ for every $i=1, 2, 3\ldots\ldots$

Thus $f(x)=\sum_{i=1}^{\infty} \frac{f_i(x)}{3^i}=\sum_{i=1}^{\infty} \frac{a_i}{3^i}=x$

and $$\nu(x)=\sum_{i=1}^{\infty} \frac{\nu_i(x)}{3^i}=\sum_{i=1}^{\infty} \frac{0}{3^i}=0.$$

Hence every $x \in C$ satisfies the equations (1) and (2).

The condition is necessary.

Let $x \in [0, 1]$ be such that $f(x)=x$ and $\nu(x)=0$.

Then we show that x is necessarily a Cantor point.

Let $f(x)=\sum_{i=1}^{\infty}\frac{f_i(x)}{3^i},\ \nu(x)=\sum_{i=1}^{\infty}\frac{\nu_i(x)}{3^i},$

where $$x=\sum_{i=1}^{\infty}\frac{\alpha_i}{3^i},\ \alpha_i=\begin{pmatrix}0\\1\\2\end{pmatrix} \text{ for every } i=1, 2, 3\ldots\ldots\ldots$$

Now, if $\alpha_i=0$, $f_i(x)=0$ and $\nu_i(x)=0$

if $\alpha_i=1$, $f_i(x)=0$ and $\nu_i(x)=2$

if $\alpha_i=2$, $f_i(x)=2$ and $\nu_i(x)=0$.

Thus $f_i(x)=\alpha_i$ if and only if $\alpha_i=0$ or 2.

and $\nu_i(x)=0$ if and only if $\alpha_i=0$ or 2.

It follows that $f(x)=x$ implies that $\sum_{i=1}^{\infty}\frac{f_i(x)}{3^i}=\sum_{i=1}^{\infty}\frac{\alpha_i}{3^i}$

which implies that $\alpha_i=f_i(x)$ for all positive integers i and this happens only when $\alpha_i=0$ or 2 (and never 1) and this implies $x \in C$.

Therefore $f(x)=x$ implies $x \in C$. Similarly $\nu(x)=0$ implies that $\sum_{i=1}^{\infty}\frac{\nu_i(x)}{3^i}=0$ which implies that $\nu_i(x)=0$ for all $i=1, 2, 3,\ldots\ldots\ldots$and this happens only when $\alpha_i=0$ or 2 (and never 1) and this implies that $x \in c$.

Hence the theorem is completely proved.

Corollary : Kinney's function $f(x)$ has the following property on the set C, viz $f\left(\frac{x_1+x_2}{2}\right) \leqslant \frac{f(x_1)+f(x_2)}{2}$ for any two points x_1 and x_2 belonging to C (This property of $f(x)$ is some what similar to that of functions convex downward).

Proof ; Let x_1 and x_2 be any two points of C.

Then $f(x_1)=x_1=\sum_{i=1}^{\infty}\frac{a_i}{3^i},\ a_i=\begin{pmatrix}0\\2\end{pmatrix}$ for every $i=1, 2, 3\ldots\ldots$

and $f(x_2)=x_2=\sum_{i=1}^{\infty}\frac{b_i}{3^i},\ b_i=\begin{pmatrix}0\\2\end{pmatrix}$ for every $i=1, 2, 3\ldots\ldots$

Hence

$$(1)\quad \frac{f(x_1)+f(x_2)}{2}=\frac{x_1+x_2}{2}=\sum_{i=1}^{\infty}\frac{\frac{a_i+b_i}{2}}{3^i}=\sum_{i=1}^{\infty}\frac{c_i}{3^i}$$

We now write $x=\frac{x_1+x_2}{2}$ where $x_1 \in C$ and $x_2 \in C$.

Then

$$(2)\quad f\left(\frac{x_1+x_2}{2}\right)=f(x)=\sum_{i=1}^{\infty}\frac{f_i(x)}{3^i}.$$

where $x=\sum_{i=1}^{\infty}\frac{c_i}{3^i}$, by (1)

If $\begin{cases} a_i=0 \\ b_i=0, \end{cases}$ $c_i=0$ and thus $f_i(x)=0=c_i$

If $\begin{cases} a_i=0 \\ b_i=2, \end{cases}$ $c_i=1$ and thus $f_i(x)=0 < c_i$

If $\begin{cases} a_i=2 \\ b_i=0, \end{cases}$ $c_i=1$ and thus $f_i(x)=0 < c_i$

If $\begin{cases} a_i=2 \\ b_i=2, \end{cases}$ $c_i=2$ and thus $f_i(x)=2=c_i$

In any case, therefore $f_i(x) \leqslant c_i$ for every $i=1, 2, 3\ldots\ldots$

Hence for (2) $f\left(\frac{x_1+x_2}{2}\right)=\sum_{i=1}^{\infty}\frac{f_i(x)}{3^i}=\frac{f(x_1)+f(x_2)}{2}$ by (1)

This proves the theorem.

Note: We know that any point $x \in C$ is the middle point of a unique pair of points x_1, x_2 where $x_1 \in C$, $x_2 \in C$ [6]

That is, $\frac{x_1+x_2}{2}=x$ where $x \in c$, $x_1 \in c$, $x_2 \in c$.

Hence by above corollary $f(x)=f\left(\frac{x_1+x_2}{2}\right) \leqslant \frac{f(x_1)+f(x_2)}{2}$ where x_1, x_2 and $\frac{x_1+x_2}{2}$ are all points of C.

REFERENCES

[1] Bose Majumdar, N. C.—"On the distance set of the Cantor middle third set"—*Bull. Cal. Math. Soc.*, **51** (1959), 93-102.

[2]———"A study of certain properties of the Cantor set and of an (SD)-set"—*Bull. Cal. Math. Soc.*, **54**, (1962), 8-20.

[3] Kinney, J. R.—"A thin set of lines"—*Israiel J. Math.* **8** (1970), 97-102.

[4] Randolph, J. F.—"Distance between points of the Cantor set"—*Amer. Math. Monthly*, **47** (1940) 549-551.

[5] Steinhaus, H.—"Sur les distance des points des exsembles de measure positive"—*Fund. Math. Month.*, **1** (1920), 93-104.

[6] Sengupta, S. B.—"On some new properties of the Cantor set"—Casopis propes to vani mathematiky roc **96** (1971), Praha, 270-277.

[7] Utz, W. R.—"The distance set for the Cantor discontinuum"—*Amer. Math. Monthly*, **58** (1951), 407-408.

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Dept. of Pure Math.
Calcutta University

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L^2-CLASSIFICATION OF A VECTOR-MATRIX DIFFERENTIAL EQUATION

PRABIR KUMAR SEN GUPTA

1. Introduction

Let M denote the formally symmetric, second-order matrix differential expression given by, for suitably differentiable real-valued vector function $f=(f_1, f_2)^T$,

$$M[f]=\begin{bmatrix} -\dfrac{d}{dx}\left(p\dfrac{d}{dx}\right)+q_1 & q_2 \\ q_2 & -\dfrac{d}{dx}\left(r\dfrac{d}{dx}\right)+q_3 \end{bmatrix} f \qquad \text{on } [a, b) \tag{1.1}$$

where the coefficients p, r and q_j $(j=1, 2, 3)$ satisfy the following conditions

(i) $p(x)$ and $r(x)$ are real-valued and absolutely continuous on all compact sub-intervals of $[a, b)$ and $p(x), r(x)>0$ $\qquad (x \in [a, b))$

(ii) q_j $(j=1, 2, 3)$ are real-valued and continuous on $[a, b)$ with $q_1>0$ and $q_1q_3-q_2^2\geqslant 0$

$-\infty<a<b\leqq\infty$.

Moreover if $\dfrac{1}{pr}$, q_1, q_2, q_3 are summable in the whole interval $[a, b)$ then the differential expression $M[f]$ is said to be regular at all points of $[a, b)$ i.e if $\xi \in [a, b)$ then the intial value problem

$$M[f]=0$$
$$f_1(\xi)=A, \quad (pf'_1)'(\xi)=C$$
$$f_2(\xi)=B, \quad (rf'_2)'(\xi)=D$$

on $[a, b)$ can be solved for arbitrary constants A, B, C, D : For this result see the existence Theorem 3.1, Sen Gupta [4] ; otherwise, $M[.]$ is said to be singular at the open end-point b(or if $b=\infty$).

The vector function $U=(u, v)^T$ is said to be a solution of (1.1) if u, v, pu' and rv' are absolutely continuous on all compact sub-intervals of $[a, b)$ and

$$\left.\begin{aligned} -(pu')'+q_1u+q_2v=0 \\ -(rv')'+q_2u+q_3v=0 \end{aligned}\right\}$$

2. Preliminaries

The Green's formula, for any two vector functions $f=(f_1, f_2)^T$ and $g=(g_1, g_2)^T$ sufficiently smooth, takes the form

$$\int_a^b \{f^T M[g]-g^T M[f]\}dx = [fg](b) - [fg](a)$$

when the bilinear form $[fg]$ (.) is given by

$$[fg]\,(x)=p(x)\,f_1(x)\,g'_1(x)-p(x)\,f'_1(x)\,g_1(x)+r(x)\,f_2(x)\,g'_2(x)-r(x)\,f'_2(x)\,g_2(x).$$

It is well known that, if f, g are the solutions of (1.1) then $[fg]$ (·) is independent of x.

3. L^2-classification

A vector function $f(x)$ which satisfies the differential system (1.1) is said to be a L^2-solution of (1.1) if

$$\int_0^\infty f^T f\,dx < \infty$$

holds. (i.e when each element of the vector function is square-integrable)

It was proved in Chakravarty [1, 2] and Sen Gupta [4, 5] that the differential system of the type (1.1) i.e a pair of second order differential systems can have at least 2 and atmost 4 L^2-solutions.

M [·] is said to be in the limit -2, 3 or 4 at infinity according as (1.1) has 2, 3 or 4 linearly independent solutions in $L^2(0, \infty)$, (the Hilbert space of vector functions with integrable square).

Theorem I. Let $N(x)$ be a positive, non-decreasing function such that

(i) $\int^\infty \frac{dx}{\sqrt{(pN)}}, \int^\infty \frac{dx}{\sqrt{(rN)}}$ diverges (3.1)

(ii) $\overline{\lim}_{x\to\infty} \frac{N'\sqrt{p}}{\sqrt{(N^3)}}$ converges, (3.2)

further, for all sufficiently large values of x

let $\frac{q_1 q_3 - q_2^2}{q_1} > -K\,N(x)$ (3.3)

(K is a positive constant)

Then the differential system (1.1) *is not limit*—4.

Proof. To prove the theorem it is sufficient to show that the differentia- system

$$M[U]=0 \tag{3.4}$$

has at least one solution not belonging to $L^2(0, \infty)$.

Multiplying the equation (3.4) by $U^T=(u, v)$ and dividing by N we get

$$-\frac{q_1u^2+2q_2uv+q_3v^2}{N}=-\frac{(pu')'u+(rv')'v}{N}$$

Integrating both sides,

$$-\int_a^x \frac{q_1u^2+2q_2uv+q_3v^2}{N}\,dt=-\left[\frac{puu'+rvv'}{N}\right]_a^x+\int_a^x \frac{pu'^2+rv'^2}{N}\,dt-\int_a^x \frac{(puu'+rvv')N'}{N^2}\,dt \tag{3.5}$$

But,

$$-\int_a^x \frac{q_1u^2+2q_2uv+q_3v^2}{N}\,dt = -\int_a^x \frac{1}{N}\left\{q_1\left(u+\frac{q_2}{q_1}v\right)^2+\frac{q_1q_3-q_2^2}{q_1}v^2\right\}dt$$

$$< K\int_a^x v^2dt < K\int_a^\infty v^2dt \qquad \text{[using (3.3)]}$$

$$=K_1 \text{ (say)} \qquad \text{[Supposing } U\in L^2[O, \infty)]$$

Hence from (3.5)

$$K_1 > -\left[\frac{puu' + rvv'}{N}\right]_a^x+\int_a^x \frac{pu'^2+rv'^2}{N}\,dt-\int_a^x \frac{(puu'+rvv')N'}{N^2}dt, \ (\forall x) \tag{3.6}$$

We now show that if the solution $(u, v) \in L^2\,[O, \infty)$,

then the integral

$$\int_a^\infty \frac{pu'^2+rv'^2}{N}\,dt \text{ converges.}$$

Conversely, suppose that this integral diverges. Then the function

$$H(x)=\int_a^x \frac{pu'^2+rv'^2}{N}\,dt \tag{3.7}$$

is positive, monotonically increasing and tends to $+\infty$ as $x \to \infty$.

Now using Cauchy—Buniakovski inequality and the condition (3.2), we have, for a sufficiently large 'a' and for $x > a$

$$\left| \int_a^x \frac{puu'+rvv'}{N^2} N' dt \right| < \int_a^x \left\{ \left(\sqrt{\frac{p}{N^3}} N' u \right)^2 + \left(\sqrt{\frac{r}{N^3}} N' v \right)^2 \right\}^{1/2} \left\{ \frac{pu'^2+rv'^2}{N} \right\}^{1/2} dt$$

$$< K_2 \int_a^x (u^2+v^2)^{1/2} \left\{ \frac{pu'^2+rv'^2}{N} \right\}^{1/2} dt$$

$$< K_2 \left\{ \int_a^x (u^2+v^2) dt \right\}^{1/2} \left\{ \int_a^x \frac{pu'^2+rv'^2}{N} dt \right\}^{1/2}$$

$$< K_3 \sqrt{H(x)}$$

where K_2 is a certain constant depending on p, r and N, and

$$K_3 = K_2 \left(\int_0^\infty (u^2+v^2) \, dt \right)^{1/2}$$

Applying these results in (3.6) we get

$$K_1 > H(x) - \left[\frac{puu'+rvv'}{N} \right]_a^x - K_3 \sqrt{H(x)} \qquad (\forall x),$$

Since $H(x) \to \infty$ as $x \to \infty$, the last inequality can hold only if

$$\frac{puu'+rvv'}{N} > 0 \qquad \text{for large } x$$

i.e. $\quad puu'+rvv' > 0 \qquad$ (as $N > 0$)

or, $\quad \frac{p}{r} uu' > -vv' \qquad (r > 0)$

Two cases may arise :

Case (i) u and u' are of opposite sign.

Then $\frac{p}{r} uu'$ is negative since $\frac{p}{r} > 0$, and hence vv' is positive, which indicates that v and v' have the same sign for sufficiently large x.

Case (ii) u and u' have the same sign.

Thus either u and u' or v and v' are of the same sign. In either case one of the two integrals

$$\int^\infty u^2 \, dx \quad \text{and} \quad \int^\infty v^2 \, dx$$

fails to exist, contradictory to the hypothesis $U \in L^2 [0, \infty)$:

Thus

$$\int_0^\infty \frac{pu'^2+rv'^2}{N}\,dx \quad \text{exist for } U=(u,v)^T \in L^2\,[0,\infty)$$

so that

$$\sqrt{\frac{p}{N}}\,u',\ \sqrt{\frac{r}{N}}\,v' \in L^2\,[0,\ \infty). \tag{3.8}$$

Now let $F_j(x,\lambda)=(f_j(x,\lambda),\ g_j(x,\lambda))^T$, $j=1,2,3,4$, be the four linearly independent square-integrable solutions of the system $M[f]=\lambda f$. It is well known that $P_{jk}=[f_j(x,\lambda)\ f_k(x,\lambda)]$, $j,k=1,\ 2,3,4$; $j\neq k$ is an integral function of λ independent of x. The Wronskian for the system is then given by

$$W(\lambda)\equiv W(F_1,F_2,F_3,F_4) = P_{12}\,P_{34}-P_{13}\,P_{24}+P_{14}\,P_{23}$$

which is equal to some constant c (not equal to zero), since the four solutions F_1, F_2, F_3, F_4 are linearly independent. Therefore at least one of the $P_{jk}\neq 0$. Say $P_{12}=k\neq 0$

i.e.
$$p\,f_1\,f_2'-p\,f_1'\,f_2+r\,g_1\,g_2'-r\,g_1'\,g_2 = k \tag{3.9}$$

case (i) if $p>r$, dividing both sides of (3.9) by $\sqrt{pN}$ and taking moduli we obtain

$$\sqrt{\frac{p}{N}}\,|f_1|\,|f_2'|+\sqrt{\frac{p}{N}}\,|f_1'|\,|f_2|+\frac{r}{\sqrt{pN}}\,|g_1|\,|g_2'|+$$
$$+\frac{r}{\sqrt{pN}}\,|g_1'|\,|g_2|\geqslant\frac{|k|}{\sqrt{pN}} \tag{3.10}$$

Since $p>r$, we have $\frac{r}{\sqrt{pN}}\,|v'|<\sqrt{\frac{r}{N}}\,|v'|$ and hence using (3.8),

$$\frac{r}{\sqrt{pN}}\,|v'|\in L^2\ [0,\infty). \tag{3.11}$$

Now integrating (3.10) over $(0,\infty)$ and utilising the results (3.8) and (3.11) we see that

$$\int^\infty \frac{|k|}{\sqrt{pN}} \text{ converges,}$$

which is not possible due to the condition given in (3.1).

case (ii) if $r>p$ we divide (3.9) by $\sqrt{rN}$ and utilise the results

$$\sqrt{\frac{r}{N}}\,|v'|,\ \frac{p}{\sqrt{rN}}\,|u'|\in L^2\,[0,\infty)$$

to show that

$$\int^{\infty} \frac{|k|}{\sqrt{rN}} \text{ converges,}$$

contradictory to the condition (3.1).

Thus the assumption $P_{12} \neq 0$ implies that both F_1 and F_2 cannot be square-integrable. Since at least one of $P_{jk} \neq 0$, all the four solutions F_j, $(j=1, 2, 3, 4)$ of the system cannot be square-integrable.

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REFERENCES

[1] Chakravarty, N. K.—Some problems in eigenfunction expansions III. Quart. J. Math. (Oxford) (2). 19(1968), 397-415.

[2] ——— Some problems in eigenfunction expansions IV. Ind. J. Pure. Appl. Math., (1970), 1, 347-353.

[3] Levinson, N.—Criteria for the limit-point case for 2nd-order linear differential operators, Chasopis Pohst. Math. Fys., 74, (1947) 17-20.

[4] Sen Gupta, Prabir Kumar—On a self-adjoint extension of a type of a matrix differential operator, J. Ind. Inst. Sci. 60(B), (1978), 225-234.

[5] ——— Integrable-square solutions of certain vector-matrix differential equations, J. Ind. Inst. Sci. 60(B), (1978), 235-245.

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Dept. Pure Math.
Calcutta University

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CHARACTERIZATION OF GROUPS

M. K. Sen

In this short note we shall give a characterization of groups in terms of two binary operations.

Let G be a set with two binary operations $*$ and o, satisfying the following two axioms :

(1) $(a \circ b) * c = b \circ (c * a)$,

(2) $a \circ (b * a) = b$.

Lemma 1. If $a \circ a = a$ and $b \circ b = b$ in G, then $a = b$.

Proof.
$$\begin{aligned} b * a &= (b \circ b) * a \\ &= b \circ (a * b) && \text{by (1)} \\ &= a. && \text{by (2)} \end{aligned}$$

Hence $a = a \circ a = a \circ (b * a) = b$.

Lemma 2. If $e_a = a * a$, then $a \circ e_a = a$.

Proof. $a \circ (a * a) = a$. by (1)

Hence $a \circ e_a = a$.

Lemma 3. If $e_a = a * a$, then $e_a \circ e_a = e_a$

Proof. $a = a \circ (a * a)$. by (2)

$$\begin{aligned} a * a &= (a \circ (a * a)) * a \\ &= (a * a) \circ (a * a). && \text{by (1)} \end{aligned}$$

This shows that $e_a \circ e_a = e_a$.

Lemma 4. e_a is independent of a, that is, $e_a = b * b$ for every b in G.

Proof. Let $e_b = b * b$. As in Lemma 3, we can show that $e_b \circ e_b = e_b$. Also we have $e_a \circ e_a = e_a$. Then from Lemma 1, it follows that $e_a = e_b$.

Let us write e for e_a.

Lemma 5. $e * b = b$ for all $b \in G$.

Proof.
$$\begin{aligned} e * b &= (e \circ e) * b \\ &= e \circ (b * e) && \text{by (1)} \\ &= b. && \text{by (2)} \end{aligned}$$

Lemma 6. $b \circ e = b$ for all $b \in G$.

Proof. $b \circ e = b \circ (b * b)$ by Lemma 4

$= b.$ by (2)

Lemma 7. $b * e = e \circ b$ for all $b \in G$.

Proof.

$$\begin{aligned} b * e &= (b \circ e) * e && \text{by Lemma 6} \\ &= e \circ (e * b) && \text{by (1)} \\ &= e \circ b. && \text{by Lemma 5} \end{aligned}$$

Lemma 8. $a * (b \circ c) = (a * b) \circ c$ for all $a, b, c \in G$.

Proof.

$$\begin{aligned} a * (b \circ c) &= (a \circ e) * (b \circ c) && \text{by Lemma 6} \\ &= e \circ ((b \circ c) * a) && \text{by (1)} \\ &= e \circ (c \circ (a * b)) && \text{by (1)} \\ &= (c \circ (a * b)) * e && \text{by Lemma 7} \\ &= (a * b) \circ (e * c) && \text{by (1)} \\ &= (a * b) \circ c. \end{aligned}$$

Theorem. If a binary operation in G is defined by $ab = (e \circ a) * b$ for all $a, b \in G$, then G is a group.

Proof. Let $a, b, c \in G$.

(1) Associativity :

$$\begin{aligned} (ab)c &= (e \circ ((e \circ a) * b)) * c && \text{by Definition} \\ &= ((b \circ e) * (e \circ a)) * c && \text{by (1)} \\ &= (b * (e \circ a)) * c && \text{by Lemma 6} \\ &= ((b * e) \circ a) * c && \text{by Lemma 8} \\ &= a \circ (c * (b * e)) && \text{by (1)} \\ &= a \circ (c * (e \circ b)) && \text{by Lemma 7} \\ &= a \circ ((c \circ e) * (e \circ b)) && \text{by Lemma 8} \\ &= a \circ (e \circ ((e \circ b) * c)) && \text{by (1)} \\ &= a \circ (((e \circ b) * c) * e) && \text{by Lemma 7} \\ &= (e \circ a) * ((e \circ b) * c) && \text{by (1)} \\ &= a(bc). && \text{by Definition} \end{aligned}$$

(2) Left identity :

$$
\begin{aligned}
eb &= (e \circ e) * b && \text{by Definition}\\
&= e \circ (b * e) && \text{by (1)}\\
&= b. && \text{by (2)}
\end{aligned}
$$

(3) Left inverse :

$$
\begin{aligned}
(e \circ a)a &= (e \circ (e \circ a)) * a && \text{by Definition}\\
&= ((e \circ a) * e) * a && \text{by Lemma 7}\\
&= (a \circ (e * e)) * a && \text{by (1)}\\
&= (a \circ e) * a && \text{by Lemma 5}\\
&= a * a && \text{by Lemma 6}\\
&= e. && \text{by Lemma 4}
\end{aligned}
$$

Hence G is a group.

Note 1. With respect to the operation $ab=(e \circ a) * b$ for all $a, b \in G$, G is a commutative group if $a * b=b \circ a$ for all $a, b \in G$.

Proof. Suppose $a * b=b \circ a$.

$$
\begin{aligned}
\text{Then } ab &= (e \circ a) * b && \text{by Definition}\\
&= b \circ (e \circ a) && \text{by Assumption}\\
&= b \circ (a * e) && \text{by Lemma 7}\\
&= (e \circ b) * a && \text{by (1)}\\
&= ba.
\end{aligned}
$$

Note 2. Let G be a group. If we define $a * b=a^{-1}b$ and $a \circ b=ab^{-1}$, then G satisfies both (1) and (2).

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ON A TYPE OF SEMI-SYMMETRIC CONNECTION ON A RIEMANNIAN MANIFOLD

M. C. CHAKI and ARABINDA KONAR

Introduction.

The present paper deals with a type of semi-symmetric metric connection ∇ on a Riemannian Manifold such that the curvature tensor R and the torsion tensor T of ∇ satisfy the conditions 1) $R(X, Y)Z=0$ and 2) $(\nabla_X T)(Y, Z) = B(X)\, T(Y, Z)$, where B is a 1-form. The nature of curvature restriction on the manifold induced by the introduction of such a connection is determined. Further, it is shown that if for a semi-symmetric metric connection the manifold is a group manifold, then the manifold is of constant curvature.

1. **Preliminaries.** Let (M, g) be an n-dimensional Riemannian manifold with Levi-civita connection $\overset{o}{\nabla}$. A linear connection on (M, g) is said to be semi-symmetric if

$$T(X, Y)=\pi(Y)X-\pi(X)Y \quad [1] \qquad \ldots \quad (1.1)$$

where π is a 1-form.

Then we have

$$\nabla_X Y=\overset{o}{\nabla}_X Y+\pi(Y)X-g(X, Y)\rho \qquad \ldots \quad (1.2)$$

where $\quad g(X, \rho)=\pi(X)$ for every vector field X. .. (1.3)

Further, if R and K denote the curvature tensors of ∇ and $\overset{o}{\nabla}$ respectively, then

$$\begin{aligned} R(X, Y)Z=K(X, Y)Z-\alpha(Y, Z)X+\alpha(X, Z)Y \\ -g(Y, Z)AX+g(X, Z)AY \end{aligned} \qquad \cdots \quad (1.4)$$

where α is a tensor field of type (0, 2) defined by

$$\alpha(X, Y)=(\overset{o}{\nabla}_X\pi)(Y)-\pi(X)\pi(Y)+\tfrac{1}{2}\pi(\rho)g(X, Y) \qquad \ldots \quad (1.5)$$

and A is a tensor field of type (1, 1) defined by

$$g(AX, Y)=\alpha(X, Y) \qquad \ldots \quad (1.6)$$

for any vector fields X, Y.

Moreover, we have

$$(\nabla_X\pi)(Y)=(\overset{o}{\nabla}_X\pi)(Y)-\pi(X)\pi(Y)+\pi(\rho)\, g(X, Y) \qquad \ldots \quad (1.7)$$

We shall use the above results in the next section.

2. A Special type of semi-symmetric connection

We consider a type of semi-symmetric metric connection ∇ whose curvature tensor R and torsion tensor T satisfy the following conditions :

$$R(X, Y)Z=0 \qquad \dots (2.1)$$

and

$$(\nabla_X T)(Y, Z)=B(X)\, T(Y, Z) \qquad \dots (2.2)$$

where B is a 1-form.

From (1.1) we have

$$(C_1^1 T)(Y)=(n-1)\, \pi(Y) \qquad \dots (2.3)$$

From (2.3) we get

$$(\nabla_X C_1^1 T)(Y)=(n-1)(\nabla_X \pi)(Y) \qquad \dots (2.4)$$

Again from (2.2) we obtain

$$(\nabla_X C_1^1 T)(Y)=B(X)(C_1^1 T)(Y)=B(X)(n-1)\, \pi(Y)$$

$$=(n-1)\, B(X)\, \pi(Y) \qquad \dots (2.5)$$

From (2.4) and (2.5) we get

$$(\nabla_X \pi)(Y)=B(X)\, \pi(Y) \qquad \dots (2.6)$$

Using (2.6) we can express (1.7) as follows :

$$B(X)\, \pi(Y)=(\overset{o}{\nabla}_X \pi)(Y)-\pi(X)\pi(Y)+\pi(\rho)\, g(X, Y)$$

Hence

$$(\overset{o}{\nabla}_X \pi)(Y)=B(X)\, \pi(Y)+\pi(X)\, \pi(Y)-\pi(\rho)\, g(X, Y) \qquad \dots (2.7)$$

Using (2.7) it follows from (1.5) that

$$\alpha(X, Y)=B(X)\, \pi(Y)-\tfrac{1}{2}\pi(\rho)\, g(X, Y) \qquad \dots (2.8)$$

Now,

$$g(AX, Y)=\alpha(X, Y)=B(X)\, \pi(Y)-\tfrac{1}{2}\pi(\rho)\, g(X, Y) \qquad \text{by } (2.8)$$

$$=B(X)\, g(\rho, Y)+g(-\tfrac{1}{2}\pi(\rho)X, Y)=g(B(X)\rho, Y)$$

$$+g(-\tfrac{1}{2}\pi(\rho)X, Y)$$

$$=g(B(X)\rho-\tfrac{1}{2}\pi(\rho)X, Y)$$

Hence

$$AX=B(X)\, \rho-\tfrac{1}{2}\pi(\rho)\, X \qquad \dots (2.9)$$

Using (2.8) and (2.9) we can write (1.4) as follows :

$$\begin{aligned} R(X, Y)Z = K(X, Y)Z &+ B(X)\,[\pi(Z)Y - g(Y, Z)\rho] \\ &+ B(Y)\,[-\pi(Z)X + g(X, Z)\,\rho] \\ &+ \pi(\rho)\,[g(Y, Z)X - g(X, Z)\,Y] \end{aligned} \qquad \ldots (2.10)$$

Since by (2.1), $R(X,Y)Z=0$ it follows from (2.10) that

$$\begin{aligned} K(X, Y)Z = B(X)\,&[g(Y, Z)\rho - \pi(Z)\,Y] \\ -B(Y)\,&[g(X, Z)\rho - \pi(Z)\,X] \\ -\pi(\rho)\,&[g(Y, Z)X - g(X, Z)\,Y] \end{aligned} \qquad \ldots (2.11)$$

Hence we can state the following theorem :

Theorem 1. If a Riemannian manifold admits a semi-symmetric metric connection whose curvature tensor R and torsion tensor T satisfy the conditions (2.1) and (2.2), then the curvature tensor of the manifold is given by (2.11).

Let us now assume that $B=0$. Then (2.2) takes the form

$$(\nabla_X T)\,(Y, Z)=0 \qquad \ldots (2.12)$$

Thus when (2.1) and (2. 12) are satisfied, (2.11) assumes the form

$$K(X, Y)Z = -\pi(\rho)\,[g(Y, Z)X - g(X, Z)\,Y] \qquad \ldots (2.13)$$

From (2.13) it follows that the manifold is of constant curvature.

Hence we have the following Theorem :

Theorem 2. If a Riemannian manifold admits a semi-symmetric metric connection for which the conditions (2.1) and (2.12) are satisfied, then the manifold is of constant curvature.

Now in virtue of (2.1) and (2.12) it follows from a known result [3] that the connection ∇ determines a simply transitive group. Since in such a case the manifold is called a group manifold, the above theorem may be stated alternatively as follows :

If a Riemannian metric admits a semi-symmetric metric connection for which the manifold is a group manifold, then the manifold is of constant curvature.

Theorem 2 expressed in its alternative form was proved by Yano in [3].

REFERENCES

[1] Eisenhart L. P.—Non-Riemannian Geometry—*Amer. Math. Soc.*, Colloq. Publications, Vol. VIII, 1927, p. 36.

[2] Eisenhart L. P,—Continuous groups of transformations Dover publications, Inc, New York, 1961, p. 197, § 49.1.

[3] Yano K.—On semi-symmetric metric connection—*Rev. Roum. Math. pures et. Appl.*, **15** (1970), 1579-1581.

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Dept. Pure Math.
Calcutta University
and
Dept. Math.,
Presidency College,
Calcutta

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CLASS OF UNIVERSAL ALGEBRAS VALUED ONTO THE SAME UPPER SEMILATTICE

S. P. BANDYOPADHYAY and S. CHANDRA

1. The valuation of a universal algebra has been introduced and studied recently [1, 2, 4]. In fact, it has been defined as a mapping of the universal algebra into an upper semilattice. It has been observed [1] that the lattice of convex subalgebras of a universal algebra is isomorphic to the lattice of convex subsemilattices of the upper semilattice onto which the universal algebra is valued epimorphically.

In the present paper, the authors have studied the class of universal algebras which are valued epimorphically onto the same upper semilattice. It has been shown that these are exactly those universal algebras which have the same upper semilattice of principal convex subalgebras.

2. Let A be a universal algebra with Ω as its domain of operators and let P be an upper semilattice.

A mapping N : A$\to$P, of A into P, is called a valuation if, and only if,

(i) $N(a_1 \dots a_n\, \omega_n) \leqslant N(a_1) \cup \dots \cup N(a_n)$, where $a_1, \dots, a_n \in A$; $\omega_n \in \Omega \mid n(\omega_n) = n \geqslant \mid$ (n is called the arness of the operator ω_n).

(ii) if $\omega_0 \in \Omega \mid n(\omega_0) = O$, then $N(O_\omega) \leqslant N(a)\ \forall\ a \in A$, where O_ω is the element of A specified by ω_0.

A subset Q of an upper semilattice P is called convex, if, and only if, $\alpha \in Q, x \in P, x \leqslant \alpha \Rightarrow x \in Q$.

A subsemilattice Q of an upper semilattice P is called a convex subsemilattice of P if, and only if, Q is a convex subset of P.

Let N : A$\to$P, be a valuation of a universal algebra A into an upper semilattice P. A subset B of A is called convex if, and only if,

(1) $b \in B, a \in A, N(a) \leqslant N(b) \Rightarrow a \in B$,

(2) $a, b \in B, N(c) = N(a) \cup N(b), C \in A \Rightarrow c \in B$.

A subalgebra B of A is called convex if, and only if, B is a convex subset of A.

The valuation N : A$\to$P is called epimorphic if, and only if, for each $\alpha \in P, \exists\ a \in A \mid N(a) = \alpha$. In this case, we say that the universal algebra A is valued epimorphically onto the upper semilattice P.

3. Let P be an upper semilattice and $\alpha \in P$.

Let $P(\alpha)=\{\beta \in P \mid \beta \leqslant \alpha\}$.

Proposition 1. $P(\alpha)$ is a convex subsemilattice of P.

Proof. Indeed, $\beta, \gamma \in P(\alpha) \Rightarrow \beta \leqslant \alpha, \gamma \leqslant \alpha.$

$\Rightarrow \beta \cup \gamma \leqslant \alpha \Rightarrow \beta \cup \gamma \in P(\alpha).$

Also, $\delta \leqslant \gamma\,;\, \gamma \in P(\alpha),\, \delta \in P$

$\Rightarrow \delta \leqslant \gamma \leqslant \alpha \Rightarrow \delta \in P(\alpha).$

Thus, $P(\alpha)$ is a convex subsemilattice of P. $P(\alpha)$ will be called the principal convex subsemilattice of P, generated by α.

Proposition 2. The set L_c^λ (P) of all principal convex subsemilattices of P of P will form an upper semilattice isomorphic to P.

Proof : Let $P(\alpha),\ P(\beta) \in L_c^\lambda$ (P).

Then $P(\alpha)=\{\gamma \in P \mid \gamma \leqslant \alpha\}$ and $P(\beta)=\{\delta \in P \mid \delta \leqslant \beta\}$.

Now $P(\alpha \cup \beta)\{\mu \in P \mid \mu \leqslant \alpha \cup \beta\}$

Obviously, $P(\alpha), P(\beta) \subseteq P(\alpha \cup \beta)$.

Now let $P(\alpha), P(\beta) \subseteq P(\nu)$.

$\Rightarrow \alpha \leqslant \nu,\ \beta \leqslant \nu \Rightarrow \alpha \cup \beta \leqslant \nu \Rightarrow \alpha \cup \beta \in P(\nu)$

$\Rightarrow P(\alpha \cup \beta) \subseteq P(\nu).$

Hence $P(\alpha \cup \beta)=P(\alpha) \cup P(\beta)$.

Further, $\alpha \to P(\alpha)$ is an isomorphism.

4. Let $N : A \to P$, be a valuation of a universal algebra (A, Ω) into an upper semilattice P.

Let $x \in A$.

Let $A(x)=\{a \in A \mid N(a) \leqslant N(x)\}$

Proposition 3 : $A(x)$ is a convex subalgebra of A.

Proof : Indeed, $a_1 \ldots\ldots a_n \in A(x),\ \omega_n \in \Omega \mid n(\omega_n)=n \geqslant 1$

$\Rightarrow N(a_1 \ldots\ldots a_n \omega_n) \leqslant N(a_1) \cup \ldots\ldots \cup N(a_n) \leqslant N\ (x).$

$\Rightarrow a_1 \ldots\ldots a_n \omega_n \in A(x).$

Also $\omega_0 \in \Omega \Rightarrow O_\omega \in A$.

$$\Rightarrow N(O_\omega) \leqslant N(x) - O_\omega \in A(x).$$

Thus, $A(x)$ is a subalgebra of A.

Further, $b \in A(x)$, $a \in A \mid N(a) \leqslant N(b)$

$$\Rightarrow N(a) \leqslant N(b) \leqslant N(x) - a \in A(x).$$

Also $a, b \in A(x)$, $N(c) = N(a) \cup N(b)$, $c \in A$

$$\Rightarrow N(c) \leqslant N(x) \Rightarrow c \in A(x)$$

Consequently, $A(x)$ is a convex subalgebra of A.

$A(x)$ will be called the principal convex subalgebra of A generated by x.

Proposition 4 : Let N, $A \to P$ be an epimorphic valuation of the universal A onto the upper semilattice P. Then the set $L^{\lambda}_{c}(A)$ of all principal convex subalgebras of A will form an upper semilattice.

Proof : Let $A(x)$, $A(y) \in L^{\lambda}_{c}$ (A)

Then $A(x) = \{a \in A \mid N(a) \leqslant N(x)\}$, $A(Y) = \{b \in A \mid N(b) \leqslant N(y)\}$.

Let $A(x, y) = \{c \in A \mid N(c) \leqslant N(x) \cup N(y)\}$

As N is epimorphic, $\exists\, z \in A \mid N(z) = N(x) \cup N(y)$.

Then, by proposition 3, $A(z)$ is a convex subalgebra A.

Evidently $A(x), A(y) \subseteq A(z)$.

Also, if $A(x), A(y) \subseteq A(w)$, then $N(x), N(y) \leqslant N(w)$.

$N(z) = N(x) \cup N(y) \leqslant N(w)$.

Thus $A(z)$ is the least upper bound of $A(x)$, and $A(y)$.

5. Let A be a universal algebra with Ω as its domain of operators and P be an upper semilattice and N : $A \to P$, be an epimorphic valuation of A onto P.

Theorem 1. L^{λ}_{c} (A) $\cong L^{\lambda}_{c}$ (P).

Proof : Let $P(\alpha) \in L^{\lambda}_{c}$ (P), where $\alpha \in P$.

As N is epimorphic, $\exists\, x \in A \mid N(x) = \alpha$.

Then $A(x) \in L^{\lambda}_{c}$ (A).

If $y \in A \mid N(y) = \alpha$, then $A(x) = \{a \in A \mid N(a) \leqslant \alpha\} = A(y)$.

Define $P(\alpha)f = A(x)$.

Thus, f is a mapping of L_o^λ (P) into L_o^λ (A).

Evidently, f is a bijection.

Further, $P(\alpha) \subseteq P(\beta)$ in L_o^λ (P)

$\Leftrightarrow \alpha \leqslant \beta$ in P.

$\Leftrightarrow P(\alpha) f \subseteq P(\beta) f$ in L_o^λ (A).

Hence the theorem. From proposition 2 and theorem 1 follows

Theorem 2. Let (A_1, Ω_1), (A_2, Ω_2) be universal algebras and P_1, P_2 be upper semilattices and let $N_1 : A_1 \to P_1$ and $N_2 : A_2 \to P_2$ be epimorphic valuations.

If $L_o^\lambda (A_1) \simeq L_o^\lambda (A_2)$, then $P_1 \simeq P_2$.

Theorems 1 and 2 completely determine the class of universal algebras which are valued epimorphically onto the same upper semilattice.

REFERENCES

[1] Bandyopaphyay, S. P. and Chandra, S.—Valuations in Universal Algebras—*Bull. Cal. Math. Soc.*, **68**, (1976), 243-252.

[2] Bandyopadhyay, S. P. and Chandra, S.—Valuations and archimedean classes in U—Algebras—*Bull. Cal. Math. Soc.*, **71**, (1979) 273-276.

[3] Birkhoff G.—Lattice theory—*Amer. Math. Soc. Coll. Pub.* XXV, 1968.

[4] Chandra, S.—Valuations in Universal Algebras I—*Bull. Col. Math. Soc.*, **68** (1976), 107-110.

[5] Kurosch, A. G.—Lectures on General Algebra, Pergamon Press, 1965.

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Dept. of Pure Math.
Calcutta University

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MY FAVOURITE PROOF OF MEHLER'S FORMULA

S. K. CHATTERJEA

The following formula of Mehler

$$(1)\quad \sum_{n=0}^{\infty} \frac{t^n}{2^n\, n\,!} \mathrm{H}_n(x)\, \mathrm{H}_n(y) = (1-t^2)^{-1/2} \exp\left[\frac{2xyt-(x^2+y^2)\,t^2}{1-t^2}\right],$$

where $\mathrm{H}_n(x)$ is the Hermite polynomial defined by

$$(2)\quad \sum_{n=0}^{\infty} \mathrm{H}_n(x)\, t^n/n\,! = \exp\,(2xt-t^2),$$

is well-known. Besides the usual proofs I prefer a proof of Mehler's formula from the linear generating relation (2). My proof is based on some integral formulas, which simultaneously serves as a key to other bilateral or even trilateral (or trilinear) generating relations involving Hermite polynomials.

The said integral formulas are the following

$$(3)\quad \mathrm{H}_n(x) \equiv \exp\,(-\mathrm{D}^2/4)\,(2x)^n = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2}\,[2(x+iv)]^n\, dv\,;\ \ \mathrm{D} \equiv d/dx$$

$$(4)\quad e^{-x^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2+2ixv}\, dv.$$

Now we have

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2} \left(\sum_{k=0}^{\infty} \frac{[2t\,(y+iv)]^k}{k\,!} \mathrm{H}_k(x)\right) dv$$

$$= \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{t^k}{k\,!} \mathrm{H}_k(x) \int_{-\infty}^{\infty} e^{-v^2}\,[2(y+iv)]^k\, dv$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{k\,!} \mathrm{H}_k(x)\, \mathrm{H}_k(y). \qquad \text{(inversion justified)}$$

Thus we can write

$$\sum_{k=0}^{\infty} \frac{t^k}{2^k\, k\,!}\, H_k(x)\, H_k(y)$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2} \left(\sum_{k=0}^{\infty} \frac{[t\,(y+iv)]^k}{k\,!}\, H_k(x) \right) dv$$

$$(5) \qquad = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\,[-v^2 + 2xt\,(y+iv) - t^2\,(y+iv)^2]\, dv,$$

which is a generating integral equivalent to Mehler's generating series. This method serves a new technique for adjoining a Hermite polynomial to any generating relation involving various special functions of mathematical physics, but it may happen that the generating integral is convergent or divergent. Some divergent generating functions were already seen in the works [3, 4, 5,] of Fred Brafman.

From (5) we obtain finally

$$\sum_{k=0}^{\infty} \frac{t^k}{2^k\, k\,!} H_k(x)\, H_k(y)$$

$$= \frac{\exp\,(2xyt - t^2y^2)}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\,[-v^2\,(1-t^2) + 2it\,(x-yt)\,v]\, dv$$

$$= \frac{\exp\,(2xyt - t^2y^2)}{\sqrt{\pi}\,\sqrt{(1-t^2)}} \int_{-\infty}^{\infty} \exp\left[-w^2 + 2i\,\frac{t\,(x-yt)}{\sqrt{(1-t^2)}}\,w\right] dw$$

$$= \frac{\exp\,(2xyt - t^2y^2)}{\sqrt{(1-t^2)}} .\ \exp\left[-\frac{t^2(x-yt)^2}{1-t^2}\right]$$

$$= (1-t^2)^{-1/2} \exp\left[\frac{2xyt - (x^2+y^2)t^2}{1-t^2}\right],$$

which is (1).

Similarly starting from the generating relation

$$(6) \qquad \sum_{n=0}^{\infty} \frac{H_{n+k}(x)\, t^n}{n\,!} = \exp\,(2xt - t^2)\, H_k\,(x-t),$$

we derive

$$\sum_{n=0}^{\infty} \frac{t^n}{2^n\, n!} H_{n+k}(x)\, H_n(y)$$

$$(7) \quad = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left[-v^2+2xt\,(y+iv)-t^2\,(y+iv)^2\right] H_k\,(x-t\,(y+iv))\, dv,$$

which is an equivalent generating integral for the generating series. We know that the generating function for the generating series is

$$(1-t^2)^{-\frac{1}{2}(k+1)} \exp\left[\frac{2xyt-(x^2+y^2)\,t^2}{1-t^2}\right] H_k\left(\frac{x-yt}{(1-t^2)^{1/2}}\right).$$

Thus we obtain the following integral formula

$$(8) \quad \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left[-v^2\,(1-t^2)+2it\,(x-yt)\,v\right] H_k\,(x-t\,(y+iv))\, dv$$

$$= (1-t^2)^{-(k+1)/2} \exp\left[-\frac{t^2\,(x-yt)^2}{1-t^2}\right] H_k\left(\frac{x-yt}{(1-t^2)^{1/2}}\right),$$

which does not seem to appear before.

Now it may be of interest to point out that starting from the generating relation of L.Carlitz [6]

$$(9) \quad \sum_{k=0}^{\infty} \frac{t^k}{2^k\, k!} H_{k+m}(x)\, H_{k+n}(y)$$

$$= (1-t^2)^{-(m+n+1)/2} \exp\left[\frac{2xyt-(x^2+y^2)\,t^2}{1-t^2}\right].$$

$$\sum_{r=0}^{\min(m,n)} 2^r\, r!\, \binom{m}{r} \binom{n}{r}\, t^r\, H_{m-r}\left(\frac{x-ty}{(1-t^2)^{1/2}}\right) H_{n-r}\left(\frac{y-xt}{(1-t^2)^{1/2}}\right),$$

we derive in the same manner

$$(10) \quad \sum_{k=0}^{\infty} \frac{(t/4)^k}{k!} H_k(x)\, H_{k+m}(y)\, H_{k+n}(z)$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2}\, (1-t^2\,(x+iv)^2)^{-(m+n+1)/2}$$

$$\cdot \exp\left[\frac{2yzt(x+iv)-(y^2+z^2)t^2\,(x+iv)^2}{1-t^2\,(x+iv)^2}\right].$$

$$\cdot \sum_{r=0}^{\min(m,n)} 2^r\, r!\, \binom{m}{r} \binom{n}{r} (t\,(x+iv))^r\, H_{m-r}\left(\frac{y-t\,(x+iv)\,z}{(1-t^2\,(x+iv)^2)^{1/2}}\right)$$

$$\cdot H_{n-r}\left(\frac{z-t\,(x+iv)\,y}{(1-t^2\,(x+iv)^2)^{1/2}}\right) dv.$$

A particular case of (10) is worthy of much notice. Indeed, using $m=n=o$ we obtain the following trilinear generating integral for the Hermite polynomials

$$(11) \qquad \sum_{k=0}^{\infty} \frac{(t/4)^k}{k!} H_k(x)\, H_k(y)\, H_k(z)$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2} (1-t^2(x+iv)^2)^{-1/2} \exp\left[\frac{2yzt(x+iv)-t^2(y^2+z^2)(x+iv)^2}{1-t^2(x+iv)^2}\right] dv,$$

which may be compared with the remark made by R. Askey [2] and W. A. Al-Salam and L. Carlitz [1] in connection with the trilinear generating function for the Hermite polynomials.

REFERENCES

[1] Al-Salam, W. A. and Carlitz, L.—On certain generating functions for the Hermite polynomials, *Portugal Math.*, **25** (1966), 35-44.

[2] Askey, R.—Orthogonal polynomials and positivity, *Math Centrum Amsterdam afd. toegepaste Wisk.* T. W. **112** (1969), 1-34.

[3] Brafman, F.—Some generating functions for Laguerre and Hermite polynomials, *Canad. J. Math.* **9** (1957), 180-187.

[4] ———— Generating functions and associated Legendre polynomials, *Quart. J. Math.* **10** (1959), 156-160.

[5] ———— Generating functions of Jacobi and related polynomials, *Proc. Amer. Math. Soc.*, **2** (1951), 942-949.

[6] Carlitz, L.—An extension of Mehler's formula, *Bull. Un. Mat. Ital.* N. **1** (1970), 43-46.

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Dept. of Pure Math.
Calcutta University

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ON A GENERATING FUNCTION OF FENG

S. K. CHATTERJEA

In a recent paper [2], C. C. Feng has derived the following main generating relation involving modified Laguerre polynomials

$$(1)\quad exp(-a_{23}xz)exp\left(-\frac{a_{12}}{y}(1+a_{13}y+a_{23}z)(1+a_{13}y+a_{23}z)^{-n-\beta}\right.\cdot$$

$$\cdot f_n^{(\beta)}\left[\left(\frac{a_{12}}{y}+\frac{a_{22}}{z}+x\right)\left(1+a_{13}y+a_{23}z\right)\right]$$

$$=\sum_{p=0}^{\infty}\frac{(a_{23})^p}{p\,!}\sum_{m=0}^{\infty}\frac{(a_{13})^m}{m\,!}\sum_{l=0}^{\infty}\frac{(a_{22})^l}{l\,!}\sum_{k=0}^{\infty}\frac{(a_{12})^k}{k\,!}(-1)^{k+m+p}(\beta-k)_m(n-l+1)_p\cdot$$

$$\cdot f_{n-l+p}^{(\beta-k+m)}(x)y^{m-k}z^{p-l},$$

by replacing β by $y\frac{\partial}{\partial y}$, n by $z\frac{\partial}{\partial z}$ and $f_n^{(\beta)}(x)$ by $u(x, y, z)$ in the linear differential relation

$$(2)\quad xD^2 f_n^{(\beta)}(x)+(1-x-n-\beta)\,Df_n^{(\beta)}(x)+nf_n^{(\beta)}(x)=0,$$

and by following the method of L. Weisner [3].

It may be of interest to remark that the result (1) of Feng follows easily with the help of the following unilateral generating relations :

$$(3)\quad e^{xt}(1-t)^{-\beta-v}f_v^{(\beta)}(x(1-t)) = \sum_{n=0}^{\infty}\frac{(v+1)_n}{n\,!}f_{v+n}^{(\beta)}(x)t^n$$

$$(4)\quad f_n^{(\beta)}(x+t) = \sum_{n=0}^{v}\frac{1}{n\,!}f_{v-n}^{(\beta)}(x)t^n$$

$$(5)\quad (1-t)^{-\beta-n}f_n^{(\beta)}((1-t)x) = \sum_{m=0}^{\infty}\frac{t^m}{m\,!}(\beta)_m f_n^{(\beta+m)}(x)$$

$$(6)\quad e^t f_n^{(\beta)}(x-t) = \sum_{k=0}^{\infty}\frac{t^k}{k\,!}f_n^{(\beta-k)}(x).$$

Furthermore using the following result of our work [1]

$$(7)\quad e^{y}(wy-1)^{\alpha}L_n^{(\alpha)}\left(\frac{(x-y)(wy-1)}{wy}\right)=\sum_{m=0}^{\infty}\frac{y^m}{m!}\sum_{p=0}^{\infty}\frac{(-n-\alpha)_p(wy)^{\alpha-p}}{p!}L_n^{(\alpha+m-p)}(x)$$

we can deduce a relation analogous to (1) by means of (3) and (4), viz.

$$(8)\quad \sum_{p=0}^{8}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\beta)_p(n-k+1)_l y^{m-p}t_1^l t_2^k}{m!\,p!\,l!\,k!}f_{n-k+l}^{(\beta-m+p)}(x)$$

$$=exp(y+t_1(x-y))(1-t_1-y^{-1})^{-\beta-n}f_n^{(\beta)}[(1-t_1-y^{-1})(x-y+t_2)].$$

The very nature of $(\beta)_p$ in the left member of (8) implies that another result analogous to (1) can be put in the form

$$(9)\quad \sum_{p=0}^{\infty}\frac{(a_{22})^p}{p!}\sum_{m=0}^{\infty}\frac{(a_{12})^m}{m!}\sum_{l=0}^{\infty}\frac{(a_{22})^l}{l!}\sum_{k=0}^{\infty}\frac{(a_{12})^k}{k!}(-1)^{k+m+p}(\beta)_m(n-l+1)_p.$$

$$\cdot f_{n-l+p}^{(\beta-k+m)}(x)\,y^{m-k}z^{p-l}$$

$$=exp\left[-a_{22}xz-\frac{a_{12}}{y}(1-a_{22}z)\right](1-a_{12}y-a_{22}z)^{-n-\beta}.$$

$$\cdot f_n^{(\beta)}\left[(1-a_{12}y-a_{22}z)\left(x+\frac{a_{12}}{y}+\frac{a_{22}}{z}\right)\right].$$

Now the generating relations (3) and (4) are mentioned in [3, p. 45].

Also the generating relations (5) and (6) can be easily deduced from the results of the present author [1]. In fact, in [1, p. 369] we notice that

$$(10)\quad (1-t)^{\alpha}L_n^{(\alpha)}(x(1-t))=\sum_{m=0}^{\infty}\frac{(-\alpha-n)_m}{m!}L_n^{(\alpha-m)}(x)\,t^m.$$

To prove (5) we make use of the relation

$$(11)\quad f_n^{(\beta)}(x)=(-1)^n L_n^{(-\beta-n)}(x),$$

so that we have

$$\sum_{m=0}^{\infty}\frac{t^m}{m!}(\beta)_m f_n^{(\beta+m)}(x)$$

$$=(-1)^n\sum_{m=0}^{\infty}\frac{t^m}{m!}(\beta)_m L_n^{(-\beta-n)-m}(x)$$

$$=(-)^n(1-t)^{-\beta-n}L_n^{-\beta-n}(x(1-t))$$

$$=(1-t)^{-\beta-n}f_n^{(\beta)}(x(1-t)).$$

Again we notice that [1, p. 370]

$$(12)\quad e^{y}\, L_n^{(\alpha)}(x-y) = \sum_{m=0}^{\infty} \frac{y^m}{m!}\, L_n^{(\alpha+m)}(x).$$

Thus we can prove (6) as follows

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} f_n^{(\beta-k)}(x) = (-1)^n \sum_{k=0}^{\infty} \frac{t^k}{k!}\, L_n^{(-\beta-n)+k}(x)$$

$$=(-1)^n e^t\, L_n^{(-\beta-n)}(x-t)=e^t\, f_n^{(\beta)}(x-t).$$

We are now in a position to prove the result (1) of Feng in a quite easy manner. Indeed, we have by virtue of (3)

$$\sum_{p=0}^{\infty} \frac{(a_{23})^p}{p!} \sum_{m=0}^{\infty} \frac{(a_{13})^m}{m!} \sum_{l=0}^{\infty} \frac{(a_{22})^l}{l!} \sum_{k=0}^{\infty} \frac{(a_{12})^k}{k!} (-1)^{k+m+p} (\beta-k)_m (n-l+1)_p \cdot$$

$$\cdot f_{n-l+p}^{(\beta-k+m)}(x)\, y^{m-k} z^{p-l}$$

$$=e^{-a_{23}xz}(1+a_{23}z)^{-\beta-n} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-a_{12}/y)^k}{k!} \frac{(-a_{13}y)^m}{m!} \frac{(a_{22}/z)^l}{l!} \cdot$$

$$\cdot (\beta-k)_m (1+a_{23}z)^{k-m+l} f_{n-l}^{(\beta-k+m)}(x(1+a_{23}z))$$

Next using (4) we obtain

Right member of (1)

$$= e^{-a_{23}xz}(1+a_{23}z)^{-\beta-n} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{[-a_{12}(1+a_{23}z)]^k}{k!} \frac{\left(\frac{-a_{13}y}{1+a_{23}z}\right)^m}{m!} \cdot$$

$$\cdot (\beta-k)_m\, f_n^{(\beta-k+m)}\left((1+a_{23}z)\left(x+\frac{a_{22}}{z}\right)\right)$$

Again using (5) we derive

Right member of (1)

$$= e^{-a_{23}xz}(1+a_{13}y+a_{23}z)^{-\beta-n} \sum_{k=0}^{\infty} \frac{(-a_{12}(1+a_{13}y+a_{23}z)/y)^k}{k!} \cdot$$

$$\cdot f_n^{(\beta-k)}\left(\left(x+\frac{a_{22}}{z}\right)\left(1+a_{13}y+a_{23}z\right)\right)$$

Lastly using (6) we obtain

Right member of (1)

$$= exp\ (-a_{23}\ xz)\ exp\ \left(-\frac{a_{12}}{y}\ (1+a_{13}\ y+a_{23}\ z)\right)\ (1+a_{13}\ y+a_{23}\ z)^{-\beta-n}.$$

$$.f_n^{(\beta)}\left[\left(1+a_{13}\ y+a_{23}\ z\right)\ \left(x+\frac{a_{12}}{y}+\frac{a_{23}}{z}\right)\right],$$

which is the left member of (1).

Next we consider the result (7). It follows from (7) by virtue of (11)

$$\sum_{m=0}^{\infty} \frac{y^m}{m\,!} \sum_{p=0}^{\infty} \frac{(\beta)_p\ (y)^{-p}}{p\,!}\ f_{n-k+l}^{(\beta-m+k)}\ (x)$$

(13)

$$= e^y \left(\frac{y-1}{y}\right)^{-\beta-n+k-l} f_{n-k+l}^{(\beta)}\ \left(\frac{(x-y)\ (y-1)}{y}\right).$$

Thus

$$\sum_{p=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\beta)_p\ (n-k+1)_l\ y^{m-p}\ t_1^l\ t_2^k}{m\,!\ p\,!\ l\,!\ k\,!}\ f_{n-k+l}^{(\beta-m+p)}\ (x)$$

$$= e^y\ \left(\frac{y-1}{y}\right)^{-\beta-n} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(n-k+1)_l}{l\,!\ k\,!}\ \left(\frac{t_1\ y}{y-1}\right)^l\ \left(\frac{t_2(y-1)}{y}\right)^k.$$

$$.f_{n-k+l}^{(\beta)}\left(\frac{(x-y)(y-1)}{y}\right)$$

Using the relations (3) and (4) successively we obtain (8) which is analogous to (1).

Finally if we consider the series

$$\sum_{p=0}^{\infty} \frac{(a_{23})^p}{p\,!} \sum_{m=0}^{\infty} \frac{(a_{13})^m}{m\,!} \sum_{l=0}^{\infty} \frac{(a_{23})^l}{l\,!} \sum_{k=0}^{\infty} \frac{(a_{12})^k}{k\,!}\ (-1)^{k+m+p}\ (\beta)_m\ (n-l+1)_p.$$

$$\cdot\ f_{n-l+p}^{(\beta-k+m)}(x)\ y^{m-k}\ z^{p-l}$$

and first sum the series over k, then we obtain

$$e^{-a_{12}/y}\sum_{p=0}^{\infty}\frac{(a_{23})^p}{p!}\sum_{m=0}^{\infty}\frac{(a_{13})^m}{m!}\sum_{l=0}^{\infty}\frac{(a_{22})^l}{l!}(-1)^{m+p}(\beta)_m(n-l+1)_p\cdot$$

$$\cdot f_{n-l+p}^{(\beta+m)}\left(x+\frac{a_{12}}{y}\right)y^m z^{p-l}$$

Then we sum the above triple series by means of (3), (4) and (5) and obtain (9).

REFERENCES

[1] Chatterjea, S. K.—Group-theoretic origins of certain generating functions of Laguerre polynomials, *Bull. Inst. Math. Acad. Sinica* **3** (1975), 369-375.

[2] Feng, C. C.—Some generating functions of modified Laguerre polynomials, *Hokkaido Math. J.* **7** (1978), 189-197.

[3] McBride, E. B.—Obtaining generating functions, Springer-Verlag (1971).

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Dept. of Pure Math.
Calcutta University

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ON AFFINELY CONNECTED GENERALISED 2-RECURRENT SPACES

KAMALAKANT SHARMA

Introduction : Generalised 2-recurrent Riemannian Spaces were studied by A. K. Roy [2] who denoted an N-space of this kind by G $\{{}^2K_N\}$. In the present paper we call an affinely connected space L_N with symmetric connection Γ^i_{jk} a generalised 2-recurrent space if

$$(1) \quad \nabla_p \nabla_m B^i_{jkl} = T_{mp} B^i_{jkl} + \beta_p \nabla_m B^i_{jkl}.$$

where ∇ denotes covariant differentiation with respect to Γ^i_{jk} and T_{mp} is a covariant tensor of second order and β_m is a covariant vector. An N-space of this kind shall be denoted by A G $\{{}^2K_N\}$. It has been shown that if such a space of symmetric connection is decomposable and $T_{ij} \neq 0$, then one of the component spaces is plane. A sufficient condition is obtained in order that such a space may be an affinely connected recurrent space.

Defining an affinely connected space as a generalised projective 2-recurrent space if (1.1) $\nabla_p \nabla_m W^i_{jkl} = T^*_{mp} W^i_{jkl} + \beta^*_p \nabla_m W^i_{jkl}$ and denoting an N-space of this kind by A G P $\{{}^2K_N\}$ it is easy to see that every A G $\{{}^2K_N\}$ is an A G P $\{{}^2K_N\}$ but the converse is not in general true. In this paper a necessary and sufficient condition has been obtained that an A G P $\{{}^2K_N\}$ may be A G $\{{}^2K_N\}$.

1. Decomposable A G $\{{}^2K_N\}$

If two spaces L_M and L_{N-M} are given with co-ordinates x^α : $(\alpha, \beta, \gamma = 1, 2, \ldots, M)$ and x^A : (A, B, C = M+1, . , N) and the connections $\Gamma^\alpha_{\beta\gamma}$ and Γ^A_{BC}, then the L_N with co-ordinates x^a : $(a, b, c = 1, 2, \ldots, N)$ and connection $\Gamma^a_{bc} \equiv \{\Gamma^\alpha_{\beta\gamma}, \Gamma^A_{BC}\}$ is called the product of L_M and L_{N-M}. An L_N that is a product space is said to be decomposable [1]. A geometric object in a decomposable L_N is decomposable if and only if its components with respect to the special co-ordinates are always zero when they have indices from both ranges and the components belonging to the sub-space L_N (L_{N-M}) are functions of x^α (x^A) only. In a decomposable L_N, B^a_{bcd}, B_{bc} and their covariant derivatives are decomposable.

We now consider a decomposable A G $\{{}^2K_N\}$.

Since (2) $\nabla_m B^i_{jkl} + \nabla_k B^i_{jlm} + \nabla_l B^i_{jmk} = 0$, we have

$$\nabla_n \nabla_m B^i_{jkl} + \nabla_n \nabla_k B^i_{jlm} + \nabla_n \nabla_l B^i_{jmk} = 0.$$

Using (1) this gives,

(3) $T_{mn} B^i_{jkl} + T_{kn} B^i_{jlm} + T_{ln} B^i_{jmk} = 0$ by virtue of (2)

Put $m=\rho,\ n=\sigma$; $i, j, k, l = \alpha, \beta, \gamma, \delta$

Then from (3) we get, $T_{\rho\sigma} B^{\alpha}_{\beta\delta\delta} + T_{\gamma\sigma} B^{\alpha}_{\beta\delta\rho} + T_{\delta\sigma} B^{\alpha}_{\beta\rho\gamma} = 0.$

Whence (4) $T_{\rho\sigma} B^{\alpha}_{\beta\gamma\delta} = 0.$

Next, we put $m = \alpha,\ n = \beta$; $i, j, k, l = \nu, \rho, \sigma, \tau$

Then, $T_{\alpha\beta} B^{\nu}_{\rho\sigma\tau} + T_{\sigma\beta} B^{\nu}_{\rho\tau\alpha} + T_{\tau\beta} B^{\nu}_{\rho\alpha\sigma} = 0.$

whence (5) $T_{\alpha\beta} B^{\nu}_{\rho\sigma\tau} = 0.$

Now, put $m = \tau,\ n=\alpha'$; $i, j. k, l = \alpha, \beta, \gamma, \delta$

Then, $T_{\tau\alpha'} B^{\alpha}_{\beta\gamma\delta} + T_{\gamma\alpha'} B^{\alpha}_{\beta\delta\tau} + T_{\delta\alpha'} B^{\alpha}_{\beta\tau\gamma} = 0$

Whence (6) $T_{\tau\alpha'} B^{\alpha}_{\beta\gamma\delta} = 0.$

Finally, we put, $m = \alpha,\ n = \tau'$; $i, j, k, l = \nu, \rho, \sigma, \tau$

Then $T_{\alpha\tau'} B^{\nu}_{\rho\sigma\tau} + T_{\sigma\tau'} B^{\nu}_{\rho\tau\alpha} + T_{\tau\tau'} B^{\nu}_{\rho\alpha\sigma} = 0$

Whence (7) $T_{\alpha\tau'} B^{\nu}_{\rho\sigma\tau} = 0$

If $T_{ij} \not\equiv 0$, one of $T_{\rho\sigma}$, $T_{\alpha\beta}$, $T_{\tau\alpha'}$, $T_{\alpha\tau'}$ must be non-null.

Hence, either $B^{\alpha}_{\beta\gamma\delta} = 0$ or $B^{\nu}_{\rho\sigma\tau} = 0.$

We can therefore, state the following theorem :

Theorem 1 : If in an A G $\{^2K_N\}$, $T_{ij} \not\equiv 0$ and the space is decomposable then one of the component spaces is plane.

Henceforth, by an A G $\{^2K_N\}$, we shall mean a non-decomposable space.

2. Recurrent A G $\{^2K_N\}$: Suppose that in an A G $\{^2K_N\}$ the tensor $T_{m\rho}$ has the form

(8) $T_{m\rho} = \nabla_\rho \chi_m + \chi_m \chi_\rho - \chi_m \beta_\rho$

where χ_m is a covariant vector field.

From (1), we get,

(9) $\nabla_\rho \nabla_m B^i_{jkl} = (\nabla_\rho \chi_m + \chi_m \chi_\rho - \chi_m \beta_\rho) B^i_{jkl} + \beta_\rho \nabla_m B^i_{jkl}$

If possible, let (10) $\nabla_m B^i_{jkl} = \chi_m B^i_{jkl} + H^i_{jklm}$

where $H^i_{jklm} \not\equiv 0.$

From (10) we get,

$$\nabla_p \nabla_m B^i_{jkl} = B^i_{jkl} \nabla_p \chi_m + \chi_m \nabla_p B^i_{jkl} + \nabla_p H^i_{jklm}$$
$$= (\nabla_p \chi_m + \chi_m \chi_p) B^i_{jkl} + \chi_m H^i_{jklp} + \nabla_p H^i_{jklm}$$

Again from (9)

$$\nabla_p \nabla_m B^i_{jkl} = \beta_p \chi_m B^i_{jkl} + \beta_p H^i_{jklm} + (\nabla_p \chi_m + \chi_m \chi_p - \chi_m \beta_{\neq}) B^i_{jkl}$$
$$= (\nabla_p \chi_m + \chi_m \chi_p) B^i_{jkl} + \beta_p H^i_{jklm}.$$

Hen ce,

$$(11) \quad \nabla_p H^i_{jklm} + \chi_m H^i_{jklp} - \beta_p H^i_{jklm} = 0$$

If the differential equations (11) have no other solution than the zero tensor, then $H^i_{jklm} \equiv 0$.

But this is contrary to the hypothesis in (10) .

Hence, from (10) it follows that the space is recurrent.

From this we get the following theorem :

Theorem 2 : If in an A G $\{^2K_N\}$ the tensor T_{mp} has the form

$$T_{mp} = \nabla_p \chi_m + \chi_m \chi_p - \chi_m \beta_p$$

and the differential equations

$$\nabla_p H^i_{jklm} + \chi_m H^i_{jklp} - \beta_p H^i_{jklm} = 0$$

have no other solutions than the zero tensor then the space is an affinely connected recurrent space.

3. Condition for A G P $\{^2K_N\}$ to be A G $\{^2K_N\}$

It is easy to see that every A G $\{^2K_N\}$ is an A G P $\{^2K_N\}$ with the same vector and tensor of recurrence but the converse is not in general true. Hence in this section we find a necessary and sufficient condition for the converse to be true.

Let us consider an A G P $\{^2K_N\}$ with symmetric Ricci Tensor.
Then,

$$W^i_{jkl} = B^i_{jkl} + \frac{1}{N+1} \delta^i_j (B_{kl} - B_{lk}) + \frac{1}{N^2-1} [\delta^i_k (N B_{jl} + B_{lj}) - \delta^i_l (N B_{jk} + B_{kj})]$$

reduces to $W^{i}_{jkl} = B^{i}_{jkl} + \frac{1}{N-1}(\delta^{i}_{k} B_{jl} - \delta^{i}_{l} B_{jk})$

putting this value in (1.1) we get,

$$(12) \quad (\nabla_{p} \nabla_{m} B^{i}_{jkl} - \beta^{*}_{p} \nabla_{m} B^{i}_{jkl} - T^{*}_{mp} B^{i}_{jkl})$$

$$+\frac{1}{N-1}[\delta^{i}_{k}(\nabla_{p} \nabla_{m} B_{jl} - \beta^{*}_{p} \nabla_{m} B_{jl} - T^{*}_{mp} B_{jk})$$

$$-\delta^{i}_{l}(\nabla_{p} \nabla_{m} B_{jk} - \beta^{*}_{p} \nabla_{m} B_{jk} - T^{*}_{mp} B_{jk})] = 0$$

Let us now suppose that this space is affinely connected generalised 2-Ricci recurrent space.

Then, $\nabla_{p} \nabla_{m} B_{jl} - \beta^{*}_{p} \nabla_{m} B_{jl} - T^{*}_{mp} B_{jl} = 0$

Hence from (12) we have

$$\nabla_{p} \nabla_{m} B^{i}_{jkl} - \beta^{*}_{p} \nabla_{m} B^{i}_{jkl} - T^{*}_{mp} B^{i}_{jkl} = 0.$$

i.e. the space is affinely connected generalised 2-recurrent space.

Conversely, it is easy to see that every affinely connected generalised 2-recurrent space is affinely connected generalised 2-Ricci-recurrent space.

Hence, we deduce the Theorem :

Theorem 3 : An affinely connected generalised projective 2-recurrent space with symmetric Ricci-tensor is an affinely connected generalised 2-recurrent space if and only if it is an affinely connected generalised 2-Ricci-recurrent space.

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REFERENCES

[1] Ficken, F. A.—The Riemannian and affine differential Geometry of Product Spaces, *Annals of Math.* (2) Vol. 40 (1939), 892-913.

[2] Ray, A. K. —On generalised 2-recurrent tensors in Riemannian Spaces, *Bull. Acad. Royalede Belg. Tome,* 58 (1972), 220-228.

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Dept. of Pure Math.
Calcutta University

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ON SOME TYPES OF AFFINE MOTIONS IN AFFINELY CONNECTED GENERALISED 2-RECURRENT SPACES

KAMALAKANT SHARMA

1. Introduction : Let L_N be an affinely connected space of N-dimensions with a symmetric affine connection Γ^i_{jk} and let B^i_{jkl} $(=-B^i_{jlk})$ be the curvature tensor. Then the space is said to be a generalised 2-recurrent space if the following condition is satisfied :

$$\nabla_n \nabla_m B^i_{jkl} = a_{mn} B^i_{jkl} + \beta_n \nabla_m B^i_{jkl} \tag{1}$$

where ∇ denotes covariant differentiation with respect to Γ^i_{jk} and β_m, a_{mn} are respectively a covariant vector and a covariant tensor. Such a space shall be denoted by $AG\{^2K_N\}$ and β_m, a_{mn} will be called its vector and tensor of recurrence respectively.

We now suppose that the space admits an infinitesimal co-ordinate transformation

$$\bar{x}^i = x^i + \xi^i(x)\,\delta t \tag{1'}$$

(δt being an infinitesimal constant) satisfying the condition

$$\pounds \Gamma^i_{jk} = \nabla_k \nabla_j \xi^i + B^i_{jkl} \xi^l = 0 \tag{2}$$

where $\pounds$ denotes Lie-derivative with respect to the above transformation. Such transformations are called affine motions.

Takano and Imai [2] considered some types of affine motions in bi-recurrent spaces. The object of this paper is to study some types of affine motions in $AG\{^2K_N\}$.

2. Some formulas in an $AG\{^2K_N\}$ admitting affine motions :

Since the space is assumed to admit affine motions the conditions (2) must be integrable. The condition of its integrability can be written as $\pounds B^i_{jkl} = 0$

or as

$$\xi^t \nabla_t B^i_{jkl} - B^t_{jkl} \nabla_t \xi^i + B^i_{tkl} \nabla_j \xi^t + B^i_{jtl} \nabla_k \xi^t + B^i_{jkt} \nabla_l \xi^t = 0. \tag{3}$$

Interchanging m and n in (1) and then subtracting it from (1) we get

$$(4)\quad A_{mn}\, B^i_{jkl} = B^t_{jkl}\, B^i_{tmn} - B^i_{tkl}\, B^t_{jmn} - B^i_{jtl}\, B^t_{kmn} - B^i_{jkt}\, B^t_{lmn} - (\beta_n \nabla_m B^i_{jkl} - \beta_m \nabla_n B^i_{jkl})$$

where $A_{mn} \equiv a_{mn} - a_{nm}$.

Putting $\nabla_j \xi^i = B^i_{jmn} f^{mn}$ where f^{mn} is a non-symmetric tensor, multiplying (4) by f^{mn} and summing over the indices m and n, we get

$$(5)\quad CB^i_{jkl} = B^t_{jkl} \nabla_t \xi^i - B^i_{tkl} \nabla_j \xi^t - B^i_{jtl} \nabla_k \xi^t - B^i_{jkt} \nabla_l \xi^t,$$

where $C = A_{mn} f^{mn}$.

With the help of (5) we can express (3) as

$$(6)\quad \pounds\, B^i_{jkl} = \xi^t \nabla_t B^i_{jkl} - C\, B^i_{jkl}$$

Since $\pounds\, B^i_{jkl} = 0$, we get

$$(7)\quad C\, B^i_{jkl} = \xi^t \nabla_t B^i_{jkl}$$

Differentiating (7) convariantly and using (1) and (7) we have

$$(\nabla_m C)\, B^i_{jkl} + C \nabla_m B^i_{jkl} = \nabla_m \xi^t \,.\, \nabla_t B^i_{jkl} + \xi^t (\nabla_m \nabla_t B^i_{jkl})$$
$$= \nabla_m \xi^t \,.\, \nabla_t B^i_{jkl} + (\xi^t a_{tm} + C \beta_m) \,.\, B^i_{jkl}$$

or $$(8)\quad C \,.\, \nabla_m B^i_{jkl} + (\nabla_m C - C \beta_m - \xi^t a_{tm})\, B^i_{jkl} = \nabla_m \xi^t \nabla_t B^i_{jkl}$$

Now, multiplying (8) by ξ^m and summing with respect to m, we get

$$(9)\quad C \,.\, \xi^m \nabla_m B^i_{jkl} + (\xi^m \nabla_m C - C\, d - A)\, B^i_{jkl} = \nabla_m \xi^t \,.\, \nabla_t B^i_{jkl}$$

where $d = \xi^t \beta_t$ and $A = \xi^l \xi^n a_{ln}$.

It is known [3] that under affine motions the operations of $\pounds$ and ∇ are interchangeable. Hence

$$0 = \pounds\, \nabla_m B^i_{jkl} = \nabla_m \xi^t \,.\, \nabla_t B^i_{jkl} + \xi^t a_{tm} B^i_{jkl} + \beta_m \xi^t \nabla_t B^i_{jkl}$$
$$- \nabla_t \xi^i \,.\, \nabla^m B^t_{jkl} + \nabla_j \xi^t \,.\, \nabla_m B^i_{tkl} + \nabla_k \xi^t \,.\, \nabla_m B^i_{jtl} + \nabla_l \xi^t \,.\, \nabla_m B^i_{jkt}$$
$$= \nabla_m \xi^t \,.\, \nabla_t B^i_{jkl} + (\xi^t a_{tm} + C \beta_m)\, B^i_{jkl} - \nabla_t \xi^i \,.\, \nabla_m B^t_{jkl}$$
$$+ \nabla_j \xi^t \,.\, \nabla_m B^i_{tkl} + \nabla_k \xi^t \,.\, \nabla_m B^i_{jtl} + \nabla_l \xi^t \,.\, \nabla_m B^i_{jkt} \qquad (\text{using } (7))$$

Transvecting this with ξ^m and using (7) we get

$$(10)\quad \xi^m \nabla_m \xi^t \,.\, \nabla_t B^i_{jkl} + AB^i_{jkl} + C \,.\, d \,.\, B^i_{jkl}$$
$$= C\, [\nabla_t \xi^i \,.\, B^t_{jkl} - \nabla_j \xi^t \,.\, B^i_{tkl} - \nabla_k \xi^t \,.\, B^i_{jtl} - \nabla_l \xi^t \,.\, B^i_{jkt}]$$
$$= C^2\, B^i_{jkl} \quad (\text{by } (3)).$$

From (9) and (10) we have

$$\xi^m \nabla_m C = 0.$$

or

(11) $$\pounds\, C = 0.$$

3. Affine motions corresponding to a concurrent vector field :

We now consider an affine motion generated by a vector field ξ^i which is a concurrent vector field. Then

(12) $\nabla_j \xi^i = k\delta^i_j$ where k is a non-zero constant.

From (12) we have $\nabla_k \nabla_j \xi^i = 0$,

whence

(13) $$\xi^h B^i_{hjk} = -\nabla_k \nabla_j \xi^i + \nabla_j \nabla_k \xi^i = 0.$$

Differentiating (13) convariantly and using (12) we get

(14) $$k\, B^i_{mjk} + \xi^h \nabla_m B^i_{hjk} = 0$$

Again differentiating (14) covariantly and using (1) we get

$$k \nabla_n B^i_{mjk} + k \nabla_m B^i_{njk} + \xi^h (a_{mn} B^i_{hjk} + \beta_n \nabla_m B^i_{hjk}) = 0$$

or, $k\,(\nabla_n B^i_{mjk} + \nabla_m B^i_{njk} - \beta_n B^i_{mjk}) = 0,$ (using (13) and (14))

whence

(15) $$\nabla_n B^i_{mjk} + \nabla_m B^i_{njk} = \beta_n B^i_{mjk}$$

Now, operating ∇_l on (15) and using (1) we get

$$a_{nl} B^i_{mjk} + a_{ml} B^i_{njk} + \beta_l (\nabla_n B^i_{mjk} + \nabla_m B^i_{njk})$$
$$= \beta_n \nabla_l B^i_{mjk} + B^i_{jnjk} \cdot \nabla_l \beta_n.$$

Next, using (15) we have

$$(a_{nl} + \beta_l \beta_n - \nabla_l \beta_n) B^i_{mjk} + a_{ml} B^i_{njk} = \beta_n \nabla_l B^i_{mjk}$$

Transvecting this with ξ^m and using (13) and (14) we get

$$\xi^m a_{ml} B^i_{njk} = \xi^m \beta_n \nabla_l B^i_{mjk} = -k\, \beta_n B^i_{ljk}$$

Again, transvecting with ξ^n we get

$$\xi^m a_{ml} \xi^n B^i_{njk} = -k\, (\xi^n \beta_n) \,.\, B^i_{jlk}$$

which reduces in virtue of (13) to

$$k \,.\, \alpha \,.\, B^i_{ljk} = 0.$$

Since $k \neq 0$ and $B^i_{ijk} \neq 0$, $d = 0$ i.e. $\xi^n \beta_n = 0$.

Hence we can state the following theorem (cf. [1]):

Theorem I : If an $AG\{^2K_N\}$ admits an affine motion generated by a concurrent vector field ξ^i then ξ^i, is pseudo-orthogonal to the vector of recurrence of the space.

4. Affine motions corresponding to a special concircular vector field :

Next, we consider an affine motion generated by a special concircular vector field ξ^i given by

(16) $$\nabla_j \xi^i = \phi(x)\, \delta^i_j$$

where $\phi(x)(\neq$ constant) is a scalar function of co-ordinates x^i.

At first we show that in this case the following relations hold

(i) $\phi_m \xi^m = 0$ $(\phi_m = \nabla_m \phi)$; (ii) $2\phi + C = 0$;

(iii) $A + 3\phi\, C = -\, \mathbf{C} \, . \, d = -\, C\, \xi^l \beta_l$.

Proof of (i) :

Operating Δ_m on (16) and putting $\phi_m = \nabla_m \phi$ we get

$$\nabla_m \nabla_j \xi^i = \phi_m \delta^i_j$$

Also from (2), we have

$$\xi^k \nabla_k \nabla_j \xi^i = -B^i_{jkl} \xi^k \xi^l = 0$$

Whence, $$\xi^m \, . \, \phi^m \, . \, \delta^i_j = 0.$$

Since $\delta^i_j \neq 0$, $\xi^m \phi_m = 0$.

Pooof of (ii) :

From (3) it follows that

(1″) $$\xi^l \nabla_l B^i_{jkl} = -\, 2\phi\, B^i_{jkl}$$

Using (7) it may be replaced by $C\, B^i_{jkl} = -\, 2\phi\, B^i_{jkl}$

Whence $2\phi + C = 0$ [since $B^i_{jkl} \neq 0$]

Proof of (iii) :

Operating ∇_m on (1″) we obtain

$$(\xi^t a_{tm} + C\beta_m + 2\phi_m)\, B^i_{jkl} + 3\phi \, . \, \nabla_m B^i_{jkl} = (\, .$$

Multiplying this condition by ξ^m and summing on m, we get

$$(3 \phi C + A + C \,.\, d) B^i_{jkl} = 0 \qquad \text{(using (7) and (i))}$$

or, $$A + 3 \phi C = -C \,.\, d, \qquad \text{where } A = \xi^t \xi^m a_{tm}.$$

This completes the proofs.

Now, we discuss the case of affine motion generated by a special concircular vector field ξ^i.

We have from Bianchi's second identity

$$\nabla_m B^i_{jkl} + \nabla_k B^i_{jlm} + \nabla_l B^i_{jmk} = 0$$

Covariant differentiation of this, use of (1) and this identity give

$$a_{mn} B^i_{jkl} + a_{kn} B^i_{jlm} + a_{ln} B^i_{jmk} = 0.$$

Multiplication with ξ^l yields

$$a_{mn} B^i_{jkl} \xi^l - a_{kn} B^i_{jml} \xi^l + a_{ln} \xi^l B^i_{jmk} = 0.$$

Applying (2) we get

(17) $$a_{ln} \xi^l B^i_{jmk} = a_{mn} (\nabla_k \nabla_j \xi^i) - a_{kn} (\nabla_m \nabla_j \xi^i)$$
$$= a_{mn} \phi_k \delta^i_j - a_{kn} \phi_m \delta^i_j = (a_{mn} \phi_k - a_{kn} \phi_m) \delta^i_j$$

Now, we have to consider the following two cases :

Case I : $a_{ln} \xi^l \neq 0$. **Case II** : $a_{ln} \xi^l = 0$.

Case I :

Since $B^i_{jmk} + B^i_{mkj} + B^i_{kjm} = 0$

we have $$a_{ln} \xi^l B^i_{mjk} + a_{ln} \xi^l B^i_{mkj} + a_{ln} \xi^l B^i_{kjm} = 0$$

Using (17) this can be expressed as

$$(a_{mn} \phi_k - a_{kn} \phi_m) \delta^i_j + (a_{kn} \phi_j - a_{jn} \phi_k) \delta^i_m (a_{jn} \phi_m - a_{mn} \phi_j) \delta^i_k = 0.$$

Whence, contraction on i & j and summation over these indices yield

$$(N-2) (a_{mn} \phi_k - a_{kn} \phi_m) = 0.$$

Hence, for $N \geqslant 3$, $a_{mn} \phi_k = a_{kn} \phi_m$,

whence, using (i) we get

$$a_{mn} \xi^m \phi_k = 0,$$

But $a_{mn} \xi^m \neq 0$ by assumption. So, $\phi_k = 0$, that is, ϕ is a constant which is contrary to our assumption.

Hence, we deduce the following theorem :

Theorem 2 : There does not exist in an $AG\{^2K_N\}$ an affine motion generated by a special concircular vector field ξ^i given by $\nabla_j \xi^i = \phi(x)\, \delta^i_j$ (ϕ being a non-constant scalar) if $a_{l n}\, \xi^l \neq 0$.

Case II :

In this case, we have form (17)

$$a_{mn}\, \phi_k = a_{kn}\, \phi_m$$

So, $\quad a_{mn}\, \xi^n\, \phi_k = a_{kn}\, \xi^n\, \phi_m$

Since $\phi_m \neq 0$, it follows that

$$a_{mn}\, \xi^n = \mu\, \phi_m$$

for a suitable scalar function μ.

However, according to (i) we have,

$$a_{mn}\, \xi^m\, \xi^n = 0$$

whence, $A=0$.

Consequently, from (ii), (iii) we have

$$3\phi + d = 0 \qquad [\text{since } \phi \neq 0 \text{ so } c \neq 0\,]$$

Whence, $\phi = -\frac{1}{3}\, d = -\frac{1}{3}\, (\xi^t\, \beta_t)$.

Hence, we deduce the following theorem :

Theorem 3 : There exists in an $AG\{^2K_N\}$ an affine motion generated by a special concircular vector field ξ^i given by $\nabla_j\, \xi^i = \phi(x)\, \delta^i_j$ if $a_{l n}\, \xi^l = 0$ and then $\phi(x) = -\frac{1}{3}\, (c^t\, \beta_t)$.

5. Affine motions corresponding to a recurrent vector field :

We now consider an affine motion generated by a recurrent vector field ξ^i. Then $\nabla_j\, \xi^i = \phi_j(x)\, \xi^i$ where ϕ_i is not a gradient vector.

In this case (8) becomes

$$C \,.\, \nabla_m B^i_{jkl} + (\nabla_m C - C\, \beta_m - \xi^t\, a_{tm})\, B^i_{jkl}$$
$$= \nabla_m \xi^t \,.\, \nabla_t B^i_{jkl} = \phi_m\, \xi^t\, \nabla_t B^i_{jkl} = \phi_m \,.\, C \,.\, B^i_{jkl} \quad \text{(using (7))}$$

Hence,

$$\nabla_m B^i_{jkl} = \frac{1}{C}\,[\phi_m\, C - \nabla_m C + C\, \beta_m + \xi^t\, a_{tm}]\, B^i_{jkl}, \qquad C \neq 0.$$

This shows that the space is a recurrent space of first order with $\frac{1}{C}[\phi_m C - \nabla_m C + C\beta_m + \xi^t a_{tm}]$, $C \neq 0$, as its vector of recurrence.

In this case the condition (2) becomes

(18) $$\xi^i \nabla_k \phi_j + \xi^i \phi_j \phi_k = -B^i_{jkl}\xi^l$$

Multiplying this by ξ^k and summing over k and using $B^i_{jkl}\xi^k\xi^l = 0$ we get

(19) $$\xi^k \nabla_k \phi_j + \alpha\phi_j = 0 \qquad [\text{Since } \xi^i \neq 0]$$

where $\alpha = \xi^k \phi_k$.

Contracting i & k in (18) we get

$B_{jt}\xi^t = \alpha\phi_j + \xi^k \nabla_k \phi_j$ which reduces in virtue of (19) to

(20) $$B_{jt}\xi^t = 0.$$

Again contracting i and l in the Bianchi's identity

$B^i_{jkl} + B^i_{klj} + B^i_{ljk} = 0$ we get

(21) $$B_{jk} - B_{kj} + {}^h_{hjk} = 0.$$

Using (20) we obtain from (21)

(22) $$B_{kj}\xi^k = B^h_{hjk}\xi^k$$

Again contracting i and j in (2) we get

(23) $$\nabla_k \alpha + B^h_{hkl}\xi^l = 0$$

From (22) and (23) we obtain

(24) $$B_{kj}\xi^k = -\nabla_j \alpha$$

Contracting i and 1 in $£ B^i_{jkl} = 0$ we get

$$0 = £ B_{jk} = \xi^t \nabla_t B_{jk} + B_{tk}\xi^t . \phi_j + B_{jt}\xi^t \phi_k$$

or (25) $$C B_{jk} - \phi_j . \nabla_k \alpha = 0. \qquad (\text{Using (20) (24) and (71)})$$

Multiplying (25) by ξ^j and summing over j and using (24) we get

$$-C . \nabla_k \alpha = \alpha . \nabla_k \alpha, \text{ that is, } (C+\alpha). \nabla_k \alpha = 0$$

Therefore either (i) $C + \alpha = 0$ or (ii) $\nabla_k \alpha = 0$

In case (i) $\alpha \neq 0$ because $C \neq 0$. Hence from (25) we get $-\alpha . B_{jk} = \phi_j \nabla_k \alpha$ Whence

(26) $$B_{jk} = -\phi_k \alpha_k \qquad \text{where } \alpha_k = \frac{1}{\alpha}\nabla_k \alpha.$$

In case (ii) it follows from (25) that $B_{jk}=0$.

Hence we can state the following theorem :

Theorem 4: If an AG $\{^{2}K_{N}\}$ admits an affine motion generated by a recurrent vector field ξ^i given by $\nabla_j \xi^i = \phi_j(x)\, \xi^i$ (ϕ_j not a gradient vector field) then the space is a recurrent space of first order and $B_{jt}\, \xi^t=0$. Further the Ricci tensor B_{ij} is either identically zero or is of the from $B_{ij} = -\phi_i \alpha_j$ where $\alpha_j = \frac{1}{\alpha} \nabla_j \alpha$.

Again, differentiating (18) covariantly, we get

(27) $$\xi^i \nabla_e \nabla_k \phi_j + \phi_e \xi^i \nabla_k \phi_j + \phi_k \xi^i \nabla_e \phi_j + \xi^i \phi_j \nabla_e \phi_k + \phi_j \phi_k \phi_e \xi^i$$
$$= - B^i_{jkt} \phi_e \xi^t - \xi^t \nabla_e B^i_{jkt}$$

Also, differentiating (19) covariantly, we get

(28) $$\xi^k \nabla_e \nabla_k \phi_j + \phi_e \xi^k \nabla_k \phi_j + \phi_j \nabla_e \alpha + \alpha \nabla_e \phi_j = 0.$$

Contraction on i & k in (27) yields

$$\xi^k \nabla_l \nabla_k \phi_j + \phi_l \xi^k \nabla_k \phi_j + \alpha \Delta_l \phi^j + \phi_j \xi^t \nabla_l \phi_t + \alpha \phi_j \phi_l$$
$$= B_{jt} \xi^t \phi_l + \xi^t \Delta_l B_{jt}$$

Combining this with (28) we get

(29) $$-\phi_j \nabla_l \alpha + \phi_j \xi^t \nabla_l \phi_t + \alpha \phi_j \phi_l = B_{jt} \xi^t \phi_l + \xi^t \nabla_l B_{jt}$$

Again, covariant differentiation of $\alpha = \xi^t \phi_t$ gives

$$\xi^t \nabla_l \phi_t = \nabla_l \alpha - \phi_t \nabla_l \xi^t = \nabla_l \alpha - \alpha \,.\, \phi_l$$

So (29) reduces to

(30) $$B_{jt} \xi^t \,.\, \phi_l + \xi^t \nabla_l B_{jt} = 0$$

Differntiating (30) covariantly, we get

(31) $$\xi^t \phi_l \nabla_m B_{jt} + B_{jt} \xi^t \nabla_m \phi_l + B_{jt} \phi_l \phi_m \xi^t + \phi_m \xi^t \nabla_l B_{jt}$$
$$+ \xi^t (a_{lm} B_{jt} + \beta_m \nabla_l B_{jt}) = 0.$$

Again from (30)

(32) $$B_{jt} \phi_l \phi_m \xi^t + \phi_m \xi^t \Delta_l B_{jt} = \phi_m [B_{jt} \xi^t \phi_l + \xi^t \Delta_l B_{jt}] = 0$$

Hence (31) reduces to

$$\xi^t \phi_l \Delta_m B_{jt} + (a_{lm} + \nabla_m \phi_l) B_{jt} \xi^t + \xi^t \beta_m \nabla_l B_{jt} = 0$$

Using (32) we get from the above condition

(33) $(a_{lm}+\nabla_m \phi_l-\phi_l \phi_m-\beta_m \phi_l) B_{jt} \xi^t=0.$

Since $B_{jt} \xi^t=0$ (by (20)) the tensor $a_{lm}+\nabla_m \phi_l-\phi_o \phi_m-\beta_m \phi_l$ may be equal to zero and may not be so. However, we suppose that

(34) $a_{lm}+\Delta_m \phi_l-\phi_l \phi_m-\beta_m \phi_l=0,$

Hence, $\xi^l a_{lm}+\xi^l \Delta_m \phi_l-\alpha (\phi_m+\beta_m)$

$$=\xi^l (a_{lm}+\nabla_m \phi_l-\phi_l \phi_m-\beta_m\phi_l)=0.$$

Therefore, the vector of recurrence

$$K_m=\frac{1}{C} [\phi_m C-\nabla_m C+C \beta_m+\xi^t a_{tm}]$$

$$=\frac{1}{C} [\phi_m (C+2\alpha)+\beta_m (C+\alpha)-\nabla_m (C+\alpha)],\ C\neq 0.$$

By virtue of case (i) i. e. $C+\alpha=0$, K_m reduces to

(35) $K_m=-\phi_m$

Again, by virtue of case (ii) i. e. $\nabla_k \alpha=0$, K_m reduces to

(36) $K_m=\frac{1}{C} [\phi_m (C+2\alpha)+\beta_m (C+\alpha)-\nabla_m C],\ C\neq 0.$

Hence, we deduce the following theorem :—

Theorem 5 : If an A G $\{^2K_N\}$ admits an affine motion generated by a recurrent vector field ξ^i given by $\nabla_j \xi^i=\phi_j (x) \xi^i$ (ϕ_j not a gradient vector field) then the space is a recurrent space of first order and $B_{jt}\xi^t=0$. Further, the Ricci tensor B_{ij} is either identically zero or is of the form $B_{ij}=-\phi_i \alpha_j$, where $\alpha_j=\frac{1}{\alpha}\nabla_j \alpha$. In the former case, the vector of recurrence of the space is $-\phi_m$, while in the later case it is $\frac{1}{C} [\phi_m (C+2\alpha)+\beta_m (C+\alpha)-\nabla_m C]$, $C\neq 0$ provided $a_{lm}+\nabla_m \phi_l-\phi_l \phi_m-\beta_m \phi_l=0$.

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REFERENCES

[1] Eisenhart L. P.—Non-Riemannian Geometry, American Mathematical Society, 1927, p. 27.
[2] Takano K. and Imai T.—On some types of affine motions in bi-recurrent spaces, Tensor N. S. 24 (1972), 93-100.
[3] Yano K.—Theory of Lie-derivatives and its applications, Noordhoff Amsterdam (1956).

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Dept. of Pure Math.
Calcutta University

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ON PARTIAL DIFFERENTIAL OPERATORS FOR $F(-n, \beta ; \gamma ; x)$

SARAMA DAS

1. Introduction : In the application of Lie algebra to a special function it is usual to find two operators (called the generators of Lie algebra) which raise and lower the index (or parameter) of the special function under consideration [1]. The object of this paper is to present two partial differential operators, viz.

$$A = x(1-x)\, yu^{-1} \frac{\partial}{\partial x} - xyzu^{-1} \frac{\partial}{\partial z} + y \frac{\partial}{\partial u} - yu^{-1}$$

$$B = x(1-x)\, yz^{-1} u^{-1} \frac{\partial}{\partial x} + xy^2 z^{-1} u^{-1} \frac{\partial}{\partial y} - xyu^{-1} \frac{\partial}{\partial z} + yz^{-1} \frac{\partial}{\partial u} - (1-x)\, yz^{-1} u^{-1},$$

such that A raises the index n and at the same time lowers the parameter γ of $F(-n, \beta ; \gamma ; x)$, while B raises the index and lowers both parameters β, γ of $F(-n, \beta ; \gamma ; x)$ at the same time.

In other words, we have,

$$(1.1) \quad \begin{aligned} &A\,[y^n z^\beta u^\gamma F(-n, \beta ; \gamma ; x)] = (\gamma-1)\, y^{n+1} z u^{\gamma-1} F(-n-1, \beta ; \gamma-1 ; x) \\ &B\,[y^n z^\beta u^\gamma F(-n, \beta ; \gamma ; x)] = (\gamma-1)\, y^{n+1} z^{\beta-1} u^{\gamma-1} F(-n-1, \beta-1 ; \gamma-1 ; x). \end{aligned}$$

The extended forms of the transformation groups generated by the operators A, B are given by

$$(1.2) \quad \exp(aA)\, f(x, y, z, u) = \frac{u}{u+ay} f\left(x \frac{u+ay}{u+axy}, y, \frac{zu}{u+axy}, u+ay\right)$$

$$\exp(bB)\, f(x, y, z, u) = \frac{zu}{zu+by\,(1-x)} f\left(x \frac{zu+by\,(1-x)}{zu}, \frac{yzu}{zu-bxy}, \frac{zu-bxy}{u}, u\, \frac{zu+by\,(1-x)}{zu-bxy}\right).$$

The introduction of such operators for $F(-n, \beta ; \gamma ; x)$ helps us to derive the following generating relations :

$$(1.3) \quad (1-t)^{\gamma-1} (1-xt)^{-\beta} F\left(-n, \beta ; \gamma ; x \frac{1-t}{1-xt}\right)$$

$$= \sum_{m=0}^{\infty} \frac{(-\gamma+1)_m}{m!} F(-n-m, \beta ; \gamma-m ; x)\, t^m,$$

where $|t| < \min(1, |x|^{-1})$.

(1.4) $(1+xt)^{\beta-n-\gamma}(1+t(x-1))^{\gamma-1}F(-n,\beta;\gamma;x+xt(x-1))$

$$=\sum_{m=0}^{\infty}\frac{(-\gamma+1)_m}{m!}F(-n-m,\beta-m;\gamma-m;x)\,t^m$$

where $x\neq 1$, $|t| < \min(|x|^{-1}, |1-x|^{-1}, |x|^{-1}|1-x|^{-1})$

Furthermore, we have proved the following general theorems on generating relations for hypergeometric polynomials $F(-n,\beta;\gamma;x)$.

Theorem I : If there exists a generating function of the form :

$$F(x,t)=\sum_{n=0}^{\infty}a_n F(-n,\beta;\gamma-n;x)\,t^n.$$

Then,

(1.5) $\displaystyle\sum_{n=0}^{\infty}F(-n,\beta;\gamma-n;x)\,\sigma_n(y)\,t^n.$

$$=(1-t)^{\gamma-1}(1-xt)^{-\beta}F\left(x\frac{1-t}{1-xt},\ \frac{yt}{1-t}\right),$$

where

$$\sigma_n(y)=\sum_{k=0}^{n}a_k\frac{(-\gamma+k+1)_{n-k}}{(n-k)!}y^k.$$

Theorem II : If there exists a generating function of the form :

$$F(x,y,t)=\sum_{n=0}^{\infty}a_n F(-n,\beta;\gamma-n;x)\,g_n(y)\,t^n$$

where $g_n(y)$ is any arbitrary polynomial, then,

(1.6) $\displaystyle\sum_{n=0}^{\infty}F(-n,\beta;\gamma-n;x)\,\sigma_n(y,z)\,t^n$

$$=(1-t)^{\gamma-1}(1-xt)^{-\beta}F\left[x\frac{1-t}{1-xt},\ y,\ \frac{zt}{1-t}\right]$$

where,

$$\sigma_n(y,z)=\sum_{k=0}^{n}a_k\frac{(-\gamma+k+1)_{n-k}}{(n-k)!}g_k(y)\,z^k.$$

Theorem III : If there exists a generating function of the form :

$$G(x, t)=\sum_{n=0}^{\infty} a_n F(-n, \beta ; \gamma ; x)\, t^n$$

then,

$$(1.7)\quad (1-y)^{\gamma-1}(1-xt)^{-\beta}\, G\left(x\frac{1-y}{1-xy},\, yt\right)=\sum_{n=0}^{\infty} \sigma_n(x, t)\, y^n,$$

where

$$\sigma_n(x, t)=\sum_{k=0}^{n} a_k \frac{(-\gamma+1)_{n-k}}{(n-k)\,!} F(-n, \beta ; \gamma-n+k ; x)\, t^k$$

Theorem IV : If there exists a generating function of the form :

$$F(x, t)=\sum_{n=0}^{\infty} a_n F(-n, \beta-n ; \gamma-n ; x)\, t^n$$

then,

$$(1.8)\quad \sum_{n=0}^{\infty} F(-n, \beta-n ; \gamma-n ; x)\, \sigma_n(y)\, t^n$$

$$=(1+xt)^{\beta-\gamma}(1+t(x-1))^{\gamma-1} F\left[x+xt(x-1),\, \frac{yt(1+xt)}{1+(x-1)t}\right],$$

where

$$\sigma_n(y)=\sum_{k=0}^{n} a_k \frac{(-\gamma+k+1)_{n-k}}{(n-k)\,!}\, y^k.$$

Theorem V : If there exists a generating function of the form :

$$F(x, y, t)=\sum_{n=0}^{\infty} a_n F(-n, \beta-n ; \gamma-n ; x)\, g_n(y)\, t^n$$

where $g_n(y)$ is any arbitrary polynomial of y, then,

$$(1.9)\quad \sum_{n=0}^{\infty} F(-n, \beta-n ; \gamma-n ; x)\, \sigma_n(y, z)\, t^n$$

$$=(1+xt)^{\beta-\gamma}(1+(x-1)t)^{\gamma-1} F\left[x+xt(x-1),\, y,\, \frac{zt(1+xt)}{1+(x-1)t}\right],$$

where

$$\sigma_n(y, z) = \sum_{k=0}^{n} a_k \frac{(-\gamma+k+1)_{n-k}}{(n-k)!} g_k(y) z^k$$

Theorem VI : If there exists a generating function of the form :

$$G(x, t) = \sum_{n=0}^{\infty} a_n F(-n, \beta ; \gamma ; x) t^n$$

then

$$(1.10) \quad (1+xy)^{\beta-\gamma} (1+(x-1) y)^{\gamma-1} G\left(x+xy(x-1), \frac{yt}{1+xy}\right)$$

$$= \sum_{n=0}^{\infty} \sigma_n(x, t) y^n,$$

where

$$\sigma_n(x, t) = \sum_{k=0}^{n} a_k \frac{(-\gamma+1)_{n-k}}{(n-k)!} F(-n, \beta-n+k ; \gamma-n+k ; x) t^k$$

2. Derivation of the operators :

We know that $F(-n, \beta, \gamma, x)$ satisfies the following relations :

$$(2.1) \quad \frac{d}{dx} F(-n, \beta ; \gamma ; x) = x^{-1} (1-x)^{-1} [(\beta x-\gamma+1) F(-n, \beta ; \gamma ; x) + (\gamma-1) F(-n-1, \beta ; \gamma-1 ; x)]$$

$$(2.2) \quad \frac{d}{dx} F(-n, \beta ; \gamma ; x) = x^{-1}(1-x)^{-1} [\{x(\beta-n-1)-\gamma-1\} F(-n, \beta ; \gamma ; x) + (\gamma-1) F(-n-1, \beta-1 ; \gamma-1 ; x)]$$

Let $A = A_1 \frac{\partial}{\partial x} + A_2 \frac{\partial}{\partial y} + A_3 \frac{\partial}{\partial z} + A_4 \frac{\partial}{\partial u} + A_0$ be an operator such that

$$A[y^n z^\beta u^\gamma F(-n, \beta ; \gamma ; x)] = a_n y^{n+1} z^\beta u^{\gamma-1} F(-n-1, \beta ; \gamma-1 ; x),$$

where each A_i is a function of x, y, z, u and independent of n, β, γ and a_n is a function of n, β, γ but independent of x, y, z, u. With the help of (2.1) we have

$$(2.3) \quad A[y^n z^\beta u^\gamma F(-n, \beta ; \gamma ; x)] = A_1 x^{-1}(1-x)^{-1} y^n z^\beta u^\gamma \{(\beta x-\gamma-1) . F(-n, \beta ; \gamma ; x) + (\gamma-1) F(-n-1, \beta ; \gamma-1 ; x)\} + y^n z^\beta u^\gamma F(-n, \beta ; \gamma ; x) . \{A_2 n y^{-1} + A_3 \beta z^{-1} + A_4 \gamma u^{-1} + A_0\}.$$

In order to make the coefficients of $F(n-1, \beta ; \gamma-1 ; x)\, y^{n+1} z^{\beta} u^{\gamma-1}$ independent of x, y, z, u, we choose $A_1 = x(1-x)\, yu^{-1}$, so that (2.3) reduces to

$$A[y^n z^{\beta} u^{\gamma} F(-n, \beta ; \gamma ; x)] = (\gamma-1) y^{n+1} z^{\beta} u^{\gamma-1} F(-n-1, \beta ; \gamma-1 ; x)$$
$$+ [(\beta x-\gamma+1)\, yu^{-1} + A_2\, ny^{-1} + A_3\, \beta z^{-1} + A_4\, \gamma u^{-1} + A_0]$$
$$. y^n z^{\beta} u^{\gamma} F(-n, \beta ; \gamma ; x)$$

In order to make the coefficients of $y^n z^{\beta} u^{\gamma} F(-n, \beta ; \gamma ; x)$ zero we choose $A_2 = O$, $A_3 = -xyzu^{-1}$, $A_4 = y$, $A = -yu^{-1}$.

Thus we get

(2.4) $$A = x(1-x)yu^{-1} \frac{\partial}{\partial x} - xyzu^{-1} \frac{\partial}{\partial z} + y \frac{\partial}{\partial u} - yu^{-1},$$

for which

(2.5) $$A[y^n z^{\beta} u^{\gamma} F(-n, \beta ; \gamma ; x)] = (\gamma-1)\, y^{n+1} z^{\beta} u^{\gamma-1} F(-n-1, \beta ; \gamma-1 ; x)$$

Similarly, we have on using (2.2)

(2.6) $$B = x(1-x)yz^{-1} u^{-1} \frac{\partial}{\partial x} + xy^2 z^{-1} u^{-1} \frac{\partial}{\partial y} - xyu^{-1} \frac{\partial}{\partial z}$$
$$+ yz^{-1} \frac{\partial}{\partial u} - (1-x)\, yz^{-1} u^{-1},$$

for which

(2.7) $$B[y^n z^{\beta} u^{\gamma} F(-n, \beta ; \gamma ; x)]$$
$$= (\gamma-1) y^{n+1} z^{\beta-1} u^{\gamma-1} F(-n-1, \beta-1 ; \gamma-1 ; x)$$

3. Extended form of the groups generated by A and B :

Let $\phi_1\,(x, y, z, u)$ be a function such that $A\,\phi_1 = 0$. Then on solving $A\,\phi_1 = 0$ we get a solution as $\phi_1 = yz^{-1}\, u(1-x)$, so that A reduces to $A' = x(1-x)\, yu^{-1} \frac{\partial}{\partial x} - xyzu^{-1} \frac{\partial}{\partial z} + y \frac{\partial}{\partial u}$. Thus $A = \phi_1^{-1} A' \phi_1$

Now let X, Y, Z, U be a set of new variables for which

(3.1) $$A'X = 1, \quad A'Y = 0, \quad A'Z = 0, \quad A'U = 0,$$

so that A reduces to $\frac{\partial}{\partial X}$.

Solving (3.1) we get, a set of solutions as

$$X = \frac{u}{y}, \quad Y = y, \quad Z = \frac{1-x}{z}, \quad U = \frac{x}{(1-x)u},$$

from which we get

$$x=\frac{XYU}{1+XYU},\ y=Y,\ z=\frac{1}{Z(1+XYU)},\ u=XY.$$

Then

$$e^{aA}f(x,y,z,u)=\phi_1^{-1}(x,y,z,u)\ e^{aA'}\left[\phi_1(x,y,z,u)f(x,y,z,u)\right]$$

$$=\phi_1^{-1}(x,y,z,u)\exp\left(a\frac{\partial}{\partial X}\right)g_1(X,Y,Z,U)$$

$$=\phi_1^{-1}(x,y,z,u)\ g_1(X+a,Y,Z,U).$$

On calculation we get

$$\text{(3.2)}\quad e^{aA}f(x,y,z,u)=\frac{u}{u+ay}f\left[x\ \frac{u+ay}{u+axy},\ y,\ \frac{zu}{u+axy},\ u+ay\right]$$

Similarly, $B\phi_2=0$ gives a solution $\phi_2=x^2yu^{-1}$, so that B reduces to B' and $B=\phi_2^{-1}B'\phi_2$. For the new set of variables X, Y, Z, U, such that

(3.3) $B'X=1,\ B'Y=0,\ B'Z=0,\ B'U=0,$

which gives a solution as,

$$x=\frac{Z+XY^2U}{Z},\ y=\frac{-Z}{XYU},\ z=-XYU,\ u=\frac{XY^2U+Z}{X^2Y^2U^2}.$$

Now

$$e^{bB}f(x,y,z,u)=\phi_2^{-1}(x,y,z,u)\ e^{bB'}\left[\phi_2(x,y,z,u)f(x,y,z,u)\right]$$

Thus

$$\text{(3.4)}\quad e^{bB}f(x,y,z,u)=\frac{zu}{zu+by(1-x)}f\left[x\ \frac{zu+by(1-x)}{zu},\ \frac{yzu}{zu-bxy},\ \frac{zu-bxy}{u},\ u\ \frac{zu+by(1-x)}{zu-bxy}\right].$$

4. Application of the operator 'A' :

I. First we notice that

$$\text{(4.1)}\quad e^{aA}\left[y^nz^\beta u^\gamma F(-n,\beta;\gamma;x)\right]=\frac{u}{u+ay}y^n\left(\frac{zu}{u+axy}\right)^\beta(u+ay)^\gamma F\left(-n,\beta;\gamma;x\ \frac{u+ay}{u+axy}\right)$$

$$=y^nz^\beta u^\gamma\left(1+\frac{ay}{u}\right)^{\gamma-1}\left(1+\frac{ay}{u}x\right)^{-\beta}F\left(-n,\beta;\gamma;x\ \frac{1+\frac{ay}{u}}{1+\frac{ay}{u}x}\right).$$

On the otherhand,

$$(4.2) \quad e^{aA}\left[y^n z^\beta u^\gamma F(-n, \beta ; \gamma ; x)\right]=\sum_{m=0}^{\infty} \frac{a^m}{m !} A^m\left[y^n z^\beta u^\gamma F(-n, \beta ; \gamma ; x)\right]$$

$$=\sum_{m=0}^{\infty} \frac{a^m}{m !}(-1)^m(-\gamma+1)_m y^{n+m} z^\beta u^{\gamma-m} F(-n-m, \beta ; \gamma-m ; x).$$

Equating and using the substitution $\frac{-ay}{u}=t$, we get

$$(4.3) \quad (1-t)^{\gamma-1}(1-xt)^{-\beta} F\left(-n, \beta ; \gamma ; x \frac{1-t}{1-xt}\right)$$

$$=\sum_{m=0}^{\infty} \frac{(-\gamma+1)_m}{m !} F(-n-m, \beta ; \gamma-m ; x) t^m$$

where $|t|<\min(1,|x|^{-1})$.

Now making use of the relation (4.3) we shall derive the following general theorems on generating functions :

Theorem I : If there exists a generating function of the form :

$$F(x, t)=\sum_{n=0}^{\infty} a_n F(-n, \beta ; \gamma-n ; x) t^n$$

then

$$(4.4) \quad \sum_{n=0}^{\infty} F(-n, \beta ; \gamma-n ; x) \sigma_n(y) t^n$$

$$=(1-t)^{\gamma-1}(1-xt)^{-\beta} F\left(x \frac{1-t}{1-xt}, \frac{yt}{1-t}\right),$$

where

$$\sigma_n(y)=\sum_{k=0}^{n} a_k \frac{(-\gamma+k+1)_{n-k}}{(n-k) !} y^k$$

Proof : We have

$$\sum_{n=0}^{\infty} F(-n, \beta ; \gamma-n ; x) \sigma_n(y) t^n$$

$$=\sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k \frac{(-\gamma+k+1)_{n-k}}{(n-k) !} y^k F(-n, \beta ; \gamma-n ; x) t^n$$

$$=\sum_{k=0}^{\infty} a_k\,(yt)^k \sum_{n=0}^{\infty} \frac{(-\gamma+k+1)_n}{n\,!}\,F(-n-k,\beta\,;\gamma-n-k\,;x)\,t^n$$

$$=(1-t)^{\gamma-1}\,(1-xt)^{-\beta}\sum_{k=0}^{\infty} a_k\,F\left(-k,\beta\,;\gamma-k\,;x\frac{1-t}{1-xt}\right)\left(\frac{yt}{1-t}\right)^k$$

$$=(1-t)^{\gamma-1}\,(1-xt)^{-\beta}\,F\left[x\frac{1-t}{1-xt},\ \frac{yt}{1-t}\right].$$

Theorem II : If there exists a generating function of the form :

$$F(x,y,t)=\sum_{n=0}^{\infty} a_n\,F(-n,\beta\,;\gamma-n\,;x)\,g_n(y)\,t^n$$

where $g_n(y)$ is any arbitrary polynomial, then

$$(4.5)\quad \sum_{n=0}^{\infty} F(-n,\beta\,;\gamma-n\,;x)\,\sigma_n(y,z)\,t^n$$

$$=(1-t)^{\gamma-1}\,(1-xt)^{-\beta}\,F\left(x\frac{1-t}{1-xt},\,y,\,\frac{zt}{1-t}\right)$$

where $\sigma_n(y,z)=\sum_{k=0}^{n} a_k\,\frac{(-\gamma+k+1)_{n-k}}{(n-k)\,!}\,g_k(y)\,z^k$

Proof : We have,

$$\sum_{n=0}^{\infty} F(-n,\beta\,;\gamma-n\,;x)\,\sigma_n(y,z)\,t_n$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k\,\frac{(-\gamma+k+1)_{n-k}}{(n-k)\,!}\,g_k(y)\,z^k\,F(-n,\beta\,;\gamma-n\,;x)\,t^n$$

$$=\sum_{k=0}^{\infty} a_k\,g_k(y)\,(zt)^k\sum_{n=0}^{\infty}\frac{(-\gamma+k+1)_n}{n\,!}\,F(-n-k,\beta\,;\gamma-n-k\,;x)\,t^n$$

$$=(1-t)^{\gamma-1}\,(1-xt)^{-\beta}\sum_{k=0}^{\infty} a_k\,g_k(y)\,F\left(-k,\beta;\gamma-k;x\,\frac{1-t}{1-xt}\right)\left(\frac{zt}{1-t}\right)^k$$

$$=(1-t)^{\gamma-1}\,(1-xt)^{-\beta}\,F\left(x\frac{1-t}{1-xt},\,y,\,\frac{zt}{1-t}\right).$$

II. Next we shall use the operator Δ to derive another general theorem on generating function.

Theorem III : If there exists a generating function of the form :

$$G(x, t) = \sum_{n=0}^{\infty} a_n\, F(-n, \beta\,;\,\gamma\,;\,x)\, t^n$$

then

$$(4.6)\quad (1-y)^{\gamma-1}\,(1-xy)^{-\beta}\, G\left(x\,\frac{1-y}{1+xy}, ty\right) = \sum_{n=0}^{\infty} \sigma_n\,(x, t)\, y^n$$

where

$$\sigma_n\,(x, t) = \sum_{k=0}^{n} a_k\, \frac{(-\gamma+1)_{n-k}}{(n-k)\,!}\, F(-n, \beta\,;\,\gamma-n+k\,;\,x)\, t^k$$

Proof : We have, $G(x, t) = \sum_{n=0}^{\infty} a_n\, F(-n, \beta\,;\,\gamma\,;\,x)\, t^n.$

Replacing t by ty and multiplying both sides by $z^\beta u^\gamma$, we have, on applying the operator $e^{a\Delta}$ to both sides,

$$e^{a\Delta}\,[G\,(x, ty)\, z^\beta\, u^\gamma] = e^{a\Delta}\left[\sum_{n=0}^{\infty} a_n\, F(-n, \beta\,;\,\gamma\,;\,x)\, t^n\, y^n\, z^\beta\, u^\gamma\right].$$

The left member becomes

$$\frac{u}{u+ay}\, G\left(x\,\frac{u+ay}{u+axy}, ty\right)\left(\frac{zu}{u+axy}\right)^\beta (u+ay)^\gamma.$$

On the other hand the right member becomes

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n\, t^n\, \frac{a^m}{m\,!}\, \Delta^m\,[y^n\, z^\beta\, u^\gamma\, F(-n, \beta\,;\,\gamma\,;\,x)]$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n\, t^n\, \frac{(-a)^m}{m\,!}\, (-\gamma+1)_m\, y^{n+m}\, z^\beta\, u^{\gamma-m}\, F(-n-m, \beta\,;\,\gamma-m\,;\,x)$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{n-m}\, t^{n-m}\, \frac{(-a)^m}{m\,!}\, (-\gamma+1)_m\, y^n\, z^\beta\, u^\gamma\, F(-n, \beta\,;\,\gamma-m\,;\,x)$$

$$=\sum_{n=0}^{\infty} y^n \sum_{m=0}^{n} a_{n-m} \frac{(-\gamma+1)_m}{m\,!} \mathrm{F}(-n, \beta ; \gamma-m ; x)\, t^{n-m}\left(\frac{-a}{u}\right)^m z^\beta u^\gamma$$

Thus

$$\left(1+\frac{a}{u} y\right)^{\gamma-1}\left(1+\frac{a}{u} xy\right)^{-\beta} \mathrm{G}\left(x \frac{1+\frac{a}{u} y}{1+\frac{a}{u} xy}, ty\right)$$

$$=\sum_{n=0}^{\infty} \sigma_n (x, t, u)\, y^n,$$

where

$$\sigma_n (x, t, u)=\sum_{k=0}^{n} a_k \frac{(-\gamma+1)_{n-k}}{(n-k)\,!} \mathrm{F}(-n, \beta ; \gamma-n+k ; x)\, t^k\left(-\frac{a}{u}\right)^{n-k}$$

Putting $-\frac{a}{u}=1$, we get

$$(1-y)^{\gamma-1} (1-xy)^{-\beta}\, \mathrm{G}\left(x \frac{1-y}{1-xy}, ty\right)=\sum_{n=0}^{\infty} \sigma_n (x, t)\, y^n,$$

where

$$\sigma_n (x, t)=\sum_{k=0}^{n} a_k \frac{(-\gamma+1)_{n-k}}{(n-k)\,!} \mathrm{F}(-n, \beta ; \gamma-n+k ; x) t^k.$$

5. Application of the operator 'B':

I. First we notice that

$$(5.1)\quad e^{bB} [y^n z^\beta u^\gamma \mathrm{F}(-n, \beta ; \gamma ; x)]=\frac{zu}{zu+by (1-x)}\left(\frac{yzu}{zu-bxy}\right)^n \left(\frac{zu-bxy}{u}\right)^\beta$$

$$\cdot \left[u \frac{zu+by (1-x)}{zu-bxy}\right]^\gamma \mathrm{F}\left(-n, \beta ; \gamma ; x \frac{zu+by (1-x)}{zu}\right).$$

On the otherhand,

$$(5.2)\quad e^{bB} [y^n z^\beta u^\gamma \mathrm{F}(-n, \beta ; \gamma ; x)]=\sum_{m=0}^{\infty} \frac{b^m}{m\,!} \mathrm{B}^m y^n z^\beta u^\gamma \mathrm{F}(-n, \beta ; \gamma ; x)$$

$$=\sum_{m=0}^{\infty} \frac{(-b)^m}{m\,!} (-\gamma+1)_m y^{n+m} z^{\beta-m} u^{\gamma-m} \mathrm{F}(-n-m, \beta-m ; \gamma-m ; x).$$

Equating (5.1) and (5.2) and using the substitution $-\frac{by}{zu}=t$, we get

(5.3) $$(1+xt)^{\beta-\gamma-n}(1-t+xt)^{\gamma-1}\, F(-n, \beta ; \gamma ; x+xt(x-1))$$

$$=\sum_{m=0}^{\infty} \frac{(-\gamma+1)_m}{m!} F(-n-m, \beta-m ; \gamma-m ; x)\, t^m$$

where $|t| < \min(|x|^{-1}, |1-x|^{-1}, |x|^{-1}|1-x|^{-1})$.

Now making use of the relations (5.3) we shall derive two new general theorems on generating function.

Theorem IV : If there exists a generating function of the form :

$$F(x, t)=\sum_{n=0}^{\infty} a_n F(-n, \beta-n ; \gamma-n ; x)\, t^n$$

then

(5.4) $$\sum_{n=0}^{\infty} F(-n, \beta-n ; \gamma-n ; x)\, \sigma_n(y)\, t^n$$

$$=(1+xt)^{\beta-\gamma}(1-t+xt)^{\gamma-1}\, F\left[x+xt(x-1), \frac{yt(1+xt)}{1+(x-1)t}\right]$$

where

$$\sigma_n(y)=\sum_{k=0}^{n} a_k \frac{(-\gamma+k+1)_{n-k}}{(n-k)!} y^k$$

Proof : We have

$$\sum_{n=0}^{\infty} F(-n, \beta-n ; \gamma-n ; x)\, \sigma_n(y)\, t^n$$

$$=\sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k \frac{(-\gamma+k+1)_{n-k}}{(n-k)!} y^k\, F(-n, \beta-n ; \gamma-n ; x)\, t^n$$

$$=\sum_{n,k=0}^{\infty} a_k \frac{(-\gamma+k+1)_n}{n!} y^k t^{n+k} F(-n-k, \beta-n-k ; \gamma-n-k ; x)$$

$$=\sum_{k=0}^{\infty} a_k (yt)^k \sum_{n=0}^{\infty} \frac{(-\gamma+k+1)_n}{n!} F(-k-n, \beta-k-n ; \gamma-k-n ; x)\, t^n$$

$$= (1+xt)^{\beta-\gamma}\ (1-t+xt)^{\gamma-1} \sum_{k=0}^{\infty} a_k\ F(-k,\ \beta-k\ ;\ \gamma-k\ ;\ x+xt\ (x-1))$$

$$\cdot \left[\frac{yt\ (1+xt)}{1+(x-1)\ t}\right]^k$$

$$= (1+xt)^{\beta-\gamma}\ (1-t+xt)^{\gamma-1}\ F\left[x+xt\,(x-1),\ \frac{yt\ (1+xt)}{1+\ (x-1)\ t}\right]$$

Theorem V : If these exists a generating function of the form :

$$F\,(x,\,y,\,t) = \sum_{n=0}^{\infty} a_n\ F(-n,\ \beta-n\ ;\ \gamma-n\ ;\ x)\ g_n\ (y)\ t^n$$

where $g_n\ (y)$ is any arbitary polynomial, then

$$(5.5)\quad \sum_{n=0}^{\infty} F(-n,\ \beta-n\ ;\ \gamma-n\ ;\ x)\ \sigma_n\ (y,\,z)\ t^n$$

$$= (1+xt)^{\beta-\gamma}\ (1-t+xt)^{\gamma-1}\ F\left[x+xt\ (x-1),\ y,\ \frac{zt\ (1+xt)}{1+\ (x-1)\ t}\right]$$

where,

$$\sigma_n\ (y,\,z) = \sum_{k=0}^{n} a_k\ \frac{(-\gamma+k+1)_{n-k}}{(n-k)\ !}\ g_k\ (y)\ z^k$$

Proof : We have

$$\sum_{n=0}^{\infty} F(-n,\ \beta-n\ ;\ \gamma-n\ ;\ x)\ \sigma_n\ (y,\,z)\ t^n$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k\ \frac{(-\gamma+k+1)_{n-k}}{(n-k)\ !}\ g_k\ (y)\ z^k\ F(-n,\ \beta-n\ ;\ \gamma-n\ ;\ x)\ t^n$$

$$= \sum_{n,k=0}^{\infty} a_k\ \frac{(-\gamma+k+1)_n}{n\ !}\ g_k\ (y)\ z^k\ t^{n+k}\ F(-n-k,\beta-n-k\ ;\ \gamma-n-k\ ;\ x)$$

$$= \sum_{k=0}^{\infty} a_k\ (zt)^k\ g_k\ (y) \sum_{n=0}^{\infty} \frac{(-\gamma+k+1)_n}{n\ !}$$

$$\cdot\ F(-n-k,\ \beta-n-k\ ;\ \gamma-n-k\ ;\ x)\ t^n$$

$$=(1+xt)^{\beta-\gamma}\ (1-t+xt)^{\gamma-1} \sum_{k=0}^{\infty} a_k\ g_k\ (y) \left[\frac{zt\ (1+xt)}{1+\ (x-1)\ t}\right]^k$$

$$.\ F\,(-k, \beta-k\ ; \gamma-k\ ; x+xt\ (x-1))$$

$$=(1+xt)^{\beta-\gamma}\ (1-t+xt)^{\gamma-1}\ F\left[x+xt\,(x-1), y, \frac{zt\ (1+xt)}{1+\ (x-1)\ t}\right]$$

II. Next we shall use the operator B to derive another new general theorem on generating function :

Theorem VI : If there exists a generating function of the form :

$$G\,(x, t)=\sum_{n=0}^{\infty} a_n\ F(-n, \beta\ ; \gamma\ ; x)\ t^n$$

then

(5.6) $$(1+xy)^{\beta-\gamma}\ (1-y+xy)^{\gamma-1}\ G\left(x+xy\,(x-1), \frac{yt}{(1+xy)}\right)$$

$$=\sum_{n=0}^{\infty} \sigma_n\ (x, t)\ y^n$$

where,

$$\sigma_n\ (x, t)=\sum_{k=0}^{n} a_k \frac{(-\gamma+1)_{n-k}}{(n-k)\ !}\ F\,(-n, \beta-n+k\ ; \gamma-n+k\ ; x)\ t^k$$

Proof : We have

$$G\,(x, t)=\sum_{n=0}^{\infty} a_n\ F\,(-n, \beta\ ; \gamma\ x)\ t^n$$

Replacing t by ty and multiplying both sides by $z^\beta u^\gamma$, we have, on applying operator e^{bB} to both sides,

$$e^{bB}\,[G\,(x, ty)\ z^\beta\ u^\gamma]=e^{bB}\left[\sum_{n=0}^{\infty} a_n\ F\,(-n, \beta\ ; \gamma\ ; x)\ t^n\ y^n\ z^\beta\ u^\gamma\right]$$

The left member becomes

$$=\frac{zu}{zu+by\ (1-x)} \left(\frac{zu-bxy}{u}\right)^\beta \left[u\ \frac{zu+by\ (1-x)}{zu-bxy}\right]^\gamma$$

$$.\ G\left[x\ \frac{zu+by\ (1-x)}{zu}, \frac{tyzu}{zu-bxy}\right]$$

On the other hand, the right member becomes

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n \frac{b^m}{m\,!}\, B^m \left[F(-n, \beta\,;\, \gamma\,;\, x)\, t^n y^n z^\beta u^\gamma\right]$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n\, t^n \frac{(-\gamma+1)^m}{m\,!} (-b)^m\, F(-n-m, \beta-m\,;\, \gamma-m\,;\, x)$$

$$. \, y^{n+m} z^{\beta-m} u^{\gamma-m}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n} a_{n-m}\, t^{n-m} \frac{(-\gamma+1)_m}{m\,!} (-b)^m y^n z^{\beta-m} u^{\gamma-m}$$

$$. \, F(-n, \beta-m\,;\, \gamma-m\,;\, x)$$

$$= z^\beta u^\gamma \sum_{n=0}^{\infty} y^n \sum_{m=0}^{n} a_{n-m} \frac{(-\gamma+1)_m}{m\,!} \left(\frac{-b}{zu}\right)^m$$

$$F(-n, \beta-m\,;\, \gamma-m\,;\, x)\, t^{n-m}$$

$$= z^\beta u^\gamma \sum_{n=0}^{\infty} \sigma_n(x, t, z, u)\, y^n,$$

where

$$\sigma_n(x, t, z, u) = \sum_{k=0}^{n} a_{n-m} \frac{(-\gamma+1)_m}{m\,!} \left(\frac{-b}{zu}\right)^m F(-n, \beta-m\,;\, \gamma-m\,;\, x)\, t^{n-m}$$

$$= \sum_{k=0}^{n} a_k \frac{(-\gamma+1)_{n-k}}{(n-k)\,!} F(-n, \beta-n+k\,;\, \gamma-n+k\,;\, x)\, t^k \left(\frac{-b}{zu}\right)^{n-k}$$

Equating and using $\frac{-b}{zu}=1$, we get,

$$(1+xy)^{\beta-\gamma}(1-y+xy)^{\gamma-1}\, G\left[x+xy\,(x-1), \frac{ty}{1+xy}\right]$$

$$= \sum_{n=0}^{\infty} \sigma_n(x, t)\, y^n,$$

where

$$\sigma_n(x, t) = \sum_{k=0}^{n} a_k \frac{(-\gamma+1)_{n-k}}{(n-k)\,!} F(-n, \beta-n+k\,;\, \gamma-n+k\,;\, x)\, t^k$$

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REFERENCES

[1] McBride, E. B.—Obtaining Generating Function, Springer-Verlag (1971), Chapter II and III.

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Calcutta University

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ON A PAIR OF GENERATING RELATIONS FOR SOME SPECIAL FUNCTIONS FROM THE VIEW OF LIE-ALGEBRA

ASIT KUMAR CHONGDAR

1. **Introduction :** Starting from the infinitesimal operators R and L, the elements of Lie-algebra for a particular special function, which raise and lower the indices of the special function, we can generate the finite operatore (exp a R), (exp b L) of the corresponding Lie-group. Now since any element of the said Lie-group operates on the function in the following ways :

(i) it shifts the argument of the function

(ii) it produces an infinite sum of functions with unchanged arguments but with shifted indices,

the desired generating function can be obtained by equating the two results.

The composition law

$$(\exp a\mathrm{R})\,(\exp b\mathrm{L}) = \exp(a\mathrm{R} + b\mathrm{L})$$

will hold or not according as R, L commute or not. In case when $[\mathrm{R}, \mathrm{L}] \neq 0$, we shall operate $(\exp a\mathrm{R})$ $(\exp b\mathrm{L})$ and $(\exp b\mathrm{L})$ $(\exp a\mathrm{R})$ successively on the function concerned in order to derive a pair of generating relations for the function. This method was already suggested by S. K. Chatterjea [1].

The object of the present paper is to follow the method of Chatterjea in order to derive a pair of generating relations for the Laguerre and Bessel polinomials seperately from a different view point.

2. **Laguerre polynomials :** From the second order differential equation

$$x\mathrm{D}^2 y + (1+\alpha-x)\,\mathrm{D}y + ny = 0, \quad (\mathrm{D} \equiv d/dx)$$

for the Laguerre polynomials, we notice that

$$\text{(2.1)} \quad \mathrm{R}\,[\mathrm{L}_n^{(\alpha)}(x)\,y^{\alpha}] = -\mathrm{L}_n^{(\alpha+1)}(x)\,y^{\alpha+1}$$

$$\mathrm{L}\,[\mathrm{L}_n^{(\alpha)}(x)\,y^{\alpha}] = (n+\alpha)\,\mathrm{L}_n^{(\alpha-1)}(x)\,y^{\alpha-1},$$

where

$$\text{(2.2)} \quad \mathrm{R} = y\frac{\partial}{\partial x} - y, \quad \mathrm{L} = xy^{-1}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$$

and $[\mathrm{R}, \mathrm{L}] = 1$ i.e., $[\mathrm{R}, \mathrm{L}] \neq 0$.

We shall now apply the operator (exp aR) (exp bL) to ($L_n^{(\alpha)}(x)\, y^{\alpha}$).

We have

(2.3) $$(\exp aR) f(x, y) = f(x + ay, y)$$

(2.4) $$(\exp bL) f(x, y) = e^{-ay} f\left(\frac{x(y+b)}{y},\ y+b\right).$$

Thus we get

$$(\exp aR)(\exp bL)(L_n^{(\alpha)}(x)\, y^{\alpha}) = e^{-ay}(y+b)^{\alpha} L_n^{(\alpha)}\left(\frac{(x+ay)(y+b)}{y}\right).$$

On the other hand,

$$(\exp aR)(\exp bL)(L_n^{(\alpha)}(x)\, y^{\alpha})$$

$$= \sum_{m=0}^{\infty} \frac{(-ay)^m}{m!} \sum_{p=0}^{\infty} \frac{b^p}{p!} (n+\alpha-p+1)_p\, y^{\alpha-p} L_n^{(\alpha-p+m)}(x).$$

Equating the above two results we get

(2.5) $$\sum_{m=0}^{\infty} \frac{(-ay)^m}{m!} \sum_{p=0}^{\infty} \frac{b^p}{p!} (n+\alpha-p+1)_p\, y^{\alpha-p} L_n^{(\alpha-p+m)}(x)$$

$$= e^{-ay}(b+y)^{\alpha} L_n^{(\alpha)}\left(\frac{(x+ay)(y+b)}{y}\right).$$

Next we shall apply the operator (exp bL) (exp aR) to ($L_n^{(\alpha)}(x)\, y^{\alpha}$).

First we observe that

$$(\exp bL)(\exp aR)(L_n^{(\alpha)}(x)\, y^{\alpha}) = e^{-a(y+b)}(y+b)^{\alpha} L_n^{(\alpha)}\left(\frac{(x+ay)(y+b)}{y}\right).$$

On the other hand,

$$(\exp bL)(\exp aR)(L_n^{(\alpha)}(x)\, y^{\alpha})$$

$$= \sum_{m=0}^{\infty} \frac{(-ay)^m}{m!} \sum_{p=0}^{\infty} \frac{b^p}{p!} (n+\alpha+m-p+1)_p\, y^{\alpha-p} L_n^{(\alpha+m-p)}(x).$$

Equiting the above two results we get

(2.6) $$\sum_{m=0}^{\infty} \frac{(-ay)^m}{m!} \sum_{p=0}^{\infty} \frac{b^p}{p!} (n+\alpha+m-p+1)_p\, y^{\alpha-p} L_n^{(\alpha+m-p)}(x)$$

$$= e^{-a(y+b)}(y+b)^{\alpha} L_n^{(\alpha)}\left(\frac{(x+ay)(y+b)}{y}\right).$$

It is interesting to remark that in particular when $b=0$, both the relations (2.5) and (2.6) reduce to the well-known generating relation [2, *p* 373] :

$$\sum_{m=0}^{\infty} \frac{y^m}{m!} L_n^{(\alpha+m)}(x) = e^y L_n^{(\alpha)}(x-y). \tag{2.7}$$

Thus the pair of generating relations (2.5) and (2.6) can be considered as the extension of (2.7).

Bessel Polynomials : From the second order differential equation :

$$x^2D^2y + (\alpha x + \beta) Dy - n(n + \alpha - 1) y = 0$$

for the Bessel polynomials, we notice that [3] :

$$R = xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + (n-1) y, \quad L = \frac{x^2}{y} \frac{\partial}{\partial x} - \frac{nx-\beta}{y} \tag{3.1}$$

and $[R, L] = -\beta$ i.e. $[R, L] \neq 0$,

such that,

$$R\left(Y_n^{(\alpha)}(x)\, y^{\alpha}\right) = (n + \alpha - 1)\, Y_n^{(\alpha+1)}(x)\, y^{\alpha+1}, \tag{3.2}$$

$$L\left(Y_n^{(\alpha)}(x)\, y^{\alpha}\right) = \beta Y_n^{(\alpha-1)}(x)\, y^{\alpha-1}.$$

We shall apply the operator $(\exp aR)(\exp bL)$ to $(Y_n^{(\alpha)}(x)\, y^{\alpha})$.

We have

$$(\exp aR) f(x, y) = (1-ay)^{-n+1} f(x/(1-ay),\ y/(1-ay)) \tag{3.3}$$

$$(\exp bL) f(x, y) = \left(1-b\frac{x}{y}\right)^n e^{b\beta/y} f\left(\frac{xy}{y-bx},\ y\right). \tag{3.4}$$

Thus we get,

$(\exp aR)(\exp bL)(Y_n^{(\alpha)}(x)\, y^{\alpha})$

$$= (1-ay)^{1-\alpha-n} (y-bx)^n\, y^{\alpha-n}\, e^{b\beta(1-ay)/y}\, Y_n^{(\alpha)}\left(\frac{xy}{(y-bx)(1-ay)}\right).$$

On the other hand

$(\exp aR)(\exp bL)(Y_n^{(\alpha)}(x)\, y^{\alpha})$

$$= \sum_{p=0}^{\infty} \sum_{m=0}^{\infty} \frac{(b\beta)^p}{p!} \frac{a^m}{m!} (n+\alpha-p-1)_m\, Y_n^{(\alpha-p+m)}(x)\, y^{\alpha-p+m}.$$

Equating the above two results we get,

(3.5) $(1-ay)^{1-\alpha-n}(y-bx)^n e^{b\beta(1-ay)/y}\,Y_n^{(\alpha)}(xy/(y-bx)(1-ay))$

$$=\sum_{p=0}^{\infty}\sum_{m=0}^{\infty}\frac{(b\beta)^p}{p\,!}\,\frac{a^m}{m\,!}\,(n+\alpha-p-1)_m\,Y_n^{(\alpha-p+m)}(x)\,y^{\alpha-p+m}.$$

Now we shall apply the operator $(\exp bL)\,(\exp aR)$ to $(Y_n^{(\alpha)}(x)\,y^{\alpha})$.

First we observe that

$(\exp bL)\,(\exp aR)\,(Y_n^{(\alpha)}(x)\,y^{\alpha})$

$$=(1-ay)^{1-\alpha-n}(y-bx)^n\,y^{\alpha-n}\,e^{b\beta/y}\,Y_n^{(\alpha)}(xy/(1-ay)(y-bx))$$

On the other hand

$(\exp bL)\,(\exp aR)\,(Y_n^{(\alpha)}(x)\,y^{\alpha})$

$$=\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\frac{a^m}{m\,!}\,\frac{(b\beta)^p}{p\,!}\,(n+\alpha-1)_m\,Y_n^{(\alpha+m-p)}(x)\,y^{\alpha+m-p}.$$

Equating the above two results we get,

(3.6) $(y-bx)^n(1-ay)^{1-\alpha-n}e^{b\beta/y}\,Y_n^{(\alpha)}(xy/(1-ay)(y-bx))$

$$=\sum_{p=0}^{\infty}\sum_{m=0}^{\infty}\frac{a^m}{m\,!}\,\frac{(b\beta)^p}{p\,!}\,(n+\alpha-1)_m\,Y_n^{(\alpha+m-p)}(x)\,y^{m+n-p}.$$

Notice that in particular when $b=0$, both the relations (3.5) and (3.6) reduce to the well known generating relation [4, p 50]

(3.7) $$(1-y)^{1-\alpha-n}\,Y_n^{(\alpha)}\left(\frac{x}{1-y}\right)=\sum_{m=0}^{\infty}\frac{(n+\alpha-1)_m}{m\,!}\,Y_n^{(\alpha+m)}(x)\,y^m.$$

Thus the pair of generating relations (3.5) and (3.5) can be considered as the extension of (3.7)

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REFERENCES

[1] Chatterjea, S. K.—On a pair of generating relations for the special functions from the view point of Lie-algebra, Comptes rendus de l' Academie bulgare des sciences, Tome 27, No 1, 1974, 11-14.

[2] Chatterjea, S. K.—Group theoretic origins of certain generating functions of Laguerre polynomials, Bulletine of the Institute of Mathematics, Academia Sinica, 3(2), 1975, 369-375.

[3] Chen, Ming-Po and Feng Chia Chiu—Group theoretic origins of certain generating functions of generalised Bessel Polynomials, Tamkang Journal of Mathematics, 6 (1975), 87-93.

[4] Grosswald, E.—Bessel polynomials, Springer Velag, Berlin (1978).

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Calcutta University

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ON ORDER AND TYPE OF AN ENTIRE FUNCTION REPRESENTED BY DOUBLE DIRICHLET SERIES

R. K. DAS

1. **Introduction** : The growth properties of an entire function represented by Dirichlet series in one complex variable have been studied by a number of mathematicians. But the study of the growth properties of an entire function in several variables represented by multiple Dirichlet series has not yet been done adequately. The main purpose of this paper is to extend the concepts of order and type of an entire function represented by Dirichlet series in one complex variable to an entire function in two variables represented by double Dirichlet series and to express them in terms of co-efficients and exponents. We also construct some entire double Dirichlet series with given pair of positive integers as an order point and with given type and discuss the convexity of a few sets involving the order of entire double Dirichlet series.

Consider the double Dirichlet series

$$(1.1)\quad f(s_1, s_2)=\sum_{m,n=1}^{\infty} a_{mn} \exp\,(s_1\lambda_m+s_2\mu_n)(s_j=\sigma_j+i\tau_j,\ j=1, 2)$$

where $a_{mn} \in c$, the field of complex numbers, λ_m's, μ_n's are

real, $0 \leqslant \lambda_1 < \lambda_2 < \ldots\ldots < \lambda_m \to \infty,\ 0 \leqslant \mu_1 < \mu_2 < \ldots\ldots < \mu_n \to \infty.$

A. I. Janusanskas in his paper [4] had shown that if

$$(1.2)\quad \lim_{m\to\infty}\frac{\log m}{\lambda_m}=0,\ \lim_{n\to\infty}\frac{\log n}{\mu_n}=0$$

then the domain of convergence of the series (1.1) coincides with its domain of absolute convergence. P. K. Sarkar in his paper [5] had shown that the necessary and sufficient condition that the series (1·1) satisfying (1·) to be entire is that

$$(1.3)\quad \lim_{(m,n)\to\infty}\frac{\log|a_{mn}|}{\lambda_m+\mu_n}=-\infty.$$

2. **Definitions and notations:** We indicate the elements (s_1, s_2) $(\mathrm{Re}\,s_1, \mathrm{Re}\,s_2)$ etc. of c^2 by their corresponding unsuffixed symbols s, $\mathrm{Re}\,s$ etc.

For $(p, r) \in R^2$ (2 dimensional Euclidean space) we say that,

(I) $p \leqslant r \Leftrightarrow p_j \leqslant r_j, j = 1, 2$

(II) $p < r \Leftrightarrow p \leqslant r$ but $p \neq r$

(III) $p \lll r \Leftrightarrow p_j < r_j, j = 1, 2.$

Let F stand for the family of all double Dirichlet series of the form (1.1) satisfying (1.2) and (1.3). Then $f \in F$ denotes an entire function over c^2.

Corresponding to a $f \in F$ we define the functions : the maximum modulus $M = M_f$ and the maximum term $\mu = \mu_f$ on R^2 by

$$M(\sigma) = M_f(\sigma) = \max \{ | f(s) | : s \in c^2, \operatorname{Re} s = \sigma \}$$

$$\mu(\sigma) = \mu_f(\sigma) = \max_{(m,n) \in N^2} \{ | a_{mn} | \exp (\sigma_1 \lambda_m + \sigma_2 \mu_n) \}$$

where N is the set of natural numbers.

We define the product order and type of an entire Dirichlet series over c^2 in the following way.

Let $f \in F$ and $P_f \subset R^2$ be the set of points $\alpha \in R^2$ such that for every $\alpha \in P_f$ there exists a $\sigma^0 = (\sigma_1^0, \sigma_2^0)$ such that

$$\log M(\sigma) \leqslant \exp (\sigma_1 \alpha_1 + \sigma_2 \alpha_2) \text{ for } \sigma \ggg \sigma^0, \sigma \in R^2$$

The closure $\overline{P}_f$ of the set P_f is called the product order set of f. The boundary δP_f of the set P_f is called the product order of f. A point $\rho \in \delta P_f$ is called a product order point of f. We say that f is of infinite or finite product order according as P_f is empty or non-empty. Evidently for any

$$\rho \in \delta P_f, \rho \ggg \overline{o} = (o, o).$$

For brevity hence forth we shall call product order simply as order throughout this paper.

It follows from the definition that the set P_f satisfies the following condition :

If $\alpha \in \delta P_f$ then $\{\alpha' : \alpha' \in R^2, \alpha' >> \alpha\} \subset P_f$ and

if $\alpha \in \delta P_f$ then $\{\alpha' : \alpha' \in R^2, \alpha' << \alpha\} \cap P_f = \phi$

Next let us take an order point $\rho\ (>> \overline{o}) \in \delta P_f$ and denote by $T_f\ (\rho)$ the set of all $T \in R$ such that

$$\log M(\sigma) \leqslant T \exp (\sigma_1 \rho_1 + \sigma_2 \rho_2) \quad \text{for} \quad \sigma \ggg \sigma^{(1)}, \sigma \in R^2$$

Then the infimum of the values of T for which the above relation is satisfied is called the type of f w. r. t the order point ρ. f is said to be of normal type if type of f is finite and $> o$ and of minimum type if type of $f = 0$. Also, f is said to be of infinite or (max) type if $T_f(\rho)$ is empty.

Theorem 2.1: Let $f \epsilon F$. Then $\rho = (\rho_1, \rho_2) \epsilon R_+^2$ is a product order point of f if

$$\limsup_{(m,n)\to\infty} \frac{\frac{\lambda_m}{\rho_1}\log\lambda_m + \frac{\mu_n}{\rho_2}\log\mu_n}{-\log|a_{mn}|} = 1.$$

Proof: Let us suppose that $0 < \epsilon < 1$. Then we have two sequences $\{\lambda_{m_p}\}$ and $\{\mu_{n_q}\}$ with $m_p \to \infty$ as $p \to \infty$ and $n_q \to \infty$ as $q \to \infty$ such that

$$\log|a_{mn}| > -(1-\epsilon)^{-1}\left\{\frac{\lambda_m}{\rho_1}\log\lambda_m + \frac{\mu_n}{\rho_2}\log\mu_n\right\} \text{for } m = m_p \text{ and } n = n_q$$

Since the inequality $M(\sigma_1, \sigma_2) \geqslant |a_{mn}| \exp(\sigma_1\lambda_m + \sigma_2\mu_n)$ holds for all σ_1, σ_2 and m, n it follows that for all σ_1 and σ_2 and $m = m_p$ and $n = n_q$ that

$$\log M(\sigma_1, \sigma_2) > \log|a_{mn}| + \sigma_1\lambda_m + \sigma_2\mu_n$$

$$> -(1-\epsilon)^{-1}\left[\frac{\lambda_m}{\rho_1}\log\lambda_m + \frac{\mu_n}{\rho_2}\log\mu_n\right] + \sigma_1\lambda_m + \sigma_2\mu_n$$

$$= \frac{\lambda_m}{\rho_1}\left[\sigma_1\rho_1 - (1-\epsilon)^{-1}\log\lambda_m\right] + \frac{\mu_n}{\rho_2}\left[\sigma_2\rho_2 - (1-\epsilon)^{-1}\log\mu_n\right]$$

Taking $\sigma_1\rho_1 = (1-\epsilon)^{-1}\log(e\lambda_m)$

$\sigma_2\rho_2 = (1-\epsilon)^{-1}\log(e\mu_n)$

we have $\sigma_1\rho_1 - (1-\epsilon)^{-1}\log\lambda_m = (1-\epsilon)^{-1}$

$\sigma_2\rho_2 - (1-\epsilon)^{-1}\log\mu_n = (1-\epsilon)^{-1}$

and $e^{\sigma_1\rho_1(1-\epsilon)} = e\lambda_m$

$e^{\sigma_2\rho_2(1-\epsilon)} = e\mu_n$

$$\therefore \quad \frac{e^{\sigma_1\rho_1(1-\epsilon)}}{e\rho_1} = \frac{\lambda_m}{\rho_1} \text{ and } \frac{e^{\sigma_2\rho_2(1-\epsilon)}}{e\rho_2} = \frac{\mu_n}{\rho_2}$$

$$\therefore \quad \log M(\sigma_1, \sigma_2) > \frac{e^{\sigma_1\rho_1(1-\epsilon)}}{e\rho_1(1-\epsilon)} + \frac{e^{\sigma_2\rho_2(1-\epsilon)}}{e\rho_2(1-\epsilon)} > \frac{e^{\sigma_1\rho_1(1-\epsilon)}}{e\rho_1(1-\epsilon)}$$

$$\therefore \quad \log\log M(\sigma) > \sigma_1\rho_1 (1-\epsilon) - \log [e\,\rho_1 (1-\epsilon)]$$

$$\therefore \quad \frac{\log\log M(\sigma)}{\sigma_1\rho_1+\sigma_2\rho_2} > \frac{\sigma_1\rho_1 (1-\epsilon)}{\sigma_1\rho_1+\sigma_2\rho_2} - \frac{\log [e\,\rho_1 (1-\epsilon)]}{\sigma_1\rho_1+\sigma_2\rho_2}$$

$$\therefore \quad \limsup_{(\sigma_1,\sigma_2)\to\infty} \frac{\log\log M(\sigma)}{\sigma_1\rho_1+\sigma_2\rho_2} \geqslant \limsup_{(\sigma_1,\sigma_2)\to\infty} \frac{\sigma_1\rho_1 (1-\epsilon)}{\sigma_1\rho_1+\sigma_2\rho_2} = 1 \text{ (See [1])} \qquad \dots \quad (A)$$

Again, we see that

$$\frac{\frac{\lambda_m}{\rho_1}\log\lambda_m + \frac{\mu_n}{\rho_2}\log\mu_n}{-\log|a_{mn}|} < \overline{1+\epsilon} \qquad \text{for } m > m_0,\ n > n_0$$

$$\therefore \quad \lambda_m^{-\frac{\lambda_m}{\rho_1}} \cdot \mu_n^{-\frac{\mu_n}{\rho_2}} > |a_{mn}|^{1+\epsilon} \qquad \text{for } m > m_0,\ n > n_0$$

$$\therefore \quad |a_{mn}| < \lambda_m^{-\frac{\lambda_m}{\rho_1(1+\epsilon)}} \cdot \mu_n^{-\frac{\mu_n}{\rho_2(1+\epsilon)}} \qquad \text{for } m > m_0,\ n > n_0$$

$$\text{Now } M(\sigma_1,\sigma_2) \leqslant \left[\sum_{m=1}^{m_0}\sum_{n=1}^{n_0} + \sum_{m=m_0+1}^{\infty}\sum_{n=1}^{n_0} + \sum_{m=1}^{m_0}\sum_{n=n_0+1}^{\infty} + \sum_{m=m_0+1}^{\infty}\sum_{n=n_0+1}^{\infty}\right] |a_{mn}| \exp(\sigma_1\lambda_m+\sigma_2\mu_n)$$

$$= \sum_1 + \sum_2 + \sum_3 + \sum_4 \text{ (say)}$$

clearly

$$\sum_1 = O[\exp(\sigma_1\lambda_{m_0}+\sigma_2\mu_{n_0})]$$

$$\sum_4 \leqslant \sum_{m=m_0+1}^{\infty}\sum_{n=n_0+1}^{\infty} \lambda_m^{-\frac{\lambda_m}{\rho_1(1+\epsilon)}},\ \mu_n^{-\frac{\mu_n}{\rho_2(1+\epsilon)}} \exp(\sigma_1\lambda_m+\sigma_2\mu_n)$$

$$= \sum_{m>m_0}\sum_{n>n_0} \exp\left[\sigma_1\lambda_m+\sigma_2\mu_n-(1+\epsilon)^{-1}\left(\frac{\lambda_m}{\rho_1}\log\lambda_m+\frac{\mu_n}{\rho_2}\log\mu_n\right)\right]$$

$$= \sum_{m>m_0}\sum_{n>n_0} \exp\left[\sigma_1\lambda_m+\sigma_2\mu_n-\frac{\frac{\lambda_m}{\rho_1}\log\lambda_m+\frac{\mu_n}{\rho_2}\log\mu_n}{1+2\epsilon}\left(1+\frac{\epsilon}{1+\epsilon}\right)\right]$$

$$= \sum_{m>m_0}\sum_{n>n_0} \exp\left[\sigma_1\lambda_m+\sigma_2\mu_n-\frac{\frac{\lambda_m}{\rho_1}\log\lambda_m+\frac{\mu_n}{\rho_2}\log\mu_n}{1+2\epsilon} - \frac{\frac{\lambda_m}{\rho_1}\log\lambda_m+\frac{\mu_n}{\rho_2}\log\mu_n}{\epsilon^{-1}(1+\epsilon)(1+2\epsilon)}\right]$$

$$(2.1) \qquad \leqslant \max_{(\lambda_m, \mu_n)} \exp\left[\sigma_1\lambda_m+\sigma_2\mu_n-(1+2\epsilon)^{-1}\left(\frac{\lambda_m}{\rho_1}\log\lambda_m+\frac{\mu_n}{\rho_2}\log\mu_n\right)\right]$$

$$\mathrm{X} \sum_{m>m_0}\sum_{n>n_0} \exp\left\{\frac{-\frac{\lambda_m}{\rho_1}\log\lambda_m-\frac{\mu_n}{\rho_2}\log\mu_n}{\epsilon^{-1}(1+\epsilon)(1+2\epsilon)}\right\}$$

Since the maximum of the expression

$\exp\left[\sigma_1\lambda_m+\sigma_2\mu_n-(1+2\epsilon)^{-1}\left(\frac{\lambda_m}{\rho_1}\log\lambda_m+\frac{\mu_n}{\rho_2}\log\mu_n\right)\right]$ is attained at

$\lambda_m=e^{-1}\exp\{\sigma_1\rho_1(1+2\epsilon)\}$ and $\mu_n=e^{-1}\exp\{\sigma_2\rho_2(1+2\epsilon)\}$ and the maximum value of the expression is

$$\exp\left[\frac{1}{e\rho_1(1+2\epsilon)}\exp\left\{\sigma_1\rho_1(1+2\epsilon)\right\}+\frac{1}{e\rho_2(1+2\epsilon)}\exp\left\{\sigma_2\rho_2(1+2\epsilon)\right\}\right]$$

$$\leqslant \exp\left[\frac{1}{e\rho(1+2\epsilon)}\left\{\exp\sigma_1\rho_1(1+2\epsilon)+\exp\sigma_2\rho_2(1+2\epsilon)\right\}\right]$$

where $\rho=\min(\rho_1, \rho_2)$

Since the series on the right of (2.1) is convergent,

$$\therefore \quad \sum_{1} < A\exp\left[\frac{1}{e\rho(1+2\epsilon)}\left\{\exp\left(\sigma_1\rho_1(1+2\epsilon)\right)+\exp\sigma_2\rho_2(1+2\epsilon)\right\}\right]$$

where A is an absolute constant.

Further to estimate Σ_2 it is noted that for all values of m and n, $\exists$ some tive const. k *s. t*

$$\frac{\log \lambda_m^{\frac{\lambda_m}{\rho_1}} \cdot \mu_n^{\frac{\mu_n}{\rho_2}}}{-\log|a_{mn}|} \leqslant k.$$

Therefore,

$$\Sigma_2 \leqslant \sum_{m>m_0}\sum_{n=1}^{n_0} \exp\left[(\sigma_1\lambda_m+\sigma_2\mu_n)-\frac{k^{-1}\lambda_m\log\lambda_m}{\rho_1}-\frac{k^{-1}\mu_n\log\mu_n}{\rho_2}\right]$$

$$=O\left[\exp(\sigma_2\mu_{n_0})\right]\sum_{m>m_0}\left[\exp\left(\sigma_1\lambda_m-\frac{k^{-1}\lambda_m\log\lambda_m}{\rho_1}\right)\right]$$

$$\leqslant O\left[\exp(\sigma_2\mu_{no})\right].\ \max_{\lambda_m}\left[\exp\left(\sigma_1\lambda_m-\frac{(k+\epsilon)^{-1}.\lambda_m\log\lambda_m}{\rho_1}\right)\right]$$

$$\text{X}\sum_{m>m_0}\exp\left[-\epsilon k^{-1}(k+\epsilon)^{-1}.\frac{\lambda_m\log\lambda_m}{\rho_1}\right]$$

But the maximum of the expression $\exp\left[\sigma_1\lambda_m-\dfrac{(k+\epsilon)^{-1}\lambda_m\log\lambda_m}{\rho_1}\right]$ is attained at $\lambda_m=e^{-1}\exp[\sigma_1\rho_1(k+\epsilon)]$

Hence $\sum_2 < O\left[\exp(\sigma_2\mu_{n_0})\right]\exp\left[\dfrac{e^{-1}(k+\epsilon)^{-1}}{\rho_1}\exp\left\{\sigma_1\rho_1(k+\epsilon)\right\}\right]$

since $\sum\limits_{m>m_0}\exp\left[-\epsilon k^{-1}(k+\epsilon)^{-1}\dfrac{\lambda_m\log\lambda_m}{\rho_1}\right]$ is convergent.

Similarly $\sum_3 \leqslant O\left[\exp(\sigma_1\lambda_{mo})\right]\exp\left[\dfrac{e^{-1}(k+\epsilon)^{-1}}{\rho_2}\exp\left\{\sigma_2\rho_2(k+\epsilon)\right\}\right]$

Substituting these values of $\sum_i (1 \leqslant i \leqslant 4)$ in $M(\sigma_1, \sigma_2)$

$$M(\sigma_1,\sigma_2) < \sum_1+\sum_2+\sum_3+\sum_4$$

$$< O\left[\exp(\sigma_1\lambda_{mo}+\sigma_2\mu_{no})\right]+O\left[\exp(\sigma_2\mu_{no})\right]\exp\left[\frac{e^{-1}(k+\epsilon)^{-1}}{\rho_1}\exp\sigma_1\rho_1(k+\epsilon)\right]$$

$$+O\left[\exp(\sigma_1\lambda_{mo})\right]\exp\left[\frac{e^{-1}(k+\epsilon)^{-1}}{\rho_2}\exp\left\{\sigma_2\rho_2(k+\epsilon)\right\}\right]$$

$$+A\exp\left[\frac{1}{e\rho'(1+2\epsilon)}\left\{\exp\left(\sigma_1\rho_1(1+2\epsilon)\right)+\exp\left(\sigma_2\rho_2(1+2\epsilon)\right)\right\}\right]$$

$$=A\exp\left\{\frac{1}{e\rho'(1+2\epsilon)}\left[\exp\sigma_1\rho_1(1+2\epsilon)+\exp\sigma_2\rho_2(1+2\epsilon)\right]\right\}[1+O(1)]$$

$$\therefore\ \log M(\sigma_1,\sigma_2) \leqslant \log A$$

$$+\frac{1}{e\rho'(1+2\epsilon)}\left[\exp\sigma_1\rho_1(1+2\epsilon)+\exp\sigma_2\rho_2(1+2\epsilon)\right] \quad \text{for } \sigma > \sigma^0$$

$$=\frac{1}{e\rho'(1+2\epsilon)}\left[\exp\sigma_1\rho_1(1+2\epsilon)+\exp\sigma_2\rho_2(1+2\epsilon)\right] \quad \text{for } \sigma > \sigma^0.$$

$$\therefore \quad \log\log M(\sigma_1, \sigma_2) \leqslant \log \frac{1}{e\rho'(1+2\epsilon)} + \log\left[e^{\sigma_1\rho_1(1+2\epsilon)} + e^{\sigma_2\rho_2(1+2\epsilon)}\right]$$

for $\sigma > \sigma^0$.

$$< \log \frac{1}{e\rho'(1+2\epsilon)} + \log 2 \, . \, e^{\sigma_1\rho_1(1+2\epsilon)} \text{ if } \rho_1 > \rho_2.$$

$$\therefore \quad \limsup_{(\sigma_1, \sigma_2)\to\infty} \frac{\log\log M(\sigma_1, \sigma_2)}{\sigma_1\rho_1 + \sigma_2\rho_2} \leqslant 1. \qquad \ldots \quad \text{(B)}.$$

Combining (A) and (B) we get the result. (See [2])

Theorem 2.2 : Let $\rho >> \vec{o}$ be an order point of f and let $\tau(>o)$ be the corresponding type of the function, then

$$\text{(i)} \quad \tau = \overline{\lim_{(\sigma_1, \sigma_1)\to\infty}} \; \overline{\lim_{(m, n)\to\infty}} \; \frac{\frac{\chi}{\rho}\left\{ | a_{mn} | \, e^{\sigma_1\lambda_m+\sigma_2\mu_n}\right\}^{\frac{1}{\chi\,|\,\rho}}}{e \, . \, e^{\sigma_1\rho_1+\sigma_2\rho_2}}$$

$$\text{(ii)} \quad \tau \leqslant \limsup_{(m, n)\to\infty} \left[\frac{\frac{\chi}{\rho} \, | \, a_{mn} \, | ^{\frac{1}{\chi\,|\,\rho}}}{e}\right]$$

where $\quad \dfrac{\chi(m, n)}{\rho} = \dfrac{\chi}{\rho} = \max . \left(\dfrac{\lambda_m}{\rho_1}, \; \dfrac{\mu_n}{\rho_2}\right).$

Proof : Let $T > \tau$ (>0). Then there exists a $\sigma\epsilon R^2_+$ such that

$$\log M(\sigma) \leqslant T \exp(\sigma_1\rho_1+\sigma_2\rho_2) \text{ for } \sigma \geqslant \sigma^\circ \qquad \ldots \quad \text{(A)}$$

By Cauchy's inequality

$$| \, a_{mn} \, | \leqslant \frac{M(\sigma)}{\exp(\sigma_1\lambda_m+\sigma_2\mu_n)} \text{ for all } \sigma \text{ and } (m, n) \, \epsilon N^2.$$

$$\therefore \quad | \, a_{mn} \, | \leqslant \inf_{\sigma\geqslant\sigma} . \; \frac{\exp(T \, e^{\sigma_1\rho_1+\sigma_2\rho_2})}{\exp(\sigma_1\lambda_m+\sigma_2\mu_n)} \text{ for } (m, n) \, \epsilon N^2$$

which by ([4] Th. 3.4) is equivalent to the fact that

$$| \, a_{mn} \, | \leqslant \frac{\left[\dfrac{e. \, T. \, e^{\sigma^0_1\rho_1 + \sigma^0_2\rho_2}}{\chi/\rho}\right]^{\chi/\rho}}{e^{\sigma^0_1\lambda_m + \sigma^0_2\mu_n}}$$

for $(m, n) \in N^2 - J$

$$\leqslant \frac{\left[\dfrac{e.\,T.\, e^{\sigma_1\rho_1 + \sigma_2\rho_2}}{\chi/\rho}\right]^{\chi/\rho}}{e^{\sigma_1\lambda_m + \sigma_2\mu_n}}$$

for $(\sigma_1, \sigma_2) \geqslant (\sigma_1^0, \rho_2^0)$ and $(m, n) \in N^2 - J$.

$$\therefore \quad \frac{\chi}{\rho}\left[\,|\,a_{mn}\,|\, e^{\sigma_1\lambda_m + \sigma_2\mu_n}\right]^{\frac{1}{\chi/\rho}} \leqslant e\, T\, e^{\sigma_1\rho_1 + \sigma_2\rho_2}$$

for $(\sigma_1, \sigma_2) \geqslant (\sigma_1^0, \sigma_2^0)$ and $(m, n) \in N^2 - J$.

$$\therefore \quad \frac{\frac{\chi}{\rho}\left[\,|\,a_{mn}\,|\, e^{\sigma_1\lambda_m + \sigma_2\mu_n}\right]^{\frac{1}{\chi/\rho}}}{e.\, e^{\sigma_1\rho_1 + \sigma_2\rho_2}} \leqslant T$$

for $\sigma \geqslant \sigma^0$ and $(m, n) \in N^2 - J$.

Since τ is the infimum of the values of T for which (A) is satisfied

$$\therefore \quad \tau = \limsup_{(\sigma_1, \sigma_2)\to\infty}\ \limsup_{(m, n)\to\infty} \frac{\frac{\chi}{\rho}\left[\,|\,a_{mn}\,|\, e^{\sigma_1\lambda_m + \sigma_2\mu_n}\right]^{\frac{1}{\chi/\rho}}}{e.\, e^{\sigma_1\rho_1 + \sigma_2\rho_2}}$$

so (i) is proved.

Again, since $\dfrac{\left(e^{\sigma_1\lambda_m + \sigma_2\mu_n}\right)^{\frac{1}{\chi/\rho}}}{e^{\sigma_1\rho_1 + \sigma_2\rho_2}} = \dfrac{e^{\sigma_1\rho_1}\left(e^{\sigma_2\mu_n}\right)^{\frac{1}{\lambda_m/\rho_1}}}{e^{\sigma_1\rho_1 + \sigma_2\rho_2}}$ if $\dfrac{\chi}{\rho} = \dfrac{\lambda_m}{\rho_1}$

$$= e^{\sigma_2\rho_1\frac{\mu_n}{\lambda_m} - \sigma_2\rho_2} = e^{\sigma_2\left(\rho_1\frac{\mu_n}{\lambda_m} - \rho_2\right)}$$

$$= e^{\frac{\sigma_2\rho_1\rho_2}{\lambda_m}\left(\frac{\mu_n}{\rho_2} - \frac{\lambda_m}{\rho_1}\right)}$$

Since $\dfrac{\mu_n}{\rho_2} - \dfrac{\lambda_m}{\rho_1} \leqslant 0$

$$\therefore \quad e^{\frac{\sigma_2\rho_1\rho_2}{\lambda_m}\left(\frac{\mu_n}{\rho_2} - \frac{\lambda_m}{\rho_1}\right)} \leqslant 1$$

Similarly $\dfrac{(e^{\sigma_1\lambda_m+\sigma_2\mu_n})^{\frac{1}{\chi/\rho}}}{e^{\sigma_1\rho_1+\sigma_2\rho_2}} \leqslant 1$ if $\dfrac{\chi}{\rho}=\dfrac{\mu_n}{\rho_2}$

$$\therefore\ \tau < \limsup_{(m,n)\to\infty} \frac{\frac{\chi}{\rho}\,|\,a_{mn}\,|^{\frac{1}{\chi/\rho}}}{e}.$$

Construction of entire functions with given integral order point and with a given type.

Theorem 2.3 : Let $(\alpha_1, \alpha_2)\in N^2$ and $a\in R_+$ be given. Then there exists an entire function f represented by a double Dirichlet series such that (α_1, α_2) is an order point and 'a' is the corresponding type of f.

Proof : Let us consider a double Dirichlet Series

$$f(s_1, s_2)=\sum_{t=1}^{\infty}\left(\frac{ea}{t}\right)^t \exp\,(ms_1+ns_2),\ m=t\alpha_1,\ n=t\alpha_2$$

Since $a_{mn}=\left(\dfrac{ea}{t}\right)^t$ when $m=t\alpha_1,\ n=t\alpha_2$

$=0$ otherwise,

f satisfies the condition (1.3) and hence it is entire.

Now (ρ_1, ρ_2) is an order point of f iff [2]

$$\limsup_{(m,n)\to\infty} \frac{\frac{\chi}{\rho}\log\frac{\chi}{\rho}}{-\log|\,a_{mn}\,|}=1$$

i.e., iff $\displaystyle\limsup_{t\to\infty} \frac{\max\left(\frac{m}{\rho_1}, \frac{n}{\rho_2}\right)\log\max\left(\frac{m}{\rho_1}, \frac{n}{\rho_2}\right)}{t\log t-t\log(ea)}=1$

i.e., iff $\displaystyle\limsup_{t\to\infty} \frac{t\max\left(\frac{\alpha_1}{\rho_1}, \frac{\alpha_2}{\rho_2}\right)\log\left\{t\max\left(\frac{\alpha_1}{\rho_1}, \frac{\alpha_2}{\rho_2}\right)\right\}}{t[\log t-\log(ea)]}=1$

i.e., iff $\displaystyle\limsup_{t\to\infty} \frac{\max\left(\frac{\alpha_1}{\rho_1}, \frac{\alpha_2}{\rho_2}\right)\left[\log t+\log\max\left(\frac{\alpha_1}{\rho_1}, \frac{\alpha_2}{\rho_2}\right)\right]}{\log t\left[1-\frac{\log(ea)}{\log t}\right]}=1$

i.e., iff $\max\left(\dfrac{\alpha_1}{\rho_1}, \dfrac{\alpha_2}{\rho_2}\right)=1.$

For given $(\alpha_1, \alpha_2) \epsilon N^2$ there exists an entire double Dirichlet series whose order pts. (ρ_1, ρ_2) satisfy the condition : $\max\left(\frac{\alpha_1}{\rho_1}, \frac{\alpha_2}{\rho_2}\right)=1$. Evidently (α_1, α_2) is an order point of f.

Next let us calculate the type of $\bar{f}$ corresponding to the order point (α_1, α_2). We know that corresponding to the order point (ρ_1, ρ_2) the type τ is given by

$$\tau = \limsup_{(\sigma_1, \sigma_2)\to\infty} \limsup_{(m, n)\to\infty} \frac{\frac{\chi}{\rho}\left[\,|\,a_{mn}\,|\,e^{\sigma_1\lambda_m+\sigma_2\mu_n}\right]^{\frac{1}{\chi/\rho}}}{e.\,e^{\sigma_1\rho_1+\sigma_2\rho_2}}$$

Now $\frac{\chi}{\rho}=\max\left(\frac{\lambda_m}{\rho_1}, \frac{\mu_n}{\rho_2}\right)=\max\left(\frac{t\alpha_1}{\alpha^1}, \frac{t\alpha_2}{\alpha_2}\right)=t.$

$$\therefore\quad \tau = \limsup_{(\sigma_1, \sigma_2)\to\infty} \limsup_{t\to\infty} \frac{t\left[\left(\frac{ea}{t}\right)^t \exp\left(\sigma_1 t\alpha_1+\sigma_2 t\alpha_2\right)\right]^{\frac{1}{t}}}{e.\exp\left(\sigma_1\alpha_1+\sigma_2\alpha_2\right)}$$

$$= \limsup_{(\sigma_1, \sigma_2)\to\infty} \limsup_{t\to\infty} \frac{t\left[\frac{ea}{t}.\exp\left(\sigma_1\alpha_1+\sigma_2\alpha_2\right)\right]}{e.\exp\left(\sigma_1\alpha_1+\sigma_2\alpha_2\right)}=a.$$

so the theorem is proved.

Definition : The set $\left[(\alpha_1, \alpha_2) : (\alpha_1, \alpha_2) \epsilon R_+^2, \left(\frac{1}{\alpha_1}, \frac{1}{\alpha_2}\right) \epsilon \rho_f\right.$ is called the reciprocal order set of $f \epsilon F$ and is denoted by ρ_f^{-1}

Theorem 2.4 : The reciprocal order set ρ_f^{-1} of some $f \epsilon$ F is convex.

Proof : Since log M (σ) is a convev function of σ [2] so for any $t \epsilon R_+^2$ and $s \epsilon R_+^2$ and any $\lambda \epsilon [0, 1]$

$$\log M(\lambda t_1+\mu s_1, \lambda t_2+\mu s_2) \leqslant \lambda \log M(t_1, t_2)+\mu \log M(s_1, s_2)$$
where $\mu=1-\lambda$.

Now let us take two arbitrary points a and $b \epsilon \rho_f$.

Now let us set,

$$t_i = \frac{\sigma_i a_i}{\lambda a_i + \mu b_i} \quad \text{and} \quad s_i = \frac{\sigma_i b_i}{\lambda a_i + \mu b_i},\ i = 1, 2$$

Then $\lambda t_1 + \mu s_1 = \dfrac{\lambda\sigma_1 a_1}{\lambda a_1 + \mu b_1} + \dfrac{\mu\sigma_1 b_1}{\lambda a_1 + \mu b_1} = \sigma_1$

Similarly, $\lambda t_2 + \mu s_2 = \sigma_2$

Since $a, b \in \rho_f$, $\log M(t_1, t_2) \leqslant \exp(t_1 b_1 + t_2 b_2)$ for $t > t_0$

and $\log M(s_1, s_2) \leqslant \exp(s_1 a_1 + s_2 a_2)$ for $s > s_0$

$$\therefore \log M(\sigma_1, \sigma_2) \leqslant \lambda \log M(t_1, t_2) + \mu \log M(s_1, s_2)$$

$$\leqslant \lambda \exp(t_1 b_1 + t_2 b_2) + \mu \exp(s_1 a_1 + s_2 a_2)$$

$$\text{for } t > t_0, \quad s > s_0$$

$$= \lambda \exp\left(\frac{\sigma_1 a_1 b_1}{\lambda a_1 + \mu b_1} + \frac{\sigma_2 a_2 b_2}{\lambda a_2 + \mu b_2}\right) + \mu \exp\left(\frac{\sigma_1 a_1 b_1}{\lambda a_1 + \mu b_1} + \frac{\sigma_2 a_2 b_2}{\lambda a_2 + \mu b_2}\right)$$

$$\text{for } \sigma > \sigma^0$$

$$= \exp\left(\frac{\sigma_1}{\dfrac{\lambda}{b_1} + \dfrac{\mu}{a_1}} + \frac{\sigma_2}{\dfrac{\lambda}{b_2} + \dfrac{\mu}{a_2}}\right) \text{ for } \sigma > \sigma^0$$

Consequently for any λ and μ, $\lambda \geqslant 0$, $\mu \geqslant 0$, $\lambda + \mu = 1$ the point $\left(\dfrac{\lambda}{b_1} + \dfrac{\mu}{a_1}, \dfrac{\lambda}{b_2} + \dfrac{\mu}{a_2}\right)$ i.e. any point of the segment joining the pts. (b_1^{-1}, b_2^{-1}) and (a_1^{-1}, a_2^{-1}) of ρ_f^{-1} is also in ρ_f^{-1}. Thus ρ_f^{-1} is convex.

The above theorem is equivalent to the following :

The set $\rho'_f \subset \rho_f$ where $\rho'_f = \{\alpha ; \alpha \in R^2_+, \alpha \in \rho_f)$ is reciprocally convex.

Theorem 2.5 : The set ρ'_f of some $f \in F$ is convex,

Let $a = (a_1, a_2)$, $b = (b_1, b_2) \in \rho'_f$ and let $C = pa + qb$ where $p \geqslant 0$, $q \geqslant 0$, $p + q = 1$. Since the set ρ'_f is reciprocally convex it follows that $d = (d_1, d_2) \in \rho'_f$ where $\dfrac{1}{d_j} = \dfrac{p}{a_j} + \dfrac{q}{b_j}$, $j = 1, 2$.

Now for $j = 1, 2$

$$(c_j - d_j)(pb_j + qa_j) = (p^2 + q^2)(a_j b_j) + pq(a_j^2 + b_j^2) - a_j b_j$$

$$= (p + q)^2 a_j b_j + pq(a_j - b_j)^2 - a_j b_j$$

$$\geqslant (p + q)^2 a_j b_j - a_j b_j = 0$$

which shows that $c \geqslant d$ and hence $c \in \rho'_f$.

This implies that the set ρ'_f is convex.

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REFERENCES

[1] Bromwich T. J. I'd—Introduction to the theory of Infinite series. Macmillan and Co. Ltd., p. 98 Prob-4.

[2] Das, R. K.—On order and type of an entire function represented by ouble Dirichlet series. Communicated for publication.

[3] Dunford Nelson and Schwartz Jacob T.—Linear operators (Part I) Interscience Publishers, Inc. New York, p. 520.

[4] Janusanskas A. I.—Elementary theorems on the convergence of Double Dirichlet series. *Dokl. Akad. Nauk. SSSR.* Tom 234 (1977) No. I, 610-614.

[5] Sarkar P. K.—On some properties of entire functions of several complex variables represented by multiple Direchlet Series. Ph. D. thesis, Cal. Univ, (1982).

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Dept. of Pure Math.
Calcutta University

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ON BIGROUPOIDS

M. K. SEN

The purpose of this note is to characterise a certain classes of monoids in terms of the corresponding classes of bigroupoids.

A monoid is a semigroup with identity. An involution of a monoid is a unary operation $t : S \to S$ satisfying the following axioms

(1) $(x^t)^t = x$ (2) $(xy)^t = y^t x^t$

A monoid is called a monoid with involution, if we assign an involution to it.

An algebraic system (B, $*$, o) of type (2, 2) which is taken into consideration here, always satisfies the following axiom B. Let us call this system bigroupoid.

(B) $x \circ (y * z) = (z \circ x) * y$.

1. Bigroupoid with identity.

Definition. An element u of a bigroupoid S is called a left (right) identity of S, if it satisfies

$(u_L) : u * a = a \; (u_R : a \circ u = a)$

Lemma 1. Let u be a left identity and v be a right identity of a bigroupoid S. Then (i) $v * u = v$ and (ii) $v \circ u = u$.

Proof: (i) $v * u = (v \circ v) * u$
$= v \circ (u * v)$ (by B)
$= v \circ v$
$= v.$

(ii) $v \circ u = v \circ (u * u)$
$= (u \circ v) * u$
$= u * u$
$= u.$

Lemma 2. Let u be a left identity and v be a right identity then $a * u = v \circ a$ for all $a \in S$.

Proof: $a * u = (a \circ v) * u$
$= v \circ (u * a)$
$= v \circ a.$

Proposition 1. If S contains a left identity u then there exists atmost one right identity.

Proof : Let v and v_1 be two right identies. Then

$$\begin{aligned} v_1 &= u * v_1 && (\text{by } u_L) \\ &= (u \circ v) * v_1 && (\text{by } u_R) \\ &= v \circ (v_1 * u) && (\text{by B}) \\ &= v \circ v_1 && (\text{by Lemma 1}) \\ &= v. && (\text{by } u_R) \end{aligned}$$

Proposition 2. If S contains a right identity v then there exists atmost one left identity in S.

Proof : Let u and u_1 be two left identities of S.

$$\begin{aligned} u_1 &= u_1 \circ v && (\text{by } u_R) \\ &= u_1 \circ (u * v) && (\text{by } u_L) \\ &= (v \circ u_1) * u && (\text{by B}) \\ &= u_1 * u && (\text{by Lemma 1}) \\ &= u. && (\text{by } u_L) \end{aligned}$$

Definition. A pair (u, v) is called an identity element of S if u is a left identity and v is a right identity.

Proposition 3. S contains atmost one identity (u, v).

Proof : This follows from Proposition 1 and Proposition 2.

Definition. $J_u(x) = x * u$ and $J_v(x) = v \circ x$.

Lemma 3. $J_u = J_v$,

Proof : Let $x \in S$. From Lemma 2,
$J_u(x) = x * u = v \circ x = J_v(x)$.
Hence $J_u = J_v$.

Lemma 4. (a) $J_u(x * y) = y * x$ (b) $J_u(x \circ y) = y \circ x$.

Proof :

$$\begin{aligned} J_u(x * y) = J_v(x * y) &= v \circ (x * y) \\ &= (y \circ v) * x && (\text{by B}) \\ &= y * x. && (\text{by } u_R) \\ J_u(x \circ y) &= (x \circ y) * u \\ &= y \circ (u * x) && (\text{by B}) \\ &= y \circ x. && (\text{by } u_L) \end{aligned}$$

Lemma 5. $J_u(J_u(x))=x.$

Proof: $J_u(J_u(x))=J_u(J_v(x))$

$$=(v \circ x) * u$$

$$=x \circ (u * v) \qquad \text{(by B)}$$

$$=x \circ v \qquad \text{(by } u_L\text{)}$$

$$=x.$$

Lemma 6. Let S be a bigroupoid with the identity (u, v). Then $(a * b) \circ c=a * (b \circ c)$

Proof: Let a, b, c be three elements of S. Then

$$(a * b) \circ c=J_u(c \circ (a * b)) \qquad \text{(by Lmma 4 } (b)\text{)}$$

$$=J_u((b \circ c) * a) \qquad \text{(by B)}$$

$$=a * (b \circ c) \qquad \text{(by Lemma 4}(a)\text{)}.$$

Theorem 1. Let S be a bigroupoid with the identity (u, v). If we define $a.b=J_u(a) * b$, then (S, J_u) is a monoid with involution, where u is the identity of this monoid.

Proof: Let a, b, c be three elements of S. Then

$$a \,.\, (b \,.\, c)=a \,.\, (J_u(b) * c)$$

$$=J_u(a) * (J_u(b) * c)$$

$$=J_u(a) * (J_v(b) * c) \qquad \text{(by Lemma 3)}$$

$$=J_u(a) * ((v \circ b) * c)$$

$$=J_u(a) * (b \circ (c * v)) \qquad \text{(by axiom B)}$$

$$=(J_u(a) * b) \circ (c * v) \qquad \text{(by Lemma 6)}$$

$$=(v \circ (J_u(a) * b)) * c \qquad \text{(by axiom B)}$$

$$=((b \circ v) * J_u(a)) * c$$

$$=(b * J_u(a)) * c$$

$$=J_u(J_u(a) * b) * c \qquad \text{(by Lemma 4)}$$

$$=(a \,.\, b) \,.\, c.$$

Hence associative property holds in (S, .).

Now for any $a \in S$, we have

$u \,.\, a = J_u(u) * a = (u * u) * a = u * a = a$, and

$a \,.\, u = J_u(a) * u = (a * u) * u = a$ (by Lemma 5)

Hence u is the identity element of the semigroup (S, .). Let us now show that J_u is an involution in (S, .). Let $x \in S$. Then $J_u(J_u(x)) = x$ (by Lemma 5)

and $J_u(a \,.\, b) = J_u(J_u(a) * b) = b * J_u(a) = J_u(J_u(b)) * J_u(a) = J_u(b) \,.\, J_u(a)$. Hence $J_u : S \to S$ is an involution in $(S, .)$. Hence the theorem.

We can also prove the following theorem :

Theorem 2. Let (u, v) be the identity of the bigroupoid S. If we define $a \,.\, b = a \circ J_v(b)$ then $(S, ., J_v)$ is a monoid with involution where v is the identity of this monoid.

Let us denote by S_u the monoid obtained in Theorem 1 and S_v the monoid obtained in Theorem 2.

Theorem 3. S_u and S_v are isomorphic to each other.

Proof : Let us define a mapping f from S_v to S_u by $f(u) = a \circ u$ for $a \in S_v$. Then,

$$\begin{aligned}
f(a \,.\, b) &= (a \,.\, b) \circ u \\
&= (a \circ J_v(b)) \circ u && \text{(by the definition of } a \,.\, b \text{ in } S_v) \\
&= (a \circ J_u(b)) \circ u && \text{(by Lemma 3)} \\
&= (a \circ (b * u)) \circ u \\
&= ((u \circ a) * b) \circ u && \text{(by axiom B)} \\
&= (u \circ a) * (b \circ u) && \text{(by Lemma 6)} \\
&= J_u(a \circ u) * (b \circ u) && \text{(by Lemma 4(b))} \\
&= (a \circ u) \,.\, (b \circ u) && \text{(by the definition of } a \,.\, b \text{ in } S_u) \\
&= f(a) \,.\, f(b).
\end{aligned}$$

Let $a, b \in S_v$ such that $f(a) = f(b)$. Then $a \circ u = b \circ u$. From this $J_u(a \circ u) = J_u(b \circ u)$. Then by Lemma 4(b), $u \circ a = u \circ b$. $(u \circ a) * v = (u \circ b) * v$.

This implies $a \circ (v * u) = b \circ (v * u)$. Then by Lemma 1, $a \circ v = b \circ v$. Hence $a = b$ if $f(a) = f(b)$. Let $a \in S$. Then $(v \circ a) * v \in S$.

Now $f((v \circ a) * v) = ((v \circ a) * v) \circ u = (v \circ a) * (v \circ u) = (v \circ a) * u$ (by Lemma 1) $= a \circ (u * v) = a \circ v = a$. Thus it follows that f is an isomorphism of S_v onto S_u.

Note. Let S be a monoid with an involution. If we define $a * b = a^t b$ and $a \circ b = ab^t$ where a^t denotes the involution of a, then we can show that S is a bigroupoid with identity.

2. Bigroupoid and Group.

In [(2), P 73] we have considered a bigroupoid S that satisfies the following axiom :

$$G : a \text{ o } (b * a) = b \text{ for all } a, b \in S$$

Some results abount this system are listed below :

Lemma 7. [(2) P 72] $a * a = b * b$ for $a, b \in S$

Let $u = a * a = b * b = c * c = \cdots$ for $a, b, c \in S$.

Lemma 8. [(2), P 72] $u * d = d$ for all $d \in S$.

Lemma 9. [(2), P 73] $d \text{ o } u = d$ for all $d \in S$.

Theorem 4. [(2), P 73]. Let S be a bigroupoid satisfying the axiom G. If we define $a \, . \, b = J_u(a) * b$ for all $a, b \in G$, then G is a group.

3. Regular bigroupoid.

Definition. An element a of a bigroupoid S is said to be regular if $a = (a \text{ o } a) * a$. If every element of S is regular then S is said to be a regular bigroupoid.

Definition. An element $a \in S$ is said to be left (right) idempotent if $a * a = a$ $(a \text{ o } a = a)$.

Leema 10. In a regular bigroupoid every left idempotent is a right idempotent and conversely.

Proof: Let a be a left idempotent then $a * a = a$. Now $a \text{ o } a = a \text{ o } (a * a) = a$. Conversely, assume that a is right idempotent.

Then $a \text{ o } a = a$. Hence $a * a = (a \text{ o } a) * a = a \text{ o } (a * a) = a$.

Lemma 11. In a regular bigroupoid $a \text{ o } a$ and $a * a$ are idempotents.

Proof: We have $a \text{ o } (a * a) = a$. Then $(a \text{ o } a) * a = a$. From this, $a \text{ o } ((a \text{ o } a) * a) = a \text{ o } a$. This implies $(a \text{ o } a) * (a \text{ o } a) = a \text{ o } a$. Hence $a \text{ o } a$ is a left idempotent. From Lemma 10, it follows that $a \text{ o } a$ is idempotent.

Again $a \text{ o } (a * a) = a$ we find that $(a \text{ o } (a * a)) * a = a * a$.

This implies $(a * a) \text{ o } (a * a) = a * a$. Hence $a * a$ is a right idemptent. Then from Lemma 10, it follows that $a * a$ is an idempotent.

Theorem 5. A bigroupoid is a group if and only if it is regular and contains only one idempotent.

Proof. Suppose that the bigroupoid S is regular and contains only one idempotent. Then from leema 11, it follows that $a \circ a = a * a = b \circ b = b * b$ for any $a, b \in S$. Hence $a \circ (b * a) = (a \circ a) * b = (b \circ b) * b = b \circ (b * b) = b$. This shows that S satisfies the axiom G. Then from Theorem 4 we find that S is a group. Conversely, suppose that S is a group. Then $a \circ (b * a) = b$ for any $b \in S$. Hence $a \circ (a * a) = a$. This implies that S is regular. Then from Lemma 8 and Lemma 9 we have $a * a$ is the identity of S. Let e be an idempotent in S. Hence $e = e \circ e = e * e$. This shows that e is the identity of S. Hence S contains only one idempotent.

A semigroup S with an involution t is called [(1), P 370] *a t*- regular semigroup if it satisfies the axiom

$$x = x x^t x$$

If we define $a * b = a^t b$ and $a \circ b = ab^t$, then $(S, * \circ)$ is a regular bigroupoid.

Definition. A bigroupoid S is said to be commutative if $a * b = b \circ a$, for any $a, b \in B$.

Theorem 6. A commutative regular bigroupoid is a disjoint union of groups.

Proof : Let E be the set of all distinct idempotents of a commutative regular bigroupoid S. Suppose $e \in E$. Let $G_e = \{a \in S : e * a = a$ and there exists $a' \in S$ with the properties (i) $a' * a = e$ (ii) $e * a' = a\}$.

Since S is commutative, we have $e * a = a \circ e = a$ for all $a \in G_e$.

Suppose $a, b \in G_e$. Then $e * (a * b) = (e \circ e) * (a * b) = e \circ ((a * b) * e)$
$= e \circ ((b \circ a) * a) = e \circ (a \circ (e * b)) = e \circ (a \circ b) = e \circ (b * a) = (a \circ e) * b$
$= (e * a) * b = a * b$. Since $a, b \in G_e$, there exist $a', b' \in S$, such that $a' * a = e$ and $b' * b = e$.

Now $(a' * b') * (a * b) = (b' \circ a') * (a * b) = a' \circ ((a * b) * b')$
$= a' \circ ((b \circ a) * b') = a' \circ (a \circ (b' * b)) = a' \circ (a \circ e) = a' \circ (e * a)$
$= a' \circ a = a * a' = (a \circ e) * a' = e \circ (a' * a) = e \circ e = e$.

Also $e * (a' * b') = (e \circ e) * (a' * b') = e \circ ((a' * b') * e)$
$= e \circ (e \circ (a' * b')) = e \circ ((b' \circ e) * a' = e \circ (b' * a') = (a' \circ e) * b'$
$= a' * b'$. Hence $a * b \in G_e$ for all $a, b \in G_e$. Since $a \circ b = b * a$, it follows that $a \circ b \in G_e$ for all $a, b \in G_e$. Hence G_e is a bigroupoid. Let f be an idempotent of S such that $f \in G_e$. Then $e * f = f$ and $g * f = e$ for some $g \in S$. Also $e * g = g$. Now $f = e * f = f \circ e = f \circ (g * f) = (f \circ f) * g = f * g = (f \circ e) * g$

$=e \circ (g * f)=e \circ e=e$. Hence G_e contains only one idempotent. Also it is true that G_e is a regular bigroupoid. Then from Theorem 5, it follows that G_e is a group. Let $a \in S$. Then $(a \circ a) * a=a$ implies that $e=a \circ a=a \circ ((a \circ a) * a)$ $=(a \circ a)$ is an idempotent and $e * a=a$. Also $e=a \circ a=a * a$. Hence $a \in G_e$. Let e, f be two dstinct idempotents of S. Suppose $a \in G_e \cap G_f$. Then there exist t and t_1 such that $t * a=e, e * t=t, t_1 * a=f$ and $f * t_1=t_1$. Now from $t * a=e$ we have $f \circ (t * a)=f \circ e$. Hence $f \circ e=(a \circ f) * t=a * t$ (since $a \in G_f$) $=(a \circ e) * t$ (since $a \in G_e)=e \circ (t * a)=e \circ e=e$. Then $e=f \circ e=e * f=$ $(f \circ e) * f=e \circ (f * f)=e \circ f=e \circ (t_1 * a)=(a \circ e) * t_1=a * t_1=(a \circ f) * t_1$ $=f \circ (t_1 * a)=f \circ f=f$. This is a contradiction. Hence $G_e \cap G_f=\phi$ when $e \neq f$.

REFERENCES

[1] Nordahi, T. E. and Scheiblich, H. E.—Regular * Semigroups. Semigroup Forum. Vol 16 (1978), 369-377.

[2] Sen M. K.—Characterization of Groups. Journal of Pure Mathematics, Vol. 1 (1981), 73-75.

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Dept. of Pure Math.
Calcutta University

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COMMON FIXED POINT THEOREM IN 2-METRIC SPACES

KANAN MAJUMDAR

Introduction: S. Gahler [2] introduced the notion of 2-metric space as follows:

A set X is defined to be 2-metric space, if there exists a real valued function $d : X \times X \times X \to R^+$ satisfying the following conditions:

(i) to each pair of points x, y $(x \neq y)$ of X there is one $z \in X$ such that $d(x, y, z) \neq 0$

(ii) $d(x, y, z) = 0$ only when at least two of three points are equal

(iii) $d(x, y, z) = d(x, z, y) = d(y, z, x) = \cdots$

(iv) $d(x, y, z) \leqslant d(x, y, u) + d(x, u, z) + d(u, y, z)$.

In the work of I. Keyoshi [3] we find the following definitions.

Definition 1: If $d(x, y, z)$ is bounded, the 2-metric space is said to be bounded. By its diameter we mean $\sup\limits_{x, y, z, \in X} d(x, y, z)$

Definition 2: If $d(x_n, x, a)$ converges to zero for all $a \in X$ we say that the sequence $\{x_n\}$ converges to x and x is a limit of $\{x_n\}$.

Definition 3: If in a 2-metric space X, $d(x_m, x_n, a) \to 0$ $(m, n \to \infty)$ for all $a \in X$, the sequence $\{x_n\}$ is called a Cauchy sequence.

If in X, every Cauchy sequence is convergent, X is called complete.

Also in the 2-metric space, the notion of continuity of a self-mapping may be given as follows:

Definition 4: If in a 2-metric space X, $d(x, x_0, a) \to 0$ implies $d(Tx, Tx_0, a) \to 0$ for all $a \in X$, then we say that $T : X \to X$ is continuous at $x = x_0$.

Here we shall prove a fixed point theorem which is proved for a 1-metric space by B. Fisher [1].

Theorem: Suppose S and T are continuous mappings of the complete and bounded 2-metric space X into itself. If S and T satisfy the condition

$$d(S^2x, T^2y, a) \leqslant C \max [d(x, Ty, a), d(y, Sx, a), d(x, y, a)]$$

for all x, y, a in X, where $0 \leqslant C < 1$. Then S and T have a unique common fixed point u.

Proof: Let x be any arbitrary point in X. Then

$$d(S^n x, T^r x, a)$$
$$\leqslant C \max [d(S^{n-2}x, T^{r-1}x, a), d(S^{n-1}x, T^{r-2}x, a), d(S^{n-2}x, T^{r-2}x, a)]$$
$$\leqslant C^2 \max [d(S^{n-4}x, T^{r-2}x, a), d(S^{n-3}x, T^{r-3}x, a), d(S^{n-4}x, T^{r-3}x, a), d(S^{n-2}x, T^{r-4}x, a), d(S^{n-3}x, T^{r-4}x, a), d(S^{n-4}x, T^{r-4}x, a)]$$

and so on, till powers of S and T do not become negative.

Since X is bounded,

$$M = \text{Sup}\,[d(x, y, a), x, y, a \in X] < \infty.$$

For arbitary $\epsilon > 0$, choose N so that $C^N M < \epsilon/3$

$\therefore\ d(S^n x, T^r x, a) < \epsilon/3$ for $n, r \geqslant 2N$

and so
$$\begin{aligned} d(S^n x, S^m x, a) &\leqslant d(S^n x, T^r x, a) + d(T^r x, S^m x, a) + d(S^n x, S^m x, T^r x) \\ &< 2\epsilon/3 + d(S^n x, T^r x, S^m x) \\ &< 2\epsilon/3 + C^N M \\ &< 2\epsilon/3 + \epsilon/3 = \epsilon \end{aligned}$$

for $m, n, r \geqslant 2N$.

Hence $\{S^n x\}$ is a Cauchy sequence in the complete 2-metric space X and so has a limit u in X. Since S is continuous $Su = u$ and so u is a fixed point of S.

Similarly $\{T^n x\}$ is a Cauchy sequence in X and since $d(S^n x, T^n x, a) < \epsilon/3$ for $n \geqslant 2N$, the sequence $\{T^n x\}$ also converges to u.

As T is continuous, $Tu = u$ and so u is a common fixed point of S and T.

If possible, let w be another common fixed point of S and T. Then

$$\begin{aligned} d(u, w, a) &= d(S^2 u, T^2 w, a) \\ &\leqslant C \max [d(u, Tw, a), d(Su, w, a), d(u, w, a)] \\ &= C\, d(u, w, a). \end{aligned}$$

As $C < 1$, $u = w$ and so the common fixed point is unique.

Corollary: Let S and T be continuous mappings of the complete and bounded 2-metric space X into itself satisfying the inequality

$$d(S^2 x, T^2 y, a) \leqslant C \max [d(x, Ty, a), d(y, Sx, a)] \text{ for all } x, y, a \in X,$$

where $0 \leqslant C < 1$.

Then S and T have a unique common fixed point.

Proof: Since $d(S^2x, T^2y, a) \leqslant C \max [d(x, Ty, a), d(y, Sx, a)]$
$\leqslant C \max [d(x, Ty, a), d(y, Sx, a), d(x, y, a)]$

for all x, y in X, the result follows from the theorem (above).

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REFERENCES

[1] Fisher, B.—Common fixed point mappings on complete and compact metric spaces, *Math. Japonica*, **22**, No. 5 (1978), 565-569.

[2] Gahler, S.—2 Metrische Raume and ihre topologische struktur, *Math. Nachr.*, **26** (1963/64), 115-148.

[3] Kiyoshi, I.—Fixed point theorem in 2-metric spaces, *Mathematics Seminar Notes*, **3** (1975), 133-136.

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ON THE DISTRIBUTION OF THE EIGENVALUES OF A DIFFERENTIAL SYSTEM

N. K. CHAKRAVARTY and SUDIP KUMAR ACHARYYA

Abstract : The object of the present paper is to investigate certain asymptotic relations connecting the infinite series expansions involving eigenvalues associated with the differential system $(-D^2+P)\,U=\lambda U$ and the integrals involving the characteristic roots of P, which is a positive definite symmetric 2×2 matrix. Tauberian theorems are next applied to obtain, interalia, the asymptotic distribution of $N(\lambda)$, the number of eigenvalues less than λ.

1. The Problem

Let the differential system be

$$M\,\phi = \lambda\phi, \qquad \ldots \quad (1.1)$$

where $M \equiv \begin{pmatrix} -D^2+p & r \\ r & -D^2+q \end{pmatrix}$, $D \equiv \dfrac{d}{dx}$

p, q, r are real valued functions of x, with derivatives, which are alsolutely continuous over any compact sub-interval of R : $[0, \infty)$ and λ is a complex parameter.

The boundary conditions considered are

$$u(0) = v(0) = 0 \qquad \ldots \quad (1.2)$$

Or

$$u'(0) = v'(0) = 0 \qquad \ldots \quad (1.3)$$

where $\phi \equiv \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$.

The Hilbert space H in which the theory associated with the operator M, is developed, is that of functions $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ for which $\int_0^\infty (|f_1|^2 + |f_2|^2)\,dx < \infty$, with usual definition of the inner product.

It is well known that the differential system (1.1) along with the boundary conditions (1.2) (or with (1.3)) gives rise to the Dirichlet (or the Neumann) eigenvalue problems. It is assumed that $\begin{pmatrix} u \\ v \end{pmatrix} \in L_2$ at infinity.

Let us further assume that $p\phi$, $q\phi$, $r\phi \in \mathrm{H}$ and that $p, q > 0$, det $\mathrm{P} \geqq 0$, in $[0, \infty)$, $\mathrm{P} \equiv \begin{pmatrix} p & r \\ r & q \end{pmatrix}$. Also let P be pseudo-monotonic over $[0, \infty)$ in the sense that for $j \geqq k$, $p_j \geqq p_k$, $q_j \geqq q_k$, det $(P_j - P_k) \geqq 0$, (p_i, q_i, P_i are the p, q, P, in which x is replaced by x_i), $j, k = 0, 1, 2, \ldots$Then the sequence of eigen values $\{\lambda_n\}$ is positive, and the spectrum is discrete with $\lim\limits_{n\to\infty} \lambda_n = \infty$, over $[0, \infty)$, both for the Dirichlet and the Neumann eigenvalue problems (see Chakravarty and Sengupta [2]).

In particular $\lambda_n \geqq \lambda_0 \geqq 0$, λ_0 being the least eigenvalue of the system.

Let $\psi_n(x) \equiv \begin{pmatrix} \psi_{1n} \\ \psi_{2n} \end{pmatrix}$ be the eigenvector corresponding to the eigenvalue λ_n, and let $\Delta \equiv \Delta(x) \equiv \frac{1}{2}\left(p+q+\sqrt{(p-q)^2+4r^2}\right)$,

$\eta \equiv \eta(x) \equiv \frac{1}{2}\left(p+q-\sqrt{(p-q)^2+4r^2}\right)$ be the characteristic roots of the matrix P.

Then Δ, η are both real and both steadily increase with x, if $p\ (> q)$, r are steadily increasing and $(p - q)q' - 2rr' \geqq 0$, is satisfied (see Chakravarty and Sengupta [2]). We take into account the following conditions :

(a) $| p(\xi)-p(x) |$, $| q(\xi)-q(x) |$, $| r(\xi)-r(x) | < \mathrm{C} \, | \xi-x | \, \eta^{a_0}(x)$ for

$0 < | \xi-x | \leqq 1$, $\quad$ C, a_0 are positive constants, $0 < a_0 < \frac{5}{4}$.

(b) $p(\xi), q(\xi), | r(\xi) | \leqq \mathrm{K}_0 \exp \left\{\frac{1}{2} \, | \xi-x | \, \eta^{a_1}(x)\right\}$ for $| \xi-x | > 1$;

K_0, a_1 are positive constants, $\quad 0 < a_1 < \frac{1}{2}$.

(c) $\eta(x) \geqq x^2 \, \eta^{2a_1}(b)$ $\quad$ for large $x \in \mathrm{R} = \left(0, \infty\right)$, $b > 0$.

(d) $\Delta(x)$ as well as $\eta(x)$ are steadily increasing with x.

Let there exist two well-behaved functions $t_j(x)$, $j = 1, 2$ on R, such that

(e) $t_j(\xi) < \mathrm{C}_j t_j(x)$, $| \xi-x | \leqq 1$; $1 < t_j \leqq \eta \leqq \Delta$, for large x, $j = 1, 2$, where C_j are positive constants.

(f) $\int_0^\infty \frac{dx}{t_j^{A_j}} < \infty$, $t_j \equiv t_j(x)$, $j = 1, 2$, for some positive numbers A_1, A_2.

(g) $t_1^{2s}\psi_{1n}^2 + t_{2n}^{2s}\psi_{2n}^2 \in L[0, \infty)$, where s is a positive integer $\geqq 2$.

We define $$a_n^{(\tau_1, \tau_2, m)} = \int_0^\infty \Big(t^\tau(x), \psi_n^2(x)\Big) \Delta^{-2m}(x)\, dx \qquad \ldots \quad (1.4)$$

and $$b_n^{(\tau_1, \tau_2, m)} = \int_0^\infty \Big(t^\tau(x), \psi_m^2(x)\Big)\, \eta^{-2m}(x)\, dx \qquad \ldots \quad (1.5)$$

where $(t^\tau, \psi_n^2) = \sum_{j=1}^{2} t_j^{\tau_j} \psi_{jn}^2$, m and τ_j are positive numbers,

satisfying

(h) : $0 \leqq \tau_j \leqq \min\{2m - A_j, 2s - A_j - 5/2\}$, $j=1, 2$

and when $m = 0$,

(h') : $0 \leqq \tau_j \leqq 2s - A_j - 1$, $j=1, 2$.

Put $$S_{\tau_1, \tau_2, s, m} = \sum_{n=0}^\infty a_n^{(\tau_1, \tau_2, m)} (\lambda_n + \mu)^{-2s-2m}, \quad \mu \geqq 1,$$

$$I_{\tau_1, \tau_2, s, m} = \frac{1.3.5 \ldots (4s-3)}{2^{2s}(2s-1)!} \int_0^\infty \frac{t_1^{\tau_1}(x) + t_2^{\tau_2}(x)}{(\mu+\Delta)^{2(s+m)-1/2}}\, dx,$$

$\Delta \equiv \Delta(x)$, and $S^*_{\tau_1, \tau_2, s, m}$, $I^*_{\tau_1, \tau_2, s, m}$ are the corresponding entities, in which $\Delta \equiv \Delta(x)$ is replaced by $\eta \equiv \eta(x)$.

Our object in the present paper is to show first that as $\mu \to \infty$,

$$S_{\tau_1, \tau_2, s, m} \sim I_{\tau_1, \tau_2, s, m}; \quad S^*_{\tau_1, \tau_2, s, m} \sim I^*_{\tau_1, \tau_2, s, m}.$$

Then by means of certain Tauberian theorems, we obtain the asymptotic estimates for $a_n^{(\tau_1, \tau_2, m)}$ and $b_n^{(\tau_1, \tau_2, m)}$, and from these for $N(\lambda)$ the number of eigenvalues not exceeding λ.

2. Some preliminary results

Let $G(\xi, y, \mu) \equiv \begin{pmatrix} G_{11}(\xi, y, \mu) & G_{21}(\xi, y, \mu) \\ G_{12}(\xi, y, \mu) & G_{22}(\xi, y, \mu) \end{pmatrix}$ be the Green's matrix for the system (1.1), with (1.2) or (1.3), where $\lambda = -\mu$, $\mu \geqq 1$, in the singular case $0 \leqq \xi < \infty$, the Green's vectors being $G_j = \begin{pmatrix} G_{j1} \\ G_{j2} \end{pmatrix}$, $j = 1, 2$.

Also let $g(\xi, y, k)$ be the corresponding Green's matrix for the system

$$M_0\, \phi = -k^2(x)\, \phi, \quad M_0 \equiv \begin{pmatrix} -D^2 & 0 \\ 0 & -D^2 \end{pmatrix}, \; D \equiv d/d\xi, \; \phi \equiv \phi(\xi)$$

satisfying the same Dirichlet (or Neumann) boundary conditions as before and $0 \leqq \xi < \infty$ with vectors $g_j \equiv \begin{pmatrix} g_{j1} \\ g_{j2} \end{pmatrix}$, $j = 1, 2$.

Then both G_j and $g_j \in L_2$ $(j = 1, 2)$, G_j, g_j satisfy the same boundary conditions at $\xi = 0$, and by a variant of the analysis adopted by Sengupta [5] [Lemma 2, P-101],

$\lim\limits_{b \to \infty} \Big[G_j\ (b, \xi, y, \mu),\ g_i\ (b, \xi, y, k) \Big] = 0$, where $G_j\ (b, \cdots)$, $g_i\ (b, \cdots)$ are the Green's vectors for the interval $[0, b]$, such that $\lim\limits_{b \to \infty} G_j\ (b, \cdots) = G_j\ (\,.\,)$ and $\lim\limits_{b \to \infty} g_i\ (b, \cdots) = g_i\ (\,.\,)$ and $[U, V]$ represents the bilinear concomitant of the vectors U, V. (For definitions see Chakravarty [1], 1965).

Then by making use of the properties of the Green's matrix, it follows after some easy manipulations, that

$$G^T\ (x, y, \mu) = g^T\ (x, y, k) + \int_0^\infty G(y, \xi, \mu)\ (P(\xi) - P(x))\ g^T\ (x, \xi, k)\ d\xi$$

$$+ \int_0^\infty G(y, \xi, \mu)\ [\, P(x) - (k^2(x) - \mu)\ I\,]\ g^T\ (x, \xi, k)\ d\xi \qquad \ldots \quad (2.1)$$

where I is the unit matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $P\ (\xi)$ is the matrix P with $x = \xi$, G^T, g^T being the transpose of G and g respectively.

Since $k \equiv k(x)$ is at our disposal, to simplify our discussions, we so choose it, that the last integral on the right hand side of (2.1) vanishes. We further assume that the two matrices $A \equiv g^T(x, \xi, k)$

and
$$B = \begin{pmatrix} p-(k^2-\mu) & r \\ r & q-(k^2-\mu) \end{pmatrix}$$

commute (this is implied in our discussion in view of the explicit expression for the elements of $g^T(x, \xi, k)$). We then have $k^2 = \mu + \Delta(x)$ or $k^2 = \mu + \eta(x)$, where $\Delta(x)$ and $\eta(x)$ are the characteristic roots of the matrix $P(x)$.

The equation (2.1) now takes the simpler form

$$G^T(x, y, \mu) = g^T(x, y, k) + \int_0^\infty G(y, \xi, \mu)\,(P(\xi) - P(x))\,g^T(x, \xi, k)\,d\xi \qquad \ldots \quad (2.2)$$

where $k^2 = \mu + \Delta(x)$ or $\mu + \eta(x)$, $\mu \geqq 1$, under the assumptions made above. (2.2) is the basis for our investigations that follow.

It is easy to deduce that

$$\frac{\psi_n(x)}{\lambda_n + \mu} = -\int_0^\infty G(x, y, \mu)\,\psi_n(y)\,dy \qquad \ldots \quad (2.3)$$

Therefore by the expansion formula

$$G_j(y, x, \mu) = -\sum_{n=0}^{\infty} \frac{\psi_{jn}(x)\psi_n(y)}{(\lambda_n + \mu)}, \quad (j = 1, 2) \qquad \ldots \quad (2.4)$$

Let $D_\mu^{(n)}$ be the symbol of differentiation n times with respect to μ.

Then from (2.4)

$$D_\mu^{(s-1)} G_j(y, x, \mu) = (-1)^s (s-1)!\, \sum_{n=0}^{\infty} \frac{\psi_{jn}(x)\,\psi_n(y)}{(\lambda_n + \mu)^s} \qquad \ldots \quad (2.5)$$

from which by the Parseval relation,

$$\frac{1}{\{(s-1)!\}^2}\, \| D_\mu^{(s-1)} G_j^T(x, y, \mu) \|_{0,\infty} = \sum_{n=0}^{\infty} \frac{\psi_{jn}^2(x)}{(\lambda_n + \mu)^{2s}}, \; (j = 1, 2) \qquad (2.6)$$

where $\| \omega \|_{0,\infty} = \int_0^\infty (\omega_1^2 + \omega_2^2)\, dy$, $\omega \equiv \omega(y) \equiv \begin{pmatrix} \omega_1(y) \\ \omega_2(y) \end{pmatrix}$

and $G^{T}_{j}(.)=\begin{pmatrix} G_{1j} \\ G_{2j} \end{pmatrix}$ is the Green's vector corresponding to $G^{T}(.)$, the transpose of the Green's matrix G (.).

In the second term of the right hand side of (2.2) substitute for the Green's vectors $G_j\ (y, \xi, \mu)$ by the relations (2.4) and then differentiate $(s-1)$ times with respect to μ, by using the Leibnitz formula. Then it follows from (2.2) that

$$\frac{1}{(s-1)\,!}\, D_{\mu}^{(s-1)} G_{1}^{T}\,(x, y, \mu)=(-1)^{s} \sum_{n=0}^{\infty} \frac{\psi_{1n}(x)\ \psi_n(y)}{(\lambda_n+\mu)^{s}}$$

$$=\frac{1}{(s-1)\,!} D_{\mu}^{(S-1)}\ g_1(y, x, k)+J_S(x, y, k) \qquad \ldots \ (2.7)$$

where $J_S\ (x, y, k)\ \equiv \begin{pmatrix} J_{1s}(x, y, k) \\ J_{2s}(x, y, k) \end{pmatrix}$

$$=\sum_{j=0}^{s-1} C_{js} \sum_{n=0}^{\infty} \left(\int_0^{\infty} \psi_{n}^{T}\,(\xi)\Big(P(\xi)-P(x)\Big)\, D_{\mu}^{(s-j-1)}\ g_1(\xi, x, k)\,d\xi\right) \frac{\psi_n(y)}{(\lambda_n+\mu)^{j+1}}$$

$$=\sum_{j=0}^{s-1} C_{js} \sum_{n=0}^{\infty} \chi_n\,(x)\ \frac{\psi_n(y)}{(\lambda_n+\mu)^{j+1}} \qquad \ldots \ (2.8)$$

C_{js} are the numerical constants (suitably adjusted) obtained in the process of application of the Leibnitz formula, and

$$\chi_n\ (x)=\int_0^{\infty} \psi_{n}^{T}\ (\xi)\ \Big(P(\xi)-P(x)\Big)\, D_{\mu}^{(s-j-1)}\ g_1(\xi, x, k)d\xi.$$

A similar result holds for $D_{\mu}^{(s-1)}\ G_{2}^{T}\ (x, y, \mu)$,

Since $g_{12}=0$, it follows from (2.6) and the relation (2.7) that

$$\sum_{n=0}^{\infty} \frac{\psi_{1n}^{2}(x)}{(\lambda_n+\mu)^{2s}}=\int_0^{\infty}\left[\frac{1}{(s-1)\,!}D_{\mu}^{(s-1)}\ g_{11}(y, x, k)+J_{1s}\ (x, y, k)\right]^{2} dy$$

$$+\int_0^{\infty}\Big[J_{2s}\ (x, y, k)\Big]^{2} dy \qquad \ldots \ (2.8a)$$

$$\leqslant\left[\left(\frac{1}{\{(s-1)!\}^2}\int_0^\infty\left(D_\mu^{(s-1)}\, g_{11}(y,x,k)\right)^2 dy\right)^{1/2}+\|J_s(x,y,k)\|_{0,\infty}^{1/2}\right]^2 \quad \dots (2.9)$$

by using the Minkowsky inequality and an obvious inequality

$(a^{\frac{1}{2}}+b^{\frac{1}{2}})^2+c\leqslant\{a^{\frac{1}{2}}+(b+c)^{\frac{1}{2}}\}^2$, where a, b, c are positive.

A similar inequality for $\sum_{n=0}^{\infty}\frac{\psi_{2n}^2(x)}{(\lambda_n+\mu)^{2s}}$ also holds.

If $\psi_\lambda(z)\equiv\begin{pmatrix}\psi_{1\lambda}(z)\\ \psi_{2\lambda}(x)\end{pmatrix}$ be a solution of the system $d^2u/dz^2+\lambda u=0$; $d^2v/dz^2+\lambda v=0$, satisfying either the Dirichlet or the Neumann boundary conditions, then it is easy to verify that

$$g_1^{T}(\xi,z,k)=-\int_0^\infty\frac{\psi_{1\lambda}(z)\,\psi_\lambda^{T}(\xi)}{k^2(x)+\lambda}\,d\rho(\lambda) \quad \dots (2.10)$$

$\rho(\lambda)$ being the spectral matrix associated with the system, where

$k^2(x)=\mu+\Delta(x)$ or $\mu+\eta(x)$..

Also $(g_{ij}(\xi,z,k))$ has the explicit representation

$$\left.\begin{aligned}\left(g_{ij}(\xi,z,k)\right)&=\pm\frac{1}{2k}\left[e^{-k(z+\xi)}\mp e^{k(z-\xi)}\right]I,\quad z\leqslant\xi\\ &=\pm\frac{1}{2k}\left[e^{-k(z+\xi)}\mp e^{k(\xi-z)}\right]I,\quad z>\xi\end{aligned}\right\} \quad \dots (2.11)$$

where I is the unit matrix defined before, the upper or the lower sign being chosen according as the problem is the Dirichlet or the Neumann and $k\equiv k(x)$.

Since $g_{12}=0$, it follows from (2.10) and (2.11) and the Parseval relation that

$$\frac{1}{\{(s-1)!\}^2}\int_0^\infty\{D_\mu^{(s-1)}\, g_{11}(y,x,k)\}^2\, dy=\frac{1}{(2s-1)!}D_\mu^{(2s-1)}\, g_{11}(x,x,k)$$

$$=\frac{C_0}{k^{4s-1}(x)}\pm\frac{1}{(2s-1)!}D_\mu^{(2s-1)}\left(\frac{e^{-2kx}}{k}\right),$$

where $k^2\equiv k^2(x)=\mu+\Delta(x)$ or $\mu+\eta(x)$,

and $C_0=\frac{1.3.5\dots\dots(4s-3)}{2^{2s}(2s-1)!}$ (the upper sign being chosen for the Dirichlet problem, and the lower for the Neumann).

Since by Leibnitz's theorem,

$$D_{\mu}^{(2s-1)}\left(e^{-2kx}/k\right)=-e^{-2kx}f(k,x)\,k^{-R},$$

where $f(k,x)>0$, is a polynomial in k, x and R is a suitably chosen positive integer, it follows from (2.9) that

$$\sum_{n=0}^{\infty}\frac{\psi_{1n}^{2}(x)}{(\lambda_n+\mu)^{2s}}\leq\left(T^{1/2}+\|J_s(x,y,k)\|_{0,\infty}^{1/2}\right)^2 \qquad \ldots(2.12)$$

where $T=C_0\,/\,k^{4s-1}(x)\pm\frac{1}{(2s-1)\,!}e^{-2kx}f(k,x)\;k^{-R}$

(the upper sign being for the Neumann, and the lower sign being for the Dirichlet problem).

Similarly $$\sum_{n=0}^{\infty}\frac{\psi_{2n}^{2}(x)}{(\lambda_n+\mu)^{2s}}\leq\left(T^{1/2}+\|L_s(x,y,k)\|_{0,\infty}^{1/2}\right)^2 \qquad \ldots(2.13)$$

where $L_s(x,y,k)\equiv\begin{pmatrix}L_{1s}(x,y,k)\\ L_{2s}(x,y,k)\end{pmatrix}$ is defined in the same way

as $J_s(x,y,k)$ with $g_1(\xi,x,k)$ replaced by $g_2(\xi,x,k)$.

3. Inequality involving $S_{\tau_1,\tau_2,s,m}$ and $I_{\tau_1,\tau_2,s,m}$.

Multiply (2.12) by $t_1^{\tau_1}(x)\;\Delta^{-2m}(x)$ and (2.13) by $t_2^{\tau_2}(x)\;\Delta^{-2m}(x)$, make use of the inequalities $\lambda_n+\mu\geq\lambda_0+\mu\geq1$ (since $\mu\geq1$) ;

$$\left((a^{1/2}+b^{1/2})^2+(c^{1/2}+d^{1/2})\right)^2\leq\left[a^{1/2}+c^{1/2}+(b+d)^{1/2}\right]^2$$

where $a,b,c,d\geq0$, and the Minkowsky inequality. Then after some reductions, we obtain

$$S^{1/2}_{\tau_1,\tau_2,s,m}$$

$$\leqslant\left[C_0\int_0^{\infty}\frac{t_1^{\tau_1}(x)+t_2^{\tau_2}(x)}{\Delta^{2m}(x)\;k^{4s-1}(x)}\,dx\right]^{1/2}$$

$$+\left[\frac{1}{(2s-1)\,!}\int_0^{\infty}\frac{e^{-2kx}f(k,x)}{k^{R}\Delta^{2m}(x)}\left(t_1^{\tau_1}(x)+t_2^{\tau_2}(x)\right)dx\right]^{1/2}$$

$$+\left[\int_0^\infty\left\{\frac{t_1^{\tau_1}(x)}{\Delta^{2m}(x)}\,\|\,J_s(x,y,k)\,\|_{0,\infty}+\frac{t_2^{\tau_2}(x)}{\Delta^{2m}(x)}\,\|\,L_s(x,y,k)\,\|_{0,\infty}\right\}dx\right]^{1/2} \quad \ldots (3.1)$$

$=I_1+I_2+I_3$, say.

Since $k^2(x)=\mu+\Delta(x)>\mu,\ \Delta(x)$, we have

$$I_1^2=I_{\tau_1,\tau_2,s,m}+C_0\int_0^\infty\frac{t_1^{\tau_1}(x)+t_2^{\tau_2}(x)}{\Delta^{2m}(x)\,k^{4s-1}(x)}\left[1-\left(\frac{\Delta(x)}{\mu+\Delta(x)}\right)^{2m}\right]dx$$

$=I_{\tau_1,\tau_2,s,m}+O(\mu^{-2s+1/2})$, as $\mu\to\infty$, by using the conditions (f) and (h).

Since $e^x>\dfrac{x^R}{R\,!}$, we have, as $\mu\to\infty$,

$$I_2^2=O\left(\int_0^\infty\frac{t_1^{\tau_1}(x)+t_2^{\tau_2}(x)}{k^{2R}(x)\,\Delta^{2m}(x)}\,dx\right)=O(\mu^{-R}) \text{ as before,}$$

by the conditions (f) and (h).

It follows from (2.8) by the application of the Minkowsky inequality, the Parseval relation and some easy reductions, that

$$\left(\int_0^\infty t_1^{\tau_1}(x)\Delta^{-2m}(x)\,\|\,J_s(x,y,k)\,\|_{0,\infty}\,dx\right)^{1/2}$$

$$\leqslant C\sum_{j=0}^{s-1}\left\{\sum_{n=0}^{\infty}\frac{1}{(\lambda_n+\mu)^{2j+2}}\int_0^\infty\frac{t_1^{\tau_1}(x)}{\Delta^{2m}(x)}\times\right.$$

$$\left.\left[\int_0^\infty D_\mu^{(s-j-1)}g_{11}(\xi,x,k)((p(\xi)-p(x))\psi_{1n}(\xi)+(r(\xi)-r(x))\psi_{2n}(\xi))\,d\xi\right]^2dx\right\}^{1/2}$$

$$\ldots (3.2)$$

where $C=\max C_{js}$, the constants involved in (2.8).

Put $\displaystyle a_n^2=\int_0^\infty\frac{t_1^{\tau_1}(x)}{\Delta^{2m}(x)}\times$

$$\left[\int_0^\infty D_\mu^{(s-j-1)}g_{11}(\xi,x,k)\{(p(\xi)-p(x))\psi_{1n}(\xi)+(r(\xi)-r(x))\psi_{2n}(\xi)\}\,d\xi\right]^2dx,$$

$b_n = \dfrac{1}{(\lambda_n+\mu)^{2j+2}}$, $\alpha = \dfrac{2j+2m+2}{2m}$, $\beta = \dfrac{2j+2m+2}{2j+2}$ in the inequality

$$\sum_{n=0}^{\infty} a_n^2 b_n \leqslant \left(\sum_{n=0}^{\infty} a_n^2\right)^{1/\alpha} \left(\sum_{n=0}^{\infty} a_n^2 b_n^{\beta}\right)^{1/\beta}, \quad \frac{1}{\alpha}+\frac{1}{\beta}=1, \quad \alpha, \beta > 0 \qquad \text{... (A)}$$

a variant of the Hölder inequality, so as to obtain from (3.2)

$$\sum_{n=0}^{\infty} \frac{a_n^2}{(\lambda_n+\mu)^{2j+2}} \leqslant \left(\sum_{n=0}^{\infty} \frac{a_n^2}{(\lambda_n+\mu)^{2j+2m+2}}\right)^{\frac{2j+2}{2j+2m+2}} \left(\sum_{n=0}^{\infty} a_n^2\right)^{\frac{2m}{2j+2m+2}} \cdot \qquad \text{... (3.3)}$$

Now $\widetilde{\mathrm{M}} \equiv \int_0^{\infty} \left[\mathrm{D}_{\mu}^{(s-j-1)} g_{11}(\xi, x, k)\right]^2 \left[p(\xi)-p(x)\right]^2 d\xi$

$$= \left(\int_0^{x-1} + \int_{x+1}^{\infty}\right) \{p^2(\xi)-2p(x)p(\xi)+p^2(x)\} \left(\mathrm{D}_{\mu}^{(s-j-1)} g_{11}(\xi, x, k)\right)^2 d\xi$$

$$+ \int_{x-1}^{x+1} \{p(\xi)-p(x)\}^2 \left(\mathrm{D}_{\mu}^{(s-j-1)} g_{11}(\xi, x, k)\right)^2 d\xi$$

$= \mathrm{M}_1+\mathrm{M}_2+\mathrm{M}_3+\mathrm{M}_4$, say $(x \geqslant 1)$.

We evaluate $\mathrm{D}_{\mu}^{(s-j-1)} g_{11}(\xi, x, k)$ by the Leibnitz formula from the explicit relations for $g_{11}(\xi, x, k)$ as given by (2.11), utilize the conditions (a), as and when necessary. Then for both the Dirichlet and the Neumann problem and for the cases $\xi > x$, $\xi < x$, it is easy to derive after some steps that

$$\mathrm{M}_4 \leqslant d(k(x))^{-4(s-j)-1+4\alpha_0}$$

for $x \geqslant 1$, d being a suitable positive constant.

An utilization of the conditions (*b*) along with the use of the explicit expression for $\mathrm{D}_{\mu}^{(s-j-1)} g_{11}(\xi, x, k)$ as derived before by the use of the Leibnitz formula, yields after some elaborate steps, $\mathrm{M}_1, \mathrm{M}_2, \mathrm{M}_3 < \mathrm{C}k^{-\sigma}$, where C is a positive constant and $\sigma > 0$ may be chosen as large as possible.

It therefore follows that $\widetilde{\mathrm{M}} \leqslant d(k(x))^{-4(s-j)-1+4\alpha_0}$.

A similar estimate holds for the expression $\widetilde{\mathrm{N}}$, obtained from $\widetilde{\mathrm{M}}$ by replacing $p(u)$ by $r(u)$, $u = x, \xi$.

CONTENTS

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ON THE STABILITY OF SOLUTIONS OF A CERTAIN DIFFERENTIAL SYSTEM WITH RESPECT TO A PERTURBATION

N. K. CHAKRAVARTY
and
DEBASISH SENGUPTA

1. Introduction

The paper deals with the stability of solutions of the system of second order differential equations with respect to certain perturbation. We consider the system

(1.1) $$d^2y(x) / dx^2 + A(x)\, y(x) = 0,$$

where $y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = (y_1(x)\; y_2(x))^T,$

$A(x) = (a_{ij}(x))\,;\ i,\ j = 1, 2,$

and the independent variable x ranges over $[\,0, \infty)$. We assume that the elements $a_{ij}(x)$ are real valued continuous functions of x.

Following Bellman [1], we define stability of solutions of (1.1) with respect to a property P and perturbation B(x) as follows :

Definition : Let $B(x) = (b_{ij}(x))\,;\ i, j = 1, 2,$ be a set of perturbations which changes (1.1) to

(1.2) $$d^2y(x) / dx^2 + \{ (A (x) + B(x) \}\, y(x) = 0.$$

Then the solutions of (1.1) which satisfy a certain property P are said to be stable with respect to the set of perturbations B(x) and the property P, if the solutions of (1.2) also satisfy the same property P for all B(x) satisfying certain specified conditions.

The following notations are used in the course of our discussion :

(i) The boundedness, the differentiability or the integrability of a matrix imply as usual the same for the elements of the matrix.
We write

$$\| A \| = \sum_{i,j} | a_{ij} |$$

(ii) (α, β) represents the scalar product

$$(\alpha, \beta) = \sum_{j=1}^{m} \alpha_j \bar{\beta}_j\,; \text{ where } \alpha = (\alpha_1, \alpha_2, \dots, \alpha_m) \text{ and } \beta = (\beta_1, \beta_2, \dots, \beta_m).$$

(iii) The determinant det (a_{ij}) of order n (n = 2, 3, ... ,) is represented in terms of the diagonal elements a_{jj} by $| a_{11}, a_{22}, \ldots , a_{nn} |$.

(iv) $[\phi_i, \phi_j] = | u_i, u'_j | + | v_j, v'_j |$ is the bilinear concomitant of two vectors

$$\phi_i = (u_i, v_i)^T \text{ and } \phi_j = (u_j, v_j)^T$$

(v) $f(x) \in L^{\alpha,\beta,\gamma,\cdots}[a, b)$, means that

$$f(x) \in L^{\alpha}[a, b),\ f(x) \in L^{\beta}[a, b),\ f(x) \in L^{\gamma}[a, b], \ldots\ a{=}0,\ b = \infty \text{ allowed.}$$

(vi) $| f |^{\alpha}_{a,x}$ stands for $\left(\int_a^x | f(t) |^{\alpha} dt \right)^{1/\alpha}$

2. Preliminaries

It can be easily verified by the usual iteration process that there exists a unique solution $\phi = (u, v)^T$ of (1.1) which along with their first derivatives take prescribed values at a point over (a, b)

Let $\phi_i = (u_i, v_i)^T$, $i = 1, 2$; $\theta_j = (u_s, v_s)^T$, $j = 1$, $s = 3$; $j = 2$, $s = 4$, be the four solution vectors of the system (1.1) satisfying the following conditions at $x = a \geqslant 0$

(2.1) (i) $[\phi_1, \phi_2] = [\theta_1, \theta_2] = 0$;

(ii) $[\phi_i, \phi_j] = \delta_{ij}$,

where δ_{ij} are the kronecker delta.

The determinant (independent of x) of the vectors ϕ_i, θ_j, i, j = 1, 2, defined by

$$w = | u_1, v_2, u'_3, v'_4 |$$

plays the same role for the present system as the usual Wronskian for the general ordinary linear equations. We call this determinant the Wronskian of our system. It follows by the Laplace Development of a determinant that

$$W = -[\phi_1 \cdot \phi_2][\theta_1, \theta_2] + [_1, \theta_1 \phi][\phi_2, \theta_2] - [\phi_1, \theta_2][\phi_2, \theta_1] = 1, \text{ by (2.1)},$$

which implies that ϕ_1, ϕ_2, θ_1, θ_2 form a fundamental set of solutions of (1.1).

Let ($\Psi(x) = (\Psi_1(x), \Psi_2(x))^T$ be a solution of the system (1.2). Then by the method of variation of the constants it follows that the component $\Psi_1(x)$ of $\Psi(x)$ is given by

(2.2) $$\Psi_1(x) = \sum_{i=1}^{4} C_i u_i(x) + \int_0^x (B_2, \Psi(t)) V_1 \, dt - \int_0^x (B_1, \Psi(t)) U_1 \, dt,$$

where, $U_1 = | u_1(x), u_2(t), v_3(t), v'_4(t) |$

$V_1 = | v_1(x), v_2(t), u_3(t), u'_4(t) |$.

$B_1 = B_1(t) = (b_{11}(t), b_{12}(t))^T,$

$B_2 = B_2(t) = (b_{21}(t), b_{22}(t))^T,$

and C_i, $i = 1, 2, 3, 4$ are constants.

Also by the Laplace development and utilisation of conditions (2.1),

$$U_1 = | u_1(x), u_3(t) | + | u_2(x), u_4(t) |$$
$$V_1 = | u_1(x), v_3(t) | + | u_2(x), v_4(t) |.$$

A similar expression for $\Psi_2(x)$ involving

$$U_2 = | v_1(x), u_3(t) | + | v_2(x), u_4(t) |$$
$$V_2 = | v_1(x), v_3(t) | + | v_2(x), v_4(t) |.$$

is also obtained.

3. Generalisation of Gronwall's lemma.

Lemma : Let (i) $\phi(x)$, $\Psi(x) \geqslant 0$, and $\Psi(x)/\phi(x)$ are integrable, (ii) $h_i(x)$, $g_i(x) \geqslant 0$, $i = 1, 2, \ldots, n$; where $h_i(x)$ are all continuous and $\phi\, g_i(x)$ are integrable over (a, x).

If (iii) $h_i(x) \leqslant \Psi(x) + \phi(x) \int_a^x (h, g)\, dy$; $i = 1, 2, \ldots, n$;

then,

$$h_i(x) \leqslant \Psi(a)/\phi(a).\ \phi(x).\ \exp\left\{\int_a^x \left(\sum_{i=1}^{n} \phi\, g_i\right) dt\right\}$$

(3.1)

$$+ \phi(x).\ d/dy(\Psi/\phi).\ \exp\left\{\int_y^x \left(\sum_{i=1}^{n} \phi\, g_i\right) dt\right\} dy$$

where $\psi(a) \neq 0$, $0 \leqslant a \leqslant x \leqslant X < \infty$.

Proof : Let $V = \int_a^x (h, g)\, dy$. Then by (iii),

(3.2) $$dv / dx \leqslant \sum_{i=1}^{n} \Psi g_i + \sum_{i=1}^{n} \phi V g_i$$

The Lemma follows by multiplying (3.2) by exp. $\left\{ - \int_a^x \left(\sum_{i=1}^{n} \phi g_i \right) dt \right.$ and then integrating by parts and using the condition (iii).

Corollary 1 : Putting $\Psi(x) = C_1, \phi(x) = C_2$, where $C_1, C_2 \geqslant 0$, we obtain Grownwall's lemma [2] for vector valued functions as follows :

Let $h_i(x), g_i(x)$; $i = 1, 2, \ldots, n$ satisfy the conditions of the lemma and

$$h_i(x) \leqslant C_1 + C_2 \int_a^x (h, g) \, dy,$$

then $h_i(x) \leqslant C_1 \exp. \left\{ C_2 \int_a^x \left(\sum_{i=1}^{n} g_i \right) \right\} dy$, $i = 1, 2, \ldots, n$.

Corollary 2 : Under the same conditions as in the lemma

$$h_i \leqslant \Psi(x) + \phi(x) \int_a^x \Psi(y) \left(\sum_{i=1}^{n} g_i \right) \left\{ \exp. \int_y^x \left(\sum_{i=1}^{n} g_i(t) \right) \phi(t) \, dt \right\} dy$$

This is a consequence of a further integration by parts of the second term on the right hand side of (3.1).

4. Certain variants of the inequalities of Hölder and Young

In our discussion we require the following inequalities easily deducible from those of Young and Holder respectively.

(a) In the first factor of young's inequality as given in Titchmarsh [(3) P. 394, Ex. 8] we apply the Holder inequality to obtain

"If $\lambda_1, \lambda_2, p_1, p_2 > 0$, $\lambda_1 \lambda_2 < 1$, $1/p_1 + 1/p_2 = 1$

and $f(x) \in L^{p_1(1+\lambda_1), (1+\lambda_1)}$, $g(x) \in L^{p_2(1+\lambda_2), (1+\lambda_2)}$,

then $\left| \int fg \, dx \right|$

$$\leqslant \left\{ \left(\int |f|^{p_1(1+\lambda_1)} dx \right)^{1/p_1} \left(\int |g|^{p_2(1+\lambda_2)} dx \right)^{1/p_2} \right\}^{\frac{1-\lambda_1 \lambda_2}{(1+\lambda_1)(1+\lambda_2)}}$$

$$\times\left(\int |f|^{(1+\lambda_1)}dx\right)^{\lambda_2/(1+\lambda_2)}\left(\int |g|^{(1+\lambda_2)}dx\right)^{\lambda_1/(1+\lambda_1)}$$

(b) Let $f(x) \epsilon L^{p,q}$, $g(x) \epsilon L^{1/2\,pq}$, where $p, q > 1$, $1/p + 1/q = 1$;

then $\left|\int fg\,dx\right| \leqslant \left(\int |f|^p dx\right)^{1/p^2} \cdot \left(\int |f|^q dx\right)^{1/q^2} \left(\int |g|^{pq/2} dx\right)^{2/pq}$

In the Hölder inequality replace f by $f^{1/p} g^{1/2}$ and g by $f^{1/q} g^{1/2}$ so as to obtain

$$\left|\int fg\,dx\right| \leqslant \left(\int |fg^{p/2}|\,dx\right)^{1/p} \cdot \left(\int |fg^{q/2}|\,dx\right)^{1/q}.$$

The result follows by again applying the Holder inequality to each factor on the right hand side.

A similar technique yields

(c) If $\alpha, \beta, p, q > 0$, $1/\alpha + 1/\beta = 1/p + 1/q = 1$ and

$f(x) \epsilon L^{\alpha,\beta}$, $g(x) \epsilon L^{p,q}$, then

$$\left|\int fg\,dx\right| \leqslant \left(\int |f|^{\alpha} dx\right)^{1/\alpha p} \cdot \left(\int |g|^{p} dx\right)^{1/\beta p} \left(\int |f|^{\beta} dx\right)^{1/\beta q} \left(\int |g|^{q} dx\right)^{1/\alpha q}$$

Also utilising the Hölder inequality for three functions f(x), g(x), h(x) [Titchmarsh (3), P, 394], we have as above

(d) If $p, q > 0$, $1/p + 1/q = 1$, where $f \epsilon L^{p,q}$; $g, h \epsilon L^{pq}$, then

$$\left|\int fgh\,dx\right| \leqslant \left(\int |f|^{p} dx\right)^{1/p^2} \cdot \left(\int |f|^{q} dx\right)^{1/q^2} \left[\left(\int |g|^{pq} dx\right), \left(\int |h|^{pq} dx\right)\right]^{1/pq}$$

and

(e) If $\alpha, \beta, \gamma, \delta, p, q > 0$; $1/p + 1/q = 1/\alpha + 1/\beta = 1/\gamma + 1/\delta = 1$

and $f \epsilon L^{\alpha,\beta}$, $g \epsilon L^{p,q\gamma}$, $h \epsilon L^{\beta\delta q}$, then

$$\left|\int fgh\,dx\right| \leqslant \left(|f|^{\alpha} dx\right)^{1/q\alpha} \cdot \left(\int |f|^{\beta} dx\right)^{1/p\beta} \cdot \left(\int |g|^{p} dx\right)^{1/p\alpha}$$

$$\times\left(\int |g|^{q\gamma}dx\right)^{1/q\,\beta\gamma}\cdot\left(\int |h|^{\beta\delta\,q}dx\right)^{1/q\,\beta\gamma}.$$

5. Certain Theorems on Stability.

In the following we obtain certain theorems on the stability of solutions of the system (1.1) which belong to suitable L-classes and the perturbed matrix B(x) satisfies certain known conditions.

Theorem 1 Let all solutions of (1.1) belong to

$$L^{p_j(1+\lambda_j),\,(1+\lambda_j),\,q_j};\ j=1,2 \text{ where } \lambda_j>0,\ p_j>\frac{1-\lambda_1\lambda_2}{1+\lambda_j}$$

$\lambda_1\lambda_2<1$, $1/p_1+1/p_2=1$, $p_j>1$ and

$$q_j\left[\frac{1-\lambda_1\lambda_2}{p_j(1+\lambda_1)(1+\lambda_2)}+\frac{\lambda_k}{1+\lambda_k}\right]=1,\ j=1,\ k=2;\ j=2,\ k=1,$$

Then if $\|B(x)\|<\infty$, the solutions are stable with respect to the perturbation B(x).

Proof : Let $\phi_i(x)$, $\theta_j(x)$, i, j = 1, 2 be the fundamental set of solutions of (1.1) and $\Psi(x)=(\Psi_1(x),\Psi_2(x))^T$ be any solution of (1.2).

Let M, N, K be some positive constants such that $\|B(x)\|\leqslant N$;

$$M=\max.\left\{\int_0^\infty|\phi_i|^{p_1(1+\lambda_1)}dt,\ \int_0^\infty|\phi_i|^{(1+\lambda_1)}dt,\right.$$

$$\left.\int_0^\infty|\phi_i|^{q_1}dt,\int_0^\infty|\theta_j|^{p_1(1+\lambda_1)}dt,\ \int_0^\infty|\theta_j|^{(1+\lambda_1)}dt,\ \int_0^\infty|\theta_j|^{q_1}dt,\right\},$$

$K=\max.\ |C_r|$, r = 1, 2, 3, 4 ; and

$$Q(x)=\max.\left\{\int_0^x|\Psi_i|^{p_1(1+\lambda_1)}dt,\ \int_0^x|\Psi_i|^{(1+\lambda_1)}dt,\right.$$

$$\left.\int_0^x|\Psi_i|^{q_1}dt\right\},\text{ say; i, j, l = 1, 2.}$$

Applying the inequality (a) of art. 4 it follows that

$$\left|\int_0^x(B_2,\Psi(t))\,V_1\,dt\right|\leqslant 2M_1N\,[Q(x)]^{1/q_1}\sum_{i=1}^4|u_i(x)|,$$

where $M_1=M^{1/q_2}$.

and $$\left|\int_0^x(B_1,\Psi(t))\,U_1\,dt\right|\leqslant 2M_1.\,N\,[Q(x)]^{1/q_1}\cdot\sum_{i=1}^4|u_i(x)|$$

From (2.2) it now follows that

$$|\Psi_1(x)| \leqslant N.\ \sum_{i=1}^{4} |u_i(x)| + N_1 [Q(x)]^{1/q_1} \ .\ \sum_{i=1}^{4} |u_i(x)|$$

where $N_1 = 4\,M_1\ N$; with a similar expression for $|\Psi_2(x)|$.

Now by the given condition $q_i > 1$. Therefore raising each side to the power q_i, we integrate between $[o, \infty)$ and then apply Minkowski's inequality. It then follows that

$$Q(x) \leqslant 4\,k.\ M + N_2.\ \int_o^x Q(t).\ \left(\sum_{i=1}^{4} |u_i(t)|^{q_1}\right) dt$$

where $N_2 = \max \left\{ N_i^{\ q_i},\ i = 1, 2 \right\}$.

Applying corollary 1 of the Lemma of art. 3, it follows that

$$Q(x) \leqslant 4\,k.\ M.\ \exp.\left\{ N_2.\ \int_o^x \left(\sum_{i=1}^{4} |u_i(t)|^{q_1} \right) dt \right\}$$

Hence the theorem.

By using Hölder's inequality for two functions, we obtain similarly,

Theorem 2 Let all the solutions of (1.1) belong to $L^{\alpha, \beta}[o, \infty)$ $\alpha, \beta > 1$, $1/\alpha + 1/\beta = 1$, then these solutions are stable with respect to the perturbation B(x), if $\| B(x) \| < \infty$.

In particular, when $\alpha = \beta = 2$, we have

Theorem 3 Let all the solutions of (1.1) be $L^2 [o, \infty)$, and $\| B(x) \| < \infty$. Then all the solutions of (1.1) are stable with respect to the perturbation B(x).

Theorem 4 Let all solutions of (1.1) belong to $L^{p,\, q,\, \frac{1}{2}pq,\, \gamma}$ $p, q, \gamma > 1$ $1/p + 1/q = i$. Then if $\gamma.\ [1/p^2 + 1/q^2] = 1$, all solutions of (1.1) are stable with respect to B(x), when $\| B(x) \| < \infty$.

The proof follows as in theoren I, by making use of the inequality (b) cf art 4 instead ot the inequality (a).

Theorem 5 Let all the solutions of (1.1) belong to

$L^{\alpha,\, \beta,\, p,\, q,\, \gamma}$; $\alpha, \beta, \gamma, p, q > 0$, $1/\alpha + 1/\beta = 1/p + 1/q = 1$.

Then if $\gamma.\ [1/\alpha p + 1/\beta q] = 1$, all solutions of (1.1) are stable with respect to B (x) provided $\| B(x) \| < \infty$,

The proof follows as in theorem - 1, but we now use the inequality (c) of art 4.

In the same order of ideas we have

Theorem 6 Let all the solutions of (1.1) belong to $L^{\alpha,\beta}[o, \infty)$, then these solutions are stable with respect to a perturbation B(x), provided that $B(x) \in L^{\gamma}[o, \infty)$, where α, β, $\gamma > 1$ with $1/\alpha+1/\beta+1/\gamma=1$.

Proof : Let $\phi_i(x)$, $\theta_j(x)$, i, j = 1, 2 be the fundamental set of solutions of (1.1). By the problem, for some positive constant M, let

$$M = \max.\left\{\int_o^\infty \| B(t) \|^{\gamma} dt, \int_o^\infty | \phi_i(t) |^{\alpha} dt, \int_o^\infty | \phi_i(t) |^{\beta} dt,\right.$$

$$\left.\int_o^\infty | \theta_j(t) |^{\alpha} dt, \int_o^\infty | \theta_j(t) |^{\beta} dt\right\}; \quad i, j = 1, 2.$$

Let $\Psi(x) = (\Psi_1 (x), \Psi_2 (x))^T$ be any solution of (1.2). From (2.2), applying Hölder's inequality for three functions [Titchmarsh-(3), P-394, Ex-8] and by the stated conditions we have

$$| \Psi_1(x) | \leqslant \sum_{i=1}^{4} | C_i | \quad | U_i(x) | + K\left[| \Psi_1 |^{\alpha}_{o,x} + | \Psi_2 |^{\alpha}_{o,x} \right] \cdot \left(\sum_{i=1}^{4} | u_i(x) | \right).$$

where $K = 2M^{(1/\gamma+1/\beta)}$, with a similar inequality for $| \Psi_2(x) |$.

Raise each side to the power α, apply the inequality

$$\left[M_n^{(1)}(a) \right]^m \leqslant M_n^{(m)}(a), \quad m > 1, \text{ where } M_n^{(\mu)}(a) = \frac{\Sigma a^{\mu}}{n}, \; a \geqslant 0,$$

and then integrate over (0, x), so as to obtain

$$\int_o^x | \Psi_1(t) |^{\alpha} dt \leqslant 12 \left[\sum_{i=1}^{4} | C_i | . \int_o^x | u_i(t) |^{\alpha} dt \right.$$

$$\left. + K^{\alpha} \sum_{i=1}^{4} \int_o^x | u_i(t) |^{\alpha} \left(\int_o^t | \Psi_1(t_1) |^{\alpha} dt_1 + \int_o^t | \Psi_2(t_1) |^{\alpha} dt_1 \right) dt \right]$$

with a similar inequality for $\int_o^x | \Psi_2(t) |^{\alpha} dt$ and $\int_o^x | \Psi_j(t) |^{\beta} dt$; j = 1, 2.

The theorem is now a consequence of the cor. 1 of Lemma of art. 3.

Theorem 7 Let all solutions of (1.1) be $L^{pq}[o, \infty)$, then these solutions are stable with respect to the perturbation B(x) provided that $B(x) \in L^{p,q}[o, \infty)$, p, q > 0, 1/p + 1/q = 1, pq > 1.

The theorem follows by applying the inequality (d) of art 4.

Theorem 8 Let all the solutions of (1.1) belong to $L^{\alpha, \beta, p, q\gamma, \eta}$.

Then the solutions are stable with respect to B(x), if $B(x) \in L^{\beta \delta q}$ where

$$p, q, \alpha, \beta, \gamma, \delta > 0 ; \qquad 1/p + 1/q = 1/\alpha + 1/\beta = 1/\gamma + 1/\delta = 1$$

and $\eta\,[\,1/q\alpha + 1/\rho\beta\,] = 1$.

The theorem follows as in theorem 1 by making use of the inequality (e) of art 4.

REFERENCES

[1] Bellman, R —Stability theory of differential equations (Mc Graw Hill) (1953)

[2] Gronwall, T, H. —Note on the derivatives with respect to a parameter of the solutions of a system of differential equations, Ann. of Math. (2), 20 (1919), 292-296.

[3] Titchmarsh, E. C.—Theory of functions (Oxford) (1962)

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Dept. of Pure Math.
Calcutta University

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SOME GENERATING FUNCTIONS OF MODIFIED LAGUERRE POLYNOMIALS

BANDANA GHOSH and S. K. CHATTERJEA

1. **Intoduction :** In a recent paper [1], C. C. Feng has derived the following main generating relation involving modified Laguerre polynomials :

$$(1.1) \quad \exp\left(-a_{23}xz\right)\exp\left(-a_{12}/y\,(1+a_{13}y+a_{23}z)\right)(1+a_{13}y+a_{23}z)^{-n-\beta}\,.$$

$$\left[f_n^{(\beta)}(x+a_{12}/y+a_{22}/z)\,(1+a_{13}y+a_{23}z)\right]$$

$$=\sum_{p=0}^{\infty}\frac{(a_{23})^p}{p!}\sum_{m=0}^{\infty}\frac{(a_{13})^m}{m!}\sum_{l=0}^{\infty}\frac{(a_{22})^l}{l!}\sum_{k=0}^{\infty}\frac{(a_{12})^k}{k!}(-1)^{k+m+p}(\beta-k)_m\,(n-l+1)_p$$

$$.\,f_{n-l+p}^{(\beta-k+m)}(x)\,y^{m-k}\,z^{p-l}\,,$$

by replacing β by $y\frac{\partial}{\partial y}$, n by $z\frac{\partial}{\partial z}$ and $f_n^{(\beta)}(x)$ by $u(x, y, z)$ in the linear differential relation

$$(1.2) \quad xD^2 f_n^{(\beta)}(x) + (1-x-n-\beta)\,D f_n^{(\beta)}(x) + n f_n^{(\beta)}(x) = o\,,$$

and by following the method of L. Weisner [2].
It may be of interest to remark that the result (3.2) of Feng is not correct owing to the fact that

$$(1.3) \quad \exp\left[\sum_{i=1}^{2}\left(a_{i3}A_{i3}+a_{i2}A_{i2}+a_{i1}A_{i1}\right)\right]$$

$\neq \exp(a_{13}A_{13})\exp(a_{12}A_{12})\exp(a_{11}A_{11}).\ \exp(a_{23}A_{23})\exp(a_{22}A_{22})\exp(a_{21}A_{21})$,

by virtue of commutator rules of A_{ij} ($i=1,\ 2$; $j=1,\ 2,\ 3$) and therefore his statement, viz. "The order of A_{i3}, A_{i2}, A_{i1} can not be changed for $i=1,\ 2$

respectively" bears no meaning in connection with his formula (3.2). Actually Feng has calculated the left hand side of (3.2) by using the operator mentioned in the right side of (1.3), which is obvious from the result (4.1) of Feng. Thus the order of A_{13}, A_{12}, A_{11} can be changed at ease without altering its effect in the left member of (1.3), while that can not be changed in the right member of (1.3). The main result (1.1) of Feng is obtained by applying the operator

$$e^{a_{23} A_{23}}\, e^{a_{13} A_{13}}\, e^{a_{22} A_{22}}\, e^{a_{12} A_{12}} \text{ on } f_n^{(\beta)}(x)\, y^{\beta}\, z^{n}.$$

The object of this paper is to derive the possible variants of (1.1) by changing the order of A_{23}, A_{13}, A_{22} and A_{12} in

$$e^{a_{23} A_{23}}\, e^{a_{13} A_{13}}\, e^{a_{22} A_{22}}\, e^{a_{12} A_{12}}.$$

Indeed, by this change we have found the following new generating functions :

$$(1.4)\quad \exp(-a_{23}xz)\exp\left(-a_{12}(1+a_{23}z)/y\right)(1+a_{13}y+a_{23}z)^{-n-\beta}.$$

$$.\, f_n^{(\beta)}\left[(x+a_{12}/y+a_{22}/z)(1+a_{13}y+a_{23}z)\right]$$

$$=\sum_{p=0}^{\infty}\frac{(a_{23})^p}{p!}\sum_{k=0}^{\infty}\frac{(a_{12})^k}{k!}\sum_{m=0}^{\infty}\frac{(a_{13})^m}{m!}\sum_{l=0}^{\infty}\frac{(a_{22})^l}{l!}(-1)^{k+m+p}(\beta)_m\,(n-l+1)_p\,.$$

$$.\, f_{n-l+p}^{(\beta-k+m)}(x)\, y^{m-k}\, z^{p-l}$$

$$(1.5)\quad \exp(-a_{12}/y)\exp\left(-a_{23}z(x+a_{12}/y+a_{22}/z)\right)(1+a_{13}y+a_{23}z)^{-n-\beta}.$$

$$.\, f_n^{(\beta)}\left[(x+a_{12}/y+a_{22}/z)(1+a_{13}y+a_{23}z)\right]$$

$$=\sum_{l=0}^{\infty}(a_{22}^l/l!)\sum_{p=0}^{\infty}(a_{23}^p/p!)\sum_{k=0}^{\infty}(a_{12}^k/k!)\sum_{m=0}^{\infty}(a_{13}^m/m!)(-1)^{k+m+p}(\beta)_m(n+1)_p\,.$$

$$.\, f_{n-l+p}^{(\beta-k+m)}(x)\, y^{m-k}\, z^{p-l}$$

$$(1.6)\quad \exp\left(-a_{23}z(x+a_{22}/z)\right)\exp\left[-a_{12}(1+a_{13}y+a_{23}z)/y\right](1+a_{13}y+a_{23}z)^{-n-\beta}.$$

$$.\, f_n^{(\beta)}\left[(x+a_{12}/y+a_{22}/z)(1+a_{13}y+a_{23}z)\right]$$

$$= \sum_{l=0}^{\infty} \frac{(a_{22})^l}{l!} \sum_{p=0}^{\infty} \frac{(a_{23})^p}{p!} \sum_{m=0}^{\infty} \frac{(a_{13})^m}{m!} \sum_{k=0}^{\infty} \frac{(a_{12})^k}{k!} (-1)^{k+m+p} (\beta-k)_m (n+1)_p \,.$$

$$. \, f_{n-l+p}^{(\beta-k+m)}(x) \, y^{m-k} \, z^{p-l} \,.$$

We like to point out that the following six operators

$$(A) \qquad \begin{cases} e^{a_{23}A_{23}} \, e^{a_{13}A_{13}} \, e^{a_{22}A_{22}} \, e^{a_{12}A_{12}} \\ e^{a_{23}A_{23}} \, e^{a_{13}A_{13}} \, e^{a_{12}A_{12}} \, e^{a_{22}A_{22}} \\ e^{a_{23}A_{23}} \, e^{a_{22}A_{22}} \, e^{a_{13}A_{13}} \, e^{a_{12}A_{12}} \\ e^{a_{13}A_{13}} \, e^{a_{12}A_{12}} \, e^{a_{23}A_{23}} \, e^{a_{22}A_{22}} \\ e^{a_{13}A_{13}} \, e^{a_{23}A_{23}} \, e^{a_{12}A_{12}} \, e^{a_{22}A_{22}} \\ e^{a_{13}A_{13}} \, e^{a_{23}A_{23}} \, e^{a_{22}A_{22}} \, e^{a_{12}A_{12}} \end{cases}$$

when applied to $f_n^{(\beta)}(x) \, y^{\beta} \, z^{n}$ will give rise to the result (1.1) of Feng.

On the other hand the following six operators

$$(B) \qquad \begin{cases} e^{a_{23}A_{23}} \, e^{a_{12}A_{12}} \, e^{a_{13}A_{13}} \, e^{a_{22}A_{22}} \\ e^{a_{23}A_{23}} \, e^{a_{12}A_{12}} \, e^{a_{22}A_{22}} \, e^{a_{13}A_{13}} \\ e^{a_{23}A_{23}} \, e^{a_{22}A_{22}} \, e^{a_{12}A_{12}} \, e^{a_{13}A_{13}} \\ e^{a_{12}A_{12}} \, e^{a_{23}A_{23}} \, e^{a_{22}A_{22}} \, e^{a_{13}A_{13}} \\ e^{a_{12}A_{12}} \, e^{a_{23}A_{23}} \, e^{a_{13}A_{13}} \, e^{a_{22}A_{22}} \\ e^{a_{12}A_{12}} \, e^{a_{13}A_{13}} \, e^{a_{23}A_{23}} \, e^{a_{22}A_{22}} \end{cases}$$

when applied to $f_n^{(\beta)}(x)\, y^{\beta} z^{n}$ will give rise to the result (1.4).

Again the following six operators

$$
(C)\quad \left\{
\begin{array}{l}
e^{a_{22}A_{22}}\; e^{a_{23}A_{23}}\; e^{a_{12}A_{12}}\; e^{a_{13}A_{13}} \\
e^{a_{22}A_{22}}\; e^{a_{12}A_{12}}\; e^{a_{13}A_{13}}\; e^{a_{23}A_{23}} \\
e^{a_{22}A_{22}}\; e^{a_{12}A_{12}}\; e^{a_{23}A_{23}}\; e^{a_{13}A_{13}} \\
e^{a_{12}A_{12}}\; e^{a_{13}A_{13}}\; e^{a_{22}A_{22}}\; e^{a_{23}A_{23}} \\
e^{a_{12}A_{12}}\; e^{a_{22}A_{22}}\; e^{a_{13}A_{13}}\; e^{a_{23}A_{23}} \\
e^{a_{12}A_{12}}\; e^{a_{22}A_{22}}\; e^{a_{23}A_{23}}\; e^{a_{13}A_{13}}
\end{array}
\right.
$$

when applied to $f_n^{(\beta)}(x)\, y^{\beta} z^{n}$ will give rise to the result (1.5).

Lastly the following six operators

$$
(D)\quad \left\{
\begin{array}{l}
e^{a_{22}A_{22}}\; e^{a_{23}A_{23}}\; e^{a_{13}A_{13}}\; e^{a_{12}A_{12}} \\
e^{a_{22}A_{22}}\; e^{a_{13}A_{13}}\; e^{a_{12}A_{12}}\; e^{a_{23}A_{23}} \\
e^{a_{22}A_{22}}\; e^{a_{13}A_{13}}\; e^{a_{23}A_{23}}\; e^{a_{12}A_{12}} \\
e^{a_{13}A_{13}}\; e^{a_{22}A_{22}}\; e^{a_{23}A_{23}}\; e^{a_{12}A_{12}} \\
e^{a_{13}A_{13}}\; e^{a_{22}A_{22}}\; e^{a_{12}A_{12}}\; e^{a_{23}A_{23}} \\
e^{a_{13}A_{13}}\; e^{a_{12}A_{12}}\; e^{a_{22}A_{22}}\; e^{a_{23}A_{23}}
\end{array}
\right.
$$

when applied to $f_n^{(\beta)}(x)\, y^{\beta} z^n$ will give rise to the result (1.6).

2. Derivation of the new generating functions :

From [1, p. 190] we notice that

$$(2.1)\quad A_{12}\left[f_n^{(\beta)}(x)\, y^{\beta} z^n\right] = -f_n^{(\beta-1)}(x)\, y^{\beta-1} z^n$$

$$(2.2)\quad A_{13}\left[f_n^{(\beta)}(x)\, y^{\beta} z^n\right] = -\beta f_n^{(\beta+1)}(x)\, y^{\beta+1} z^n$$

$$(2.3)\quad A_{22}\left[f_n^{(\beta)}(x)\, y^{\beta} z^n\right] = f_{n-1}^{(\beta)}(x)\, y^{\beta} z^{n-1}$$

$$(2.4)\quad A_{23}\left[f_n^{(\beta)}(x)\, y^{\beta} z^n\right] = -(n+1) f_{n+1}^{(\beta)}(x)\, y^{\beta} z^{n+1}$$

where

$$A_{12} = y^{-1}\partial/\partial x - y^{-1}$$

$$A_{13} = xy\,\partial/\partial x - y^2\,\partial/\partial y - yz\,\partial/\partial z$$

$$A_{22} = z^{-1}\,\partial/\partial x$$

$$A_{23} = xz\,\partial/\partial x - yz\,\partial/\partial y - z^3\,\partial/\partial z - xz.$$

Also the groups corresponding to A_{12}, A_{13}, A_{22} and A_{23} are given by [1, p. 192] :

$$(2.5)\quad e^{a_{12}A_{12}}\, u(x, y, z) = e^{a_{12}/y}\, u(a_{12}/y + x, y, z)$$

$$(2.6)\quad e^{a_{13}A_{13}}\, u(x, y, z) = u(x(1 + a_{13}y),\ y/(1 + a_{13}y),\ z/(1 + a_{13}y))$$

$$(2.7)\quad e^{a_{22}A_{22}}\, u(x, y, z) = u(a_{22}/z + x, y, z)$$

$$(2.8)\quad e^{a_{23}A_{23}}\, u(x, y, z) = e^{-a_{23}xz}\, u(x(1 + a_{23}z),\ y/(1 + a_{23}z),\ z/(1+a_{23}z))$$

Now in order to prove (1.4) we can choose without any loss of generality the following operator from the set (B) :

$$e^{a_{23}A_{23}}\, e^{a_{12}A_{12}}\, e^{a_{13}A_{13}}\, e^{a_{22}A_{22}}.$$

In fact, we have

$$e^{a_{23}A_{23}}\, e^{a_{12}A_{12}}\, e^{a_{13}A_{13}}\, e^{a_{22}A_{22}} \left[f_n^{(\beta)}(x)\, y^{\beta} z^{n} \right]$$

$$= e^{a_{23}A_{23}}\, e^{a_{12}A_{12}}\, e^{a_{13}A_{13}} \sum_{l=0}^{\infty} (a_{22}^{l}/l!)\, f_{n-l}^{(\beta)}(x)\, y^{\beta} z^{n-l}$$

$$= e^{a_{23}A_{23}}\, e^{a_{12}A_{12}} \sum_{m=0}^{\infty} (a_{13}^{m}/m!) \sum_{l=0}^{\infty} (a_{22}^{l}/l!)\, (-1)^{m} (\beta)_{m} \cdot f_{n-l}^{(\beta+m)}(x)\, y^{\beta+m} z^{n-l}$$

$$= e^{a_{23}A_{23}} \sum_{k=0}^{\infty} (a_{12}^{k}/k!) \sum_{m=0}^{\infty} (a_{13}^{m})/m! \sum_{l=0}^{\infty} (a_{22}^{l})/l!\, (\beta)_{m} (-1)^{k+m}\, f_{n-l}^{(\beta+m-k)}(x)\, y^{\beta+m-k} z^{n-l}$$

$$(2.9) \quad = \sum_{p=0}^{\infty} (a_{23}^{p})/p! \sum_{k=0}^{\infty} (a_{12}^{k})/k! \sum_{m=0}^{\infty} (a_{13}^{m}/m!) \sum_{l=0}^{\infty} (a_{22}^{l})/l!\, (-1)^{k+m+p}\, (\beta)_{m}\, (n-l+1)_{p}\, f_{n-l+p}^{(\beta-k+m)}(x)\, y^{\beta-k+m} z^{n-l+p}$$

On the otherhand we have,

$$e^{a_{23}A_{23}}\, e^{a_{12}A_{12}}\, e^{a_{13}A_{13}}\, e^{a_{22}A_{22}} \left[f_n^{(\beta)}(x)\, y^{\beta} z^{n} \right]$$

$$= e^{a_{23}A_{23}}\, e^{a_{12}A_{12}}\, e^{a_{13}A_{13}} \left[f_n^{(\beta)}(x + a_{22}/z)\, y^{\beta} z^{n} \right]$$

$$= e^{a_{23}A_{23}}\, e^{a_{12}A_{12}} \left[f_n^{(\beta)}((x + a_{22}/z)(1 + a_{13}y))\, \frac{y^{\beta} z^{n}}{(1 + a_{13}y)^{\beta+n}} \right]$$

$$= e^{a_{23}A_{23}} \left[e^{-a_{12}/y}\, f_n^{(\beta)}((x + a_{12}/y + a_{22}/z)(1 + a_{13}y)) \cdot \frac{y^{\beta} z^{n}}{(1 + a_{13}y)^{\beta+n}} \right]$$

$$= e^{-a_{23}xz}\, e^{-a_{12}/y}\, (1 + a_{23}z) \cdot f_n^{(\beta)} \left[(x + a_{12}/y + a_{22}/z)(1 + a_{13}y + a_{23}z) \right] \frac{y^{\beta} z^{n}}{(1 + a_{13}y + a_{23}z)^{\beta+n}}$$

$$(2.10) \quad = e^{-a_{23}xz}\, e^{-a_{12}(1+a_{23}z)/y}\,(1+a_{13}y+a_{23}z)^{-\beta-n}$$
$$.\, f_n^{(\beta)}\left[(x+a_{12}/y+a_{22}/z)\;(1+a_{13}y+a_{23}z)\right] y^{\beta} z^{n}$$

Equating (2.9) and (2.10) we get

$$(2.11) \quad \exp(-a_{23}xz)\,\exp(-a_{12}/y\,(1+a_{23}z))\,(1+a_{13}y+a_{23}z)^{-n-\beta}$$
$$.f_n^{(\beta)}\left[(x+a_{12}/y+a_{22}/z)\;(1+a_{12}y+a_{23}z)\right]$$
$$= \sum_{p=0}^{\infty}(a_{23})^p/p!\sum_{k=0}^{\infty}(a_{12})^k/k!\sum_{m=0}^{\infty}(a_{13})^m/m!\sum_{l=0}^{\infty}(a_{22})^l/l!\,(-1)^{k+m+p}$$
$$(\beta)_m\,(n-l+1)_p\,.\,f_{n-l+p}^{(\beta-k+m)}(x)\,y^{m-k}\,z^{p-l},$$

which is (1.4)

Secondly in order to prove (1.5) we can choose without any loss of generality the following operator from the set (C) ;

$$e^{a_{22}A_{22}}\;e^{a_{12}A_{12}}\;e^{a_{13}A_{13}}\;e^{a_{23}A_{23}}.$$

Since the calculation shown in deriving (1.4) is a routine one, we only mention the main steps in deriving (1.5).

Indeed, we have

$$e^{a_{22}A_{22}}\,e^{a_{12}A_{12}}\,e^{a_{13}A_{13}}\,e^{a_{23}A_{23}}\left[f_n^{(\beta)}(x)\,y^{\beta}z^{n}\right]$$
$$(2.12) \quad = \sum_{l=0}^{\infty}(a_{22})^l/l!\sum_{k=0}^{\infty}(a_{12})^k/k!\sum_{m=0}^{\infty}(a_{13})^m/m!\sum_{p=0}^{\infty}(a_{23})^p/p!\;(-1)^{k+m+p}.$$
$$.\,(\beta)_m\,(n+1)_p\,f_{n+p-l}^{(\beta+m-k)}(x)\,y^{\beta+m-k}\,z^{n+p-l}$$

On the otherhand, we have

$$e^{a_{22}A_{22}}\,e^{a_{12}A_{12}}\,e^{a_{13}A_{13}}\,e^{a_{23}A_{23}}\left[f_n^{(\beta)}(x)\,y^{\beta}z^{n}\right]$$
$$(2.13) \quad = e^{-a_{12}/y}\,e^{-a_{23}z(x+a_{12}/y+a_{22})/z}\,(1+a_{13}y+a_{23}z)^{-\beta-n}\,.$$
$$.f_n^{(\beta)}\left[(x+a_{12}/y+a_{22}/z)\;(1+a_{13}y+a_{23}z)\right] y^{\beta} z^{n}$$

Equating (2.12) and (2.13) we get

$$\exp\left(-a_{12}/y\right)\ \exp\left(-a_{23}z\ \left(x+a_{12}/y+a_{22}/z\right)\right)\ \left(1+a_{13}y+a_{23}z\right)^{-\beta-n}.$$

$$.\, f_n^{(\beta)}\left[\left(x+a_{12}/y+a_{22}/z\right)\ \left(1+a_{13}y+a_{23}z\right)\right]$$

$$(2.14)\ =\sum_{l=0}^{\infty}\left(a_{22}^{l}/l!\right)\ \sum_{k=0}^{\infty}\left(a_{12}^{k}/k!\right)\ \sum_{m=0}^{\infty}\left(a_{13}^{m}/m!\right)\ \sum_{p=0}^{\infty}\left(a_{23}^{p}/p!\right)(-1)^{k+m+p}.$$

$$.\,(\beta)_m\ (n+1)_p\ f_{n-l+p}^{\beta-k+m}(x)\ y^{m-k}\ z^{p-l},$$

which is (1.5).

Lastly in order to prove (1.6) we can choose without any loss of generality the following operator from the set (D) :

$$e^{a_{22}A_{22}}\ e^{a_{23}A_{23}}\ e^{a_{13}A_{13}}\ e^{a_{12}A_{12}}.$$

Hence we have

$$e^{a_{22}A_{22}}\ e^{a_{23}A_{23}}\ e^{a_{13}A_{13}}\ e^{a_{12}A_{12}}\left[f_n^{(\beta)}(x)\, y^{\beta}z^{n}\right]$$

$$(2.15)\ =\sum_{l=0}^{\infty}\left(a_{22}^{l}/l!\right)\ \sum_{p=0}^{\infty}\left(a_{23}^{p}/p!\right)\ \sum_{m=0}^{\infty}\left(a_{13}^{m}/m!\right)\ \sum_{k=0}^{\infty}\left(a_{12}^{k}/k!\right)\ (-1)^{k+m+p}.$$

$$.\,(\beta-k)_m\ (n+1)_p\ f_{n-l+p}^{\beta-k+m}(x)\ y^{\beta-k+m}\ z^{n-l+p}$$

On the otherhand, we have

$$e^{a_{22}A_{22}}\ e^{a_{23}A_{23}}\ e^{a_{13}A_{13}}\ e^{a_{12}A_{12}}\left[f_n^{(\beta)}(x)\, y^{\beta}z^{n}\right]$$

$$(2.16)\qquad -e^{a_{23}z\,(x+a_{22}/z)}\ -e^{a_{12}(1+a_{13}y+a_{23}z)/y}\ \left(1+a_{13}y+a_{23}z\right)^{-\beta-n}.$$

$$.\ f_n^{(\beta)}\left[\left(x+a_{12}/y+a_{22}/z\right)\ \left(1+a_{13}y+a_{33}z\right)\right]y^{\beta}z^{n}$$

Equating (2.15) and (2.16) we get,

$$\exp\left(-a_{22}z\ \left(x+a_{22}/z\right)\right)\ \exp\left(-a_{12}\left(1+a_{13}y+a_{23}z\right)/y\right)\ \left(1+a_{13}y+a_{23}z\right)^{-\beta-n}.$$

$$.\, f_n^{(\beta)}\left[\left(x+a_{12}/y+a_{22}/z\right)\ \left(1+a_{13}y+a_{23}z\right)\right]$$

$$(2.17) \quad = \sum_{l=0}^{\infty} (a_{22}^{l} / l\,!) \sum_{p=0}^{\infty} (a_{23}^{p} / p\,!) \sum_{m=0}^{\infty} (a_{13}^{m} / m\,!) \sum_{k=0}^{\infty} (a_{13}^{k} / k\,!) \, (-1)^{k+m+p} .$$

$$\cdot \, (\beta - k)_m \, (n + 1)_p \, f_{n-l+p}^{\beta-k+m} (x) \, y^{m-k} \, z^{p-l} ,$$

which is (1.6).

REFERENCES ;

[1] Peng C. C.—Some generating functions of modified Laguerre polynomials, Hakkaido Math. J. 7 (1978), 189-197

[2] Weisner L.—Group theoretic origin of certain generating functions, Pacific J. Math. 5 (1955) 1033-1039.

Dakshin Barasat Sri Sri Saradamani Balika Vidyalaya
24 Parganas
and
Dept. of Pure Math.
Calcutta University.

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FIXED POINT THEOREMS

K. M. GHOSH

Recently M. Dutta, M. K. Das and M. Majumder [2] and subsequently M. Majumder [4] have formulated the contraction principle in weak metric space considering the classical contraction i. e. Banach's contraction, Kannan's contraction [3] and contraction of Chatterjea [1]. Here we are going to formnlate a more general contraction principle in weak metric space.

For ready reference we first state some definitions.

Definition 1. A weak metric space (M, ρ) is a set M with a metric ρ satisfying the following conditions :

(1.1) $\rho(x, y) = 0$, if $x = y$,

(1.2) $\rho(x, z) \leqslant \rho(x, y) + \rho(y, z)$, $x, y, z \in M$.

In this space, the symmetric property need not hold and moreover $\rho(x, y) = 0$ does not necessarily imply $x = y$.

Definition 2. A sequence $\{x_n\}$ in (M, ρ) is said to be d-fundamental (r-fundamental) if for an arbitrarily given positive ϵ, there exits a positive integer $N_d(\epsilon)$ (an $N_r(\epsilon)$) such that $\rho(x_n, x_m) < \epsilon$, for $m > n > N_d(\epsilon)$, $((n > m > N_r(\epsilon))$ respectively.

Definition 3. A sequence $\{x_n\}$ in (M, ρ) in said to be d-convergent (r-convergent) to a d-limit (r-limit) $x \in M$ if for an arbitrary ϵ, there exists a positive integer $N_d(\epsilon)$, $(N_r(\epsilon))$ such that $\rho(x_n, x) < \epsilon$, for $n > N_d(\epsilon)$

$$(\rho(x, x_n) < \epsilon, \text{ for } n > N_r(\epsilon)) \text{ respectively.}$$

Definition 4. A weak metric space (M, ρ) is said to be d-complete (r-complete) if every d-fundamental (r-fundamental) sequence is d-convergent (r-convergent) respectively.

Definition 5. A mapping T of a weak metric space (M, ρ) into itself is said to be orbitally continuous if for every $x_o \in M$, $T(T^{n_i} x_o) \to Tp$ whenever $T_o^{n_i} x_o \to p \in M$ in d-convergent (r-convergent) sense.

Definition 6. Let T be a mapping of a weak metric space (M, ρ) into itself. Metric space M is said to be 'T-orbitally d-complete' (T-orbitally r-complete') if every sequence $\{T^{n_i} x\}$, $x \in M$, which is a d-fundamental sequence (r-fundamental sequence) is d-convergent (r-convergent) and converges to a point in M.

Now we are in a position to prove the following fixed point theorems which are generalizations of those of [1] and [2].

Theorem 1. Let (M, ρ) be a weak metric space which is a "T_2-space" and let T be a mapping of M into itself. If T be orbitally continuous, M is T-orbitally r-complete and for any x, y $\in$ M, there exists real numbers a_i $(i=1, 2, 3, 4)$ with $0 \leqslant a_i < 1$ and $a_1 + a_2 + a_3 + a_4 < 1$ such that

$$(1.3) \quad \rho(Tx, Ty) \leqslant a_1 \rho(x, y) + a_2 \rho(Tx, x) + a_3 \rho(Ty, y) + a_4 (\rho(Tx, y) + \rho(Ty, x))$$

holds, then T has a fixed point in M.

Proof. Let $x_0 \in M$. Put $x_n = Tx_{n-1}$ $(n = 1, 2, 3, \dots\dots)$

Now $\rho(x_2, x_1) = \rho(Tx_1, Tx_0)$

$$\leqslant a_1 \rho(x_1, x_0) + a_2 \rho(x_2, x_1) + a_3 \rho(x_1, 0) + a_4 (\rho(x_2, 0) + \rho(x_1, x_1))$$

$\therefore \rho(x_2, x_1) \leqslant (a_1 + a_3 + a_4)(1 - a_2 - a_4)^{-1} \rho(x_1, x_0) = a \rho(x_1, x_0)$.

where $a = (a_1 + a_3 + a_4)(1 - a_2 - a_4)^{-1} < 1$, by the hypothesis of the theorem.

Similarly, we have, $\rho(x_3, x_2) \leqslant a \rho(x_2, x_1) \leqslant a^2 \rho(x_1, x_0)$.

By induction, we have $\rho(x_{n+1}, x_n) \leqslant a^n \rho(x_1, x_0)$

Now we shall show that this sequence $\{x_n\}$ is r-fundamental. In fact we have for $m > n$,

$$\rho(x_m, x_n) \leqslant \rho(x_m, x_{m-1}) + \rho(x_{m-1}, x_{m-2}) + \dots + \rho(x_{n-1}, x_n).$$
$$\leqslant (a^{m-1} + a^{m-2} + \dots + a^n)\ \rho(x_1, x_0).$$

Therefore, $\rho(x_m, x_n) \leqslant a^n (a^{m-n} - 1)(a - 1)^{-1} \rho(x_1, x_0) \to 0$ as $n \to \infty$.

Hence, $\rho(x_m, x_n) < \epsilon$, whenever $\epsilon > a^n (a^{m-n} - 1)(a - 1)^{-1} \rho(x_1, x_0)$, for $m > n > N_r(\epsilon)$, then clearly $\{T^n x_0\}$ (since $x_n = T^n x_0$) is r-fundamental.

Since (M, ρ) is T-orbitally r-complete, then $x_n \to x^* \in M$ in the sense of r-limit.

Now it remains to show that x^* is a fixed point of T.

Since, T is orbitally continuous,

$$Tx^* = T \lim_{n \to \infty} x_n = \lim_{n \to \infty} x_{n+1} = x^*,$$ which shows that x^* is a fixed point of T.

Theorem 2. Let (M, ρ) be a weak metric space which is "T_2-space" and T be a self-mapping of M into itself. If T be orbitally continuous, M is T-orbitally d-complete and for any $x, y \in M$, there exists real numbers a_i $(i = 1, 2, 3, 4)$ with $0 \leqslant a_i < 1$ and $a_1 + a_2 + a_3 + 2a_4 < 1$ such that

$$(1.4) \quad \rho(Tx, Ty) \leqslant a_1 \rho(x, y) + a_2 \rho(x, Tx) + a_3 \rho(y, Ty) + a_4 (\rho(x, Ty) + \rho(y, Tx))$$

holds, then T has a fixed point in M.

Proof. As the proof of this theorem is exactly similar to that of Theorem 1, so we omit it.

Remark : Theorem 1 of [2] follows as a particular case of our theorem 1 by setting $a_1 = a_4 = 0$ and $a_2 = a_3 = \lambda$. Also theorem 2 of [2] is a particular case of our theorem 2.

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REFERENCES :

[1] Chatterjea S. K., C. R. Acad. Bulgar. Sci. 25 (1972), 727-730. Also vide Research Report No. 2 (1971), Centre of Adv. Studies in Appl. Math. Calcutta University p 12.

[2] Dutta M., Das M. K. and Majumder M., Glasnik Mathematicki 9 (29) (1974), 155-159.

[3] Kannan R., Bull. Cal. Math. Soc. 60 (1968) 71-76,

[4] Majumder M., Bull. Math. Soc. Sci. Math. R. S. Roumanie, 21 (69), nr 3-4, (1977)

H. M. G. College
J-206, Paharpur Road,
Calcutta-700024.

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ON THE INVARIANCE OF THE NATURE OF SINGULARITIES OF THE m-COEFFICIENTS ASSOCIATED WITH AN ORDINARY LINEAR FOURTH-ORDER DIFFERENTIAL EQUATION.

MANJU MAJUMDER and JOYTI DAS (nee CHAUDHURI)

1. Introduction :

To define the m-coefficients of the title we start with the boundary value problem (I) consisting of the differential equation

(1.1) $Ly \equiv (r(x) y^{(2)} (x))^{(2)} - (p(x) y^{(1)} (x))^{(1)} + q(x)y(x) = \lambda y(x), \quad o \leqslant x < \infty$

and the boundary conditions :

(1.2) $U_i y \equiv \sum_{s=o}^{3} a_{is} y^{(s)} (o), \ (i=1, 2)$

where

(i) $y^{(k)} \equiv y^{(k)} (x) = d^k y/dx^k, \ (k = o, 1, 2, 3)$

(ii) r, p, q are real valued on $[o, \infty)$, and $q \in L [o, x]$ for all $x > 0$,

(iii) $r (x) > o$ for all $x \geqslant 0$,

(iv) $r^{(1)}$ and p are absolutely continuous on any compact subinterval of $[o, \infty)$,

(v) a_{is} ($i=1, 2$; $s=o, 1, 2, 3$) are all real,

(vi) $(a_{13} a_{20} - a_{23} a_{10}) + r^{(1)} (o)/r(o) (a_{13} a_{21} - a_{23} a_{11}) +$
$+ p(o)/r(o) (a_{13} a_{22} - a_{23} a_{12}) + (a_{11} a_{22} - a_{12} a_{21}) = 0.$

Two solutions $\phi_i = \phi_i (x, \lambda)$, ($i = 1, 2$) of (1.1) are determined so that

$$
(1.3) \quad \begin{cases} r (o) \phi_i (o, \lambda) = a_{i3} \\ \{ r(o) \}^2 \phi_i^{(1)} (o, \lambda) = r^{(1)} (o) a_{i3} - r(o) a_{i2}, \\ \{ r(o)\}^2 \phi_i^{(2)} (o, \lambda) = p(o) a_{i3} + r(o) a_{i1}, \\ \{ r(o) \}^2 \phi_i^{(3)} (o, \lambda) = - r^{(1)} (o) \{ r(o) a_{i1} + p(o) a_{i3} \} \\ \qquad + p(o) \{ r^{(1)} (o) a_{i3} - r(o) a_{i2} \} - \{ r(o) \}^2 a_{i0} \end{cases}
$$

ϕ_1, ϕ_2 are known as boundary condition functions, since the boundary conditions (1.2) can be exhibited in terms of them as

(1.4) $[y \phi_1] = o = [y \phi_2]$

while the self adjointness condition (vi) of the BVP (I) can be exhibited as

(1.5) $[\phi_1 \phi_2]=o$; in (1.4) and (1.5).

$$[uv](x) = r^{(1)}(x)\,[\,u^{(2)}(x)\,\bar{v}(x) - u(x)\,\bar{v}^{(2)}(x)\,] +$$

$$+\, r(x)\,[\,u^{(3)}(x)\,\bar{v}(x) - u^{(2)}(x)\,\bar{v}^{(1)}(x) + u^{(1)}(x)\,\bar{v}^{(2)}(x) - u(x)\,\bar{v}^{(3)}(x)\,] - \tag{1.6}$$

$$- p(x)\,[\,u^{(1)}(x)\,\bar{v}(x) - u(x)\,\bar{v}^{(1)}(x)\,]\,;$$

$[uv]$ is independent of x when both u and v satisfy (1.1).

Two more solutions $\theta_i \equiv \theta_i(x, \lambda)$, (i = 1, 2) of (1.1) are then determined by

$$\left\{\begin{array}{l} [\,\phi_i\,\theta_s\,] = \delta_{is} \text{ (Kronecker's delta)} \\ [\,\theta_1\,\theta_2\,] = o \end{array}\right. \tag{1.7}$$

(1.7) does not determine θ_j's uniquely, but this is immaterial as far as our analysis is concerned.

It has been proved by W. N. Everitt [1] that (1.1) always has at least two linearly independent solutions $\Psi_i \equiv \Psi_i(x, \lambda)$, (i = 1, 2), belonging to $L^2[o, \infty)$ (the space of all Lebesgue square-integrable functions), which can be expressed in terms of θ's and ϕ's as

$$\Psi_r(x, \lambda) \equiv \theta_r(x, \lambda) + \sum_{s=1}^{2} m_{rs}(\lambda)\,\phi_s(x, \lambda), \quad (r=1, 2), \tag{1.8}$$

and that each $m_{rs}(\lambda)$ is regular in each of the half-planes $\mathrm{Im}\,\lambda > o$, $\mathrm{Im}\,\lambda < o$.
In other words all the singularities of $m_{rs}(\lambda)$ lie on the real λ-axis.

The differential equation (1.1) is classified to be in the limit-2, limit-3 or limit-4 case at infinity according as (1.1) has exactly two, three or four linearly independent solutions belonging to $L^2[o, \infty)$.

We first assume that (1.1) is in the limit-2 case at infinity ; then (1.8) determines ($m_{rs}(\lambda)$) uniquely. These $m_{rs}(\lambda)$, (r, s = 1, 2) are the m-coefficients of the title. Clearly ($m_{rs}(\lambda)$) is dependent on the coefficients r, p, q, as well as, on the boundary conditions (1.2). The object of this note is to show that the nature of the singularities of ($m_{rs}(\lambda)$) is independent of the boundary conditions (1.2). In other words, if the boundary conditions (1.2) is replaced by some other boundary conditions of similar type, then the positions of the singularities of ($m_{rs}(\lambda)$) may change but their nature will remain invariant. For example, if ($m_{rs}(\lambda)$) is meromorphic corresponding to one set of boundary conditions then the m-coefficients corresponding to any other set of boundary conditions, are also meromorphic. As the singularities of ($m_{rs}(\lambda)$) are responsible for the determination of the

spectrum of the boundary value problem, the independence of the nature of the singularities of ($m_{rs}(\lambda)$) as regards boundary conditions will no doubt simplify the problem of determination of the nature of the spectrum of the boundary value problem.

To establish the assertion we consider a second boundary value problem (II) consisting of the same differential equation (1.1) but associated with the boundary conditions

(1.9) $U_r y = \sum_{s=0}^{3} A_{rs}\, y^{(s)}(o) = o\ ;\ (\ r = 1, 2\ )$

where $(A_{13} A_{20} - A_{23} A_{10}) + r^{(1)}(o)/r(o)\ (A_{13} A_{21} - A_{23} A_{11}) +$

$+\ p(o)/r(o)\ (A_{13} A_{22} - A_{23} A_{12}) + (A_{11} A_{22} - A_{21} A_{12}) = o.$

Let us denote the entities with reference to this BVP(II) with k_1, k_2 τ_1, τ_2 Ψ_1, Ψ_2 and M_{rs} (r, s = 1, 2,) respectively.

The corresponding problems, where (1.1) is in the limit-3 or limit-4 case are discussed in brief in section 4. § 2 exhibits the relations between the entities of the two BVPs (I) and (II). § 3 gives the statement of the theorem and sketches its proof.

2. As the solutions θ_1, θ_2, ϕ_1, ϕ_2 of (1.1) form a fundamental set of (1.1), the solutions k_1, k_2, τ_1, τ_2 can be expressed in terms of them.

It can be proved that,

(2.1) $\tau_1 = [\tau_1\,\theta_1]\,\phi_1 + [\tau_1\,\theta_2]\,\phi_2 - [\tau_1\,\phi_1]\,\theta_1 - [\tau_1\,\phi_2]\,\theta_2$

(2.2) $\tau_2 = [\tau_2\,\theta_1]\,\phi_1 + [\tau_2\,\theta_2]\,\phi_2 - [\tau_2\,\phi_1]\,\theta_1 - [\tau_2\,\phi_2]\,\theta_2$

(2.3) $k_1 = [k_1\,\theta_1]\,\phi_1 + [k_1\,\theta_2]\,\phi_2 - [k_1\,\phi_1]\,\theta_1 - [k_1\,\phi_2]\,\theta_2$

(2.4) $k_2 = [k_2\,\theta_1]\,\phi_1 + [k_2\,\theta_2]\,\phi_2 - [k_2\,\phi_1]\,\theta_1 - [k_2\,\phi_2]\,\theta_2$

3. Statement of the Theorem :

If the differential equation (1.1) is in the limit-2 case at infinity, the elements $M_{rs}(\lambda)$ (r, s = 1, 2) are related to the elements $m_{rs}(\lambda)$ (r, s = 1, 2) by the relations given below :

(3.1) $\triangle\, M_{11}(\lambda) = -[\tau_1\,\psi_1]\ [k_2\,\psi_2] + [k_2\,\psi_1]\ [\tau_1\,\psi_2]$

(3.2) $\triangle\, M_{12}(\lambda) = -[k_1\,\psi_1]\ [\tau_1\,\psi_2] + [\tau_1\,\psi_1]\ [k_1\,\psi_2]$

(3.3) $= -[\tau_2\,\psi_1]\ [k_2\,\psi_2] + [k_2\,\psi_1]\ [\tau_2\,\psi_2]$

(3.4) $\triangle\, M_{22}(\lambda) = -[k_1\,\psi_1]\ [\tau_2\,\psi_2] + [\tau_2\,\psi_1]\ [k_1\,\psi_2]$

where

(3.5) $\triangle = [k_1\,\psi_1]\ [k_2\,\psi_2] - [k_2\,\psi_1]\ [k_1\,\psi_2].$

Proof of the Theorem :

If the differential equation (1.1) is in the limit-2 case at infinity, it has exactly

two linearly independent solutions in $L^2\,[\,o, \infty\,)$. Hence the solutions are obtained solely in terms of ψ_1 and ψ_2. We may write

(3.6) $\Psi_1\,(x, \lambda) = J_{11}\,\psi_1\,(x, \lambda) + J_{12}\,\psi_2\,(x, \lambda)$

(3.7) $\Psi_2\,(x, \lambda) = J_{21}\,\psi_1\,(x, \lambda) + J_{22}\,\psi_2\,(x, \lambda)$

where J_{rs} ($r, s = 1, 2$) are uniquely determined functions of λ. These equations also determine $M_{rs}(\lambda)$ uniquely. Thus (3.6), (3.7) may be used to determine the eight elements J_{11}, J_{12}, J_{21}, J_{22}, M_{11}, M_{12}, M_{21}, M_{22} uniquely as follows.

From (1.8) and similar expressions for $\Psi_r\,(x, \lambda)$ ($r = 1, 2$), using the relations

$[\,\theta_1\,\theta_2\,] = o$; $[\,\phi_1\,\phi_2\,] = o$; $[\,\phi_r\,\theta_s\,] = \delta_{rs}$ ($r, s = 1, 2$); $[\,\tau_1\,\tau_2\,] = o$:

$[\,k_1\,k_2\,] = o$; $[\,k_r\,\tau_s\,] = \delta_{rs}$ ($r, s = 1, 2$) we obtain,

(3.8) $[\,\phi_r\,(x, \lambda)\,\psi_s\,(x, \lambda)\,] = \delta_{rs}$ ($r, s = 1, 2$)

(3.9) $[\,\psi_r(x, \lambda)\,\theta_s\,(x, \lambda)\,] = m_{rs}\,(\lambda)$ ($r, s = 1, 2$)

(3.10) $[\,k_r\,(x, \lambda)\,\Psi_s\,(x, \lambda)\,] = \delta_{rs}$ ($r, s = 1, 2$)

and

(3.11) $[\,\Psi_r\,(x, \lambda)\,\tau_s\,(x, \lambda)\,] = M_{rs}\,(\lambda)$ ($r, s = 1, 2$).

Using these relations (3.8 — 3.11) in (3.6) and (3.7) we obtain the following eight linear equations for the determination of J_{rs}, M_{rs} ($r, s = 1, 2$)

$[\,\Psi_1\,\phi_r\,] = J_{1r}$ ($r = 1, 2$)

$[\,\Psi_1\,\theta_r\,] = J_{11}\,m_{1r} + J_{12}\,m_{r2}$ ($r = 1, 2$)

$[\,\Psi_2\,\phi_r\,] = J_{2r}$ ($r = 1, 2$)

$[\,\Psi_2\,\theta_r\,] = J_{21}\,m_{2r} + J_{22}\,m_{r2}$ ($r = 1, 2$)

Since these eight functions are determined uniquely, the rank of the coefficient matrix of the eight linear equations should be eight. Hence the determinant of the coefficient matrix of the eight linear equations should be non zero ; whence $\triangle \neq o$. Then solving the equations by Cramer's rule we obtain (3.1) — (3.4).

4. We have similar type of linear relations between $M_{rs}(\lambda)$ and $m_{rs}(\lambda)$ when (1.1) is in the limit-3 or limit-4 case at infinity. The main difficulty in these two cases is that the number of equations is less than the number of unknown functions of λ. Accordingly we have an infinite number of solutions of the system of linear equations (a set of equations in the limit-3 and equations in the limit-4 case) relating $M_{rs}(\lambda)$ and $m_{rs}(\lambda)$, ($r, s = 1, 2$). However, any one set of solutions will serve our purpose. We shall prove below that,

in these cases also $\triangle \neq o$. The non-vanishing nature of $\triangle$ helps us to determine a set of solutions in each case and so our purpose is served. The number of equations in each of these systems being too large we refrain ourselves from depicting them in this paper. We simply indicate how to show $\triangle \neq o$ in both the cases.

In the limit-3 case at ∞ of (1.1) the three L^2—solutions of (1.1) with reference to BVP(I) and BVP(II) may be written as :

$R_1 \phi_1 + R_2 \phi_2, \ \psi_1, \psi_2 ; \ (R_1, R_2) \neq (o, o)$

and $s_1 k_1 + s_2 k_2, \ \Psi_1, \ \Psi_2 ; \ (s_1, s_2) \neq (o, o)$.

Now the solutions $s_1 k_1 + s_2 k_2, \ \Psi_1, \ \Psi_2$ can be uniquely expressed in terms of $R_1 \phi_1 + R_2 \phi_2, \ \psi_1, \psi_2$ and vice versa. We can therefore write

(4.1) $\qquad \psi_1 = B_1 (s_1 k_1 + s_2 k_2) + C_1 \Psi_1 + D_1 \Psi_2$

(4.2) $\qquad \psi_2 = B_2 (s_1 k_1 + s_2 k_2) + C_2 \psi_1 + D_2 \Psi_2$.

Using (3.10) we have, from (4.1) and (4.2),

(4.3) $\quad C_1 = [k_1 \psi_1] ; \ D_1 = [k_2 \psi_1] ; \ C_2 = [k_1 \psi_2] ; \ D_2 = [k_2 \psi_2]$

Since ψ_1, ψ_2 are linearly independent $M\psi_1 + N\psi_2 = o$ should imply $M = N = 0$.

Again $s_1 k_1 + s_2 k_2$, Ψ_1 and Ψ_2 being linearly independent, $M\psi_1 + N\psi_2 = 0$ will imply

$MC_1 + NC_2 = 0, \ MD_1 + ND_2 = 0$ that is $M [k_1 \psi_1] + N [k_1 \psi_2 = 0$,

$M [k_1 \psi_2 + N [k_2 \psi_2] = 0$

But these should lead to $M = N = 0$ and therefore we get

$\triangle = [k_1 \psi_1] \ [k_2 \psi_2] - [k_1 \psi_2] \ [k_2 \psi_1] \neq 0$

If (1.1) is in the limit-4 case at infinity, all the four linearly independent solutions of (1.1) are in $L^2 [o, \infty)$ and we have

(4.5) $\quad k_1 = [k_1 \psi_1] \phi_1 + [k_1 \psi_2] \phi_2 - [k_1 \phi_1] \psi_1 - [k_1 \phi_2] \psi_2$

(4.6) $\quad k_2 = [k_2 \psi_2] \phi_1 + [k_2 \psi_2] \phi_2 - [k_2 \phi_1] \psi_1 - [k_2 \phi_2] \psi_2$

k_1, k_2 being linearly independent, $P k_1 + Q k_2 = o$ should imply $P = Q = 0$

Again $\phi_1, \phi_2, \psi_1, \psi_2$ are linearly independent, so $PK_1 + QK_2 = o$ will imply

$$\left.\begin{array}{l} P[k_1\psi_1] + Q[k_2\psi_1] = 0 \\ P[k_1\psi_2] + Q[k_2\psi_2] = 0 \end{array}\right\} \text{ and } \left.\begin{array}{l} P[k_1\phi_1] + Q[k_2\phi_1] = 0 \\ P[k_1\phi_2] + Q[k_2\phi_2] = 0 \end{array}\right\}$$

But these should lead to $P = Q = 0$. So we have

$\triangle = [k_1\psi_1]\ [k_2\psi_2] - [k_1\psi_2]\ [k_2\psi_1] \neq 0.$

5. The same procedure can be employed for higher order self-adjoint differential equations. So, for the determination of the nature of the spectrum of a self-adjoint differential operator associated with a BVP, it is enough to consider the simplest type of boundary conditions.

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REFERENCES

[1] EVERITT, W. N. : Fourth Order Singular Differential Equations. Math. Annalen. 149, (1963), 320-340.

[2] MIRSKY, L. : An introduction to linear algebra, Oxford : Oxford University Press, 1955.

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Dept. of Pure Math.
Calcutta University.

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A HILBERT SPACE ASSOCIATED WITH A SINGULAR SECOND-ORDER BOUNDARY VALUE PROBLEM

JYOTI DAS (nee CHOUDHURI) and GOPINATH LAHA

1. Let us consider the Sturn-Liouville boundary value problem (BVP)

(1.1) $y^{(2)}(x) + \{ \lambda - q(x) \} y(x) = 0$

(1.2) $y(0) \cos \alpha + y^{(1)}(0) \sin \alpha = 0$

(1.3) $y(b) \cos \beta + y^{(1)}(b) \sin \beta = 0$

where $0 \leqslant x \leqslant b < \infty$, $\lambda = \mu + i\gamma$, $y^{(r)}(x) \equiv d^r y / dx^r$

The set of values of λ for which the above BVP comprising of (1.1)—(1.3) has a non-trivial solution are called the eigen values of the BVP. It is known that (1.1) has a monotonically increasing sequance of eigenvalues $\{ \lambda_{nb} ; n = 1, 2. \ldots\ldots\ldots \}$

The non-trivial solution of (1.1) corresponding to the eigen valve λ_{nb} is denoted by $\psi_{nb}(\,.\,)$ and is known as the eigenfunction corresponding to the eigenvalue λ_{nb}. If $\phi(\,.\,,\lambda)$ and $\theta(\,.\,,\lambda)$ denote the solutions of (1.1) satisfying

$$\phi(0,\lambda) = \sin\alpha , \quad \phi^{(1)}(0,\lambda) = -\cos\alpha ,$$

$$\theta(0,\lambda) = \cos\alpha , \quad \theta^{(1)}(0,\lambda) = \sin\alpha ,$$

then it has been proved in [1], § 2. 1. 9 that there exists a function of λ , say $l_b(\cdot)$, regular in each of the half planes $\mathrm{Im}\,\lambda > 0$ and $\mathrm{Im}\,\lambda < 0$, such thar $\psi_b(\,.,\lambda)$ where $\psi_b(x,\lambda) = \theta(x,\lambda) + l_b(\lambda)\,\phi(x,\lambda)$ is square-integrable on [0, b), for all $b > 0$. Further it is known that $l_b(.)$ has simple poles at each of the eigenvalues λ_{nb} , $n = 1, 2, \ldots \cdots$ Let r_{nb} denote the residue of $l_b(.)$ at λ_{nb} .

A non-decreasing step function $\rho_b(.)$ is now defined having discontinuities only at each of the points $\{ \lambda_{nb} \}_{n=1}^{\infty}$, and taking constant value between them, so that

$$\rho_b(0) = 0 ,$$

$$\rho_b(\lambda_{nb}) = \tfrac{1}{2} \{ \rho_b(\lambda_{nb} - 0) + \rho_b(\lambda_{nb} + 0) \} ,$$

and $\rho_b(t_2) - \rho_b(t_1) = \rho_{nb}$ for all t_1, t_2 with $\lambda_{n-1b} < t_1 < \lambda_{nb} < t_2 < \lambda_{n+1b}$.

It can be proved that $\mathrm{Lt}_{b \to \infty} \rho_b(u) = \rho(u)$ exists for all u in [0, ∞) if $b \to \infty$ through some suitable sequence, and $\rho(\,.\,)$ is a bounded non-decresing function [1, § 6. 3].

Let $L^2 [0, \infty) \equiv L^2$ denote the set of all Lebesgue-square integrable functions on $[0, \infty)$. Let

$\mathcal{F} = \{ f :$ (i) $f(x) = 0$ for all $x > c$ (some fixed positive number), (ii) f is the integral of some absolutely continuous function, (iii) $f(0) \cos \alpha + f^{(1)}(0) \sin \alpha = 0 \}$,

and for $b > c$, $F_b(u) = \int_0^b f(x) \phi(x, u)\, dx$ for all $f \in \mathcal{F}$.

Writing the Parseval formula for the BVP (1.1)—(1.3), viz.

$$\int_0^b f^2(x)\, dx = \sum_{n=1}^{\infty} c^2_{nb},$$

where $c_{nb} = \int_0^b f(x) r^{\frac{1}{2}}_{nb} \phi(x, \lambda_{nb})\, dx = r^{\frac{1}{2}}_{nb} F_b(\lambda_{nb})$,

in the form $\int_0^b f^2(x)\, dx = \int_{-\infty}^{\infty} F^2_b(u)\, d\rho_b(u)$ for all $f \in \mathcal{F}$,

since $\rho_b(u) \to \rho(u)$ (for all $u \in [0, \infty)$) through some sequence, using Helly-Bray Theorem we have $\int_{u_1}^{u_2} f(u)\, d\rho_b(u) \to \int_{u_1}^{u_2} f(u)\, d\rho(u)$ as $b \to \infty$ through some suitable sequence.

Hence, as in [1, § 6.3], we get $\int_{-\infty}^{\infty} F_b(u)\, d\rho_b(u) \to \int_{-\infty}^{\infty} F(u)\, d\rho(u)$,

as $b \to \infty$ through the same sequence, where $F(u) = \int_0^{\infty} f(x) \phi(x, u)\, dx$. Consequently

$$(1.4) \qquad \int_0^{\infty} f^2(x)\, dx = \int_{-\infty}^{\infty} F^2(u)\, d\rho(u)$$

2. Let $\mathcal{F}_1 \equiv \left\{ F(\cdot) : F(u) = \int_0^{\infty} f(x) \phi(z, u)\, dx,\ f \in \mathcal{F} \right\}$.

A mapping 'd' from $\mathcal{F}_1 \times \mathcal{F}_1$ to the set R of all non-negative real numbers is defined by

$$d(F, G) = + \left[\int_{-\infty}^{\infty} \{ F(u) - G(u) \}^2\, d\rho(u) \right]^{\frac{1}{2}}$$

we shall show that $(\mathcal{F}_1, d)$ is a metric space.

An obvious consequence of the relation (1.4) is the following

Lemma 2.1 $\int_{-\infty}^{\infty} F^2(u)\, d\rho(u) = 0 \Longleftrightarrow F \equiv 0.$

Lemma 2.2 $D(F, G) \leqslant d(F, H) + d(H, G) \quad \forall\, F, G, H \in \mathcal{F}_1$

Proof
$$d^2(F, G) = \int_{-\infty}^{\infty} \{F(u) - G(u)\}^2\, d\rho(u)$$
$$= \int_{-\infty}^{\infty} \{F(u) - H(u)\}^2\, d\rho(u) + \int_{-\infty}^{\infty} \{H(u) - G(u)\}^2\, d\rho(u)$$
$$+ 2\int_{-\infty}^{\infty} \{F(u) - H(u)\}\{H(u) - G(u)\}\, d\rho(u)$$
$$\leqslant \int_{-\infty}^{\infty} \{F(u) - H(u)\}^2\, d\rho(u) + \int_{-\infty}^{\infty} \{H(u) - G(u)\}^2\, d\rho(u)$$
$$+ 2\left\{\int_{-\infty}^{\infty} [F(u) - H(u)]^2\, d\rho(u)\right\}^{\frac{1}{2}} \left\{\int_{-\infty}^{\infty} [H(u) - G(u)]^2\, d\rho(u)\right\}^{\frac{1}{2}}$$
$$= \{d(F, H) + d(H, G)\}^2$$

and the required result follows.

Theorem 1. $(\mathcal{F}_1, d)$ is a metric space.

Proof : By definition $d(F, G) \geqslant 0$ and $d(F, G) = d(G, F)$.

Using Lemma 2.1, we get $d(F, G) = 0 \Longrightarrow F = G$. Lemma 2.2 indicates that 'd' is transitive. Hence $(\mathcal{F}_1, d)$ is a metric space.

3. **Theorem 2.** If $\mathcal{F}$ denotes the completion of the metric space $\mathcal{F}_1$, then for any $f \in L^2[0, \infty)$, there is an F in $\mathcal{F}$ such that

$$\int_0^{\infty} f^2(x)\, dx = \int_{-\infty}^{\infty} F^2(u)\, d\rho(u) \tag{3.1}$$

Proof : Consider any $f \in L^2$. Since T is dense in L^2, there is a sequence $\{f_n\}_n$

in $\mathcal{F}$, so that $\underset{n\to\infty}{\text{l. i. m}}\, f_n = f$, where l. i. m. denotes the limit in the mean in L^2.

Let $F_n(u) = \int_0^\infty f_n(x)\,\phi(x, u)\,dx$. Then

$$\int_0^\infty \{ f_n(x) - f_m(x) \}^2\, dx = \int_{-\infty}^\infty \{ F_n(u) - F_m(u) \}^2\, d\rho(u) = d^2(F_n, F_m)$$

As $\{ f_n \}_n$ is fundamental in L^2, it follows that $\{ F_n \}_n$ is fundamental in $\mathcal{F}_1$. Since $\mathcal{F}$ is the completion of $\mathcal{F}_1$, there exists an $F(\,.\,)$ in $\mathcal{F}$ such that

$$d(F_n, F) \to 0 \quad \text{as } n \to \infty$$

i.e. $\displaystyle\int_{-\infty}^\infty \{ F_n(u) - F(u) \}^2\, d\rho(u) \to 0$ as $n \to \infty$

Thus, there is a function F in $\mathcal{F}$ such that $\int_{-\infty}^\infty F^2(u)\, d\rho(u)$ exists and

$$\int_{-\infty}^\infty F_n^2(u)\, d\rho(u) - \int_{-\infty}^\infty F^2(u)\, d\rho(u)$$

$$= \int_{-\infty}^\infty \{ F_n(u) - F(u) \}^2\, d\rho(u) + 2\int_{-\infty}^\infty F_n(u)\, F(u)\, d\rho(u) - 2\int_{-\infty}^\infty F^2(u)\, d\rho(u)$$

$$\leqslant \int_{-\infty}^\infty \{ F_n(u) - F(u) \}^2\, d\rho(u) + 2\left\{ \int_{-\infty}^\infty (F_n(u) - F(u))^2\, d\rho(u) \right\}^{\frac{1}{2}} \left\{ \int_{-\infty}^\infty F^2(u)\, d\rho(u) \right\}^{\frac{1}{2}}$$

$$\to 0 \text{ as } n \to \infty$$

So, $$\int_0^\infty F^2(u)\, dx = \underset{n\to\infty}{\text{l. i. m}} \int_0^\infty f_n^2(x)\, dx = \underset{n\to\infty}{\text{l. i. m}} \int_{-\infty}^\infty F_n^2(u)\, d\rho(u) = \int_{-\infty}^\infty F^2(u)\, d\rho(u)$$

§ 4. **Theorem 3.** T is a Hilbert space

Proof : All that we need to do is to define an inner product $(\,,\,)$ in $\mathcal{F}$. Let

$$(F, G) = \int_{-\infty}^\infty F(u)\, G(u)\, d\rho(u), \quad \forall\, F, G \text{ in } \mathcal{F}.$$

This existence of the integral on the right follows by Schwarz's inequality. It is

easy to verify that (,) defines an inner product in $\mathcal{F}$. Since $\mathcal{F}$ is a complete space, it follows that T is a Hilbert space.

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REFERENCE ;

[1] Titchmarsh, E. C. : Eigen function expansions associated with 2nd order Differential Equations Part I, second edition, Oxford University Press 1962.

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Dept. of Pure Math.
Calcutta University.

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FIXED POINT MAPPINGS SATISFYING A RATIONAL INEQUALITY

KANAN MAJUMDAR

In a recent work [1] B. Fisher has studied a fixed point theorem by means of the following rational inequality :

$$(1) \qquad d(T_1x, T_2y) \leqslant \frac{b\,d(x, T_1x)\, d(x, T_2y) + c\,d(y, T_2y)\, d(y, T_1x)}{d(x, T_1x) + d(y, T_2y)}$$

where T_1 and T_2 are self mapping of a complete metric space (X, d).

The object of the present work is to consider a more general rational inequality, viz.

$$(2) \qquad d(T_1x, T_2y) \leqslant \frac{a\,d(x, T_1x)\, d(y, T_2y)}{d(x, T_2y) + d(x, y)} + \frac{b\,d(x, T_1x)\, d(x, T_2y)}{d(y, T_1x) + d(x, y)},$$

$$+ \frac{c\,d(y, T_2y)\, d(x, T_2y)}{d(x, T_1x) + d(y, T_2y)} + \frac{e\,d(y, T_2y)\, d(x, y)}{d(x, T_1x) + d(x, T_2y)}$$

$$+ f\,d(x, T_2y) + g\,d(x, y)$$

in order to obtain a common fixed point of T_1 and T_2.

First we prove the following main theorem.

Theorem : Let T_1 and T_2 be two mappings of the complete metric space X into itself satisfying the rational inequality (2) for all $x, y \in X$ for which $x \neq y$,
$d(x, T_1x) + d(y, T_2y) \neq 0$,
$d(x, T_1x) + d(x, T_2y) \neq 0$ and $a + 2b + c + 2f + e + g < 1$ where
$a, b, c, e, f, g \in 0, 1)$ then T_1 and T_2 have a common fixed point.

Proof : We define a sequence of elements $\{x_n\}$ of X as follows :
Let x be any element of X. Let $x_1 = T_1x$, $x_2 = T_2x_1$, $x_3 = T_1x_2$, $x_4 = T_2x_3$ and so on.
$d(x_1, x_2) = d(T_1x, T_2x_1)$

$$\leqslant \frac{a\,d(x, T_1x)\, d(x_1, T_2x_1)}{d(x, T_2x_1) + d(x, x_1)} + \frac{b\,d(x, T_1x)\, d(x, T_2x_1)}{d(x_1, T_1x) + d(x, x_1)}$$

$$+ \frac{c\,d(x_1, T_2x_1)\, d(x, T_2x_1)}{d(x, T_2x) + d(x_1, T_2x_1)} + \frac{e\,d(x_1, T_2x_1)\, d(x, x_1)}{d(x, T_1x) + d(x, T_2x_1)} + f\,d(x, T_2x_1) + g\,d(x, x_1)$$

$$\leqslant \frac{ad(x,x_1)\ d(x_1,x_2)}{d(x,x_2)+d(x,x_1)} + \frac{bd(x,x_1)\ d(x,x_2)}{d(x_1,x_1)+d(x,x_1)} + \frac{cd(x_1,x_2)\ d(x,x_2)}{d(x,x_1)+d(x_1,x_2)}$$

$$+ \frac{ed(x_1,x_2)\ d(x,x_1)}{d(x,x_1)+d(x,x_2)} + fd(x,x_2) + gd(x,x_1)$$

$$\leqslant \frac{ad(x,x_1)\ [d(x,x_1)+d(x,x_2)]}{d(x,x_2)+d(x,x_1)} + bd(x,x_2)$$

$$+ \frac{cd(x_1,x_2)\ [d(x,x_1)+d(x_1,x_2)]}{d(x,x_1)+d(x_1,x_2)} + \frac{ed(x,x_1)[d(x,x_1)+d(x,x_2)]}{d(x,x_1)+d(x,x_2)}$$

$$+ fd(x,x_2) + gd(x,x_1)$$

$$\leqslant ad(x,x_1) + bd(x,x_1) + bd(x_1,x_2) + cd(x_1,x_2) + ed(x,x_1)$$

$$+ fd(x,x_1) + fd(x_1,x_2) + gd(x,x_1)$$

$$\therefore\ (1-b-c-f)\, d(x_1,x_2) \leqq (a+b+e+f+g)\, d(x,x_1)$$

or,
$$d(x_1,x_2) \leqslant \frac{a+b+e+f+g}{1-b-c-f}\, d(x,x_1)$$

$$= \frac{a+b+e+f+g}{1-b-c-f}\, d(x,T_1x).$$

Again

$$d(x_2,x_3) = d(T_2x_1, T_1x_2)$$

$$\leqslant \frac{ad(x_1,T_2x_1)\, d(x_2,T_1x_2)}{d(x_1,T_1x_2)+d(x_1,x_2)} + \frac{bd(x_1,T_2x_1)\ d(x_1,T_1x_2)}{d(x_2,T_2x_1)+d(x_1,x_2)}$$

$$+ \frac{cd(x_2,T_1x_2)\, d(x_1,T_1x_2)}{d(x_1,T_2x_1)+d(x_2,T_1x_2)} + \frac{ed(x_2,T_1x_2)\ d(x_1,x_2)}{d(x_1,T_2x_1)+d(x_1,T_1x_2)}$$

$$+ fd(x_1,T_1x_2) + gd(x_1,x_2)$$

$$\leqslant \frac{ad(x_1,x_2)\ d(x_2,x_3)}{d(x_1,x_2)+d(x_1,x_3)} + \frac{bd(x_1,x_2)\ d(x_1,x_3)}{d(x_2,x_2)+d(x_1,x_2)} + \frac{cd(x_2,x_3)\, d(x_1,x_3)}{d(x_1,x_2)+d(x_2,x_3)}$$

$$+ \frac{ed(x_2,x_3)\ d(x_1,x_2)}{d(x_1,x_2)+d(x_1,x_3)} + fd(x_1,x_3) + gd(x_1,x_2)$$

$$\leqslant \frac{ad(x_1,x_2)[d(x_1,x_2)+d(x_1,x_3)]}{d(x_1,x_2)+d(x_1,x_3)} + \frac{bd(x_1,x_2)\, d(x_1,x_2)}{d(x_1,x_2)}$$

$$+ \frac{cd(x_2, x_3)[d(x_1, x_2) + d(x_2, x_3)]}{d(x_1, x_2) + d(x_2, x_3)} + \frac{ed(x_1, x_2)[d(x_1, x_2) + d(x_1, x_3)]}{d(x_1, x_2) + d(x_1, x_3)}$$

$$+ f[d(x_1, x_2) + d(x_2, x_3)] + gd(x_1, x_2)$$

$$\leqslant ad(x_1, x_2) + b[d(x_1, x_2) + d(x_2, x_3)] + cd(x_2, x_3)$$

$$+ ed(x_1, x_2) + f[d(x_1, x_2) + d(x_2, x_3)] + + gd(x_1, x_2)$$

$$\therefore \quad (1 - b - c - f)\, d(x_2, x_3) \leqslant (a + b + e + f + g)\, d(x_1, x_2)$$

or $$d(x_2, x_3 \leqslant \frac{a + b + e + f + g}{1 - b - c - f} d(x_1, x_2)$$

$$\leqslant \frac{(a + b + e + f + g)^2}{1 - b - c - f} d(x, T_1x)$$

Similarly,

$$d(x_3, x_4) \leqslant \left(\frac{a + b + e + f + g}{1 - b - c - f}\right)^3 d(x, T_1x).$$

In general we have

$$d(x_n, x_{n+p}) \leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + \dots$$
$$+ d(x_{n+p-1}, x_{n+p})$$

$$\leqslant (\alpha^n + \alpha^{n+1} + \dots + \alpha^{n+p-1})\, d(x, T_1x)$$

$$\left(\text{where } \alpha = \frac{a + b + e + f + g}{1 - b - c - f}\right)$$

$$< \frac{\alpha^n}{1-\alpha} d(x, T_1x)$$

Since $0 < \frac{a + b + e + f + g}{1 - b - c - f} < 1$, we have, $0 < \alpha < 1$.

$$d(x_n, x_{n+p}) \to 0 \text{ as } n \to \infty.$$

Thus the sequence $\{x_n\}$ is fundamental. Again X is complete.

Therefore there exists a point $x_0 \in x$ such that $\lim\limits_{n \to \infty} x_n = x_0$.

Now to show $T_1x_0 = x_0 = T_2x_0$, suppose $T_1x_0 \neq x_0$.

$\therefore$ $d(x_0, T_1x_0) \leqslant d(x_0, x_n) + d(x_n, T_1x_0)$, where n is a even positive integer.

$$\leqslant d(x_0, x_n) + d(T_2x_{n-1}, T_1x_0)$$

$$\leqslant d(x_0, x_n) + \frac{ad(x_{n-1}, T_2x_{n-1})\, d(x_0, T_1x_0)}{d(x_{n-1}, T_1x_0)\, d(x_0, T_1x_0)}$$

$$+ \frac{bd(x_{n-1}, T_2x_{n-1})\, d(x_{n-1}, T_1x_0)}{d(x_0, T_2x_{n-1}) + d(x_{n-1}, x_0)} + \frac{cd(x_0, T_1x_0)\, d(x_{n-1}, T_1x_0)}{d(x_{n-1}, T_2x_{n-1}) + d(x_0, T_1x_0)}$$

$$+ \frac{ed(x_0, T_1x_0)\, d(x_{n-1}, x_0)}{d(x_{n-1}, T_2x_{n-1}) + d(x_{n-1}, T_1x_0)} + fd(x_{n-1}, T_1x_0) + gd(x_{n-1}, x_0)$$

which implies that

$d(x_0, T_1x_0) \leqslant (c + f)\, d(x_0, T_1x_0)$ as $n \to \infty$, which is impossible since $(c + f) < 1$.

$\therefore\ d(x_0, T_1x_0) = 0$, i. e. $T_1x_0 = x_0$.

Similarly we can show $T_2x_0 = x_0$.

Again we shall show that the fixed point is unique. If possible, let y_0 be another common fixed point of T_1 and T_2 i. e. $T_1y_0 = y_0 = T_2y_0$.

$$\therefore\ d(x_0, y_0) = d(T_1x_0, T_2y_0)$$

$$\leqslant \frac{ad(x_0, T_1x_0)\, d(y_0, T_2y_0)}{d(x_0, T_2y_0) + d(x_0, y_0)} + \frac{bd(x_0, T_1x_0)\, d(x_0, T_2y_0)}{d(y_0, T_1x_0) + d(x_0, y_0)}$$

$$+ \frac{cd(y_0, T_2y_0)\, d(x_0, T_2y_0)}{d(x_0, T_1x_0)\, d(y_0, T_2y_0)} + \frac{ed(y_0, T_2y_0)\, d\, x_0, y_0)}{d(x_0, T_1x_0) + d(x_0, T_2y_0)}$$

$$+ fd(x_0, T_2y_0) + gd(x_0, y_0)$$

$\leqslant (f + g)\, d(x_0, y_0)$ which is a contradiction since $(f + g) < 1$.

$\therefore\ d(x_0, y_0) = 0$ implies $x_0 = y_0$ and this proves our main theorem.

It may be of interest to remark that our theorem yields the following corollaries :

Corollary 1 : If T be an itself mapping of a complete metric space X satisfying the inequality

$$d(Tx, Ty) \leqslant \frac{ad(x, Tx)\, d(y, ty)}{d(x, Ty) + d(x, y} + \frac{bd(x, Tx)\, d(x, Ty)}{d(y, Tx) + d(x, y)} + \frac{cd(y, Ty)\, d(x, Ty)}{d(x, Tx) + d(y, Ty)}$$

$$+ \frac{ed(y, Ty)\, d(x, y)}{d(x, Tx) + d(x, Ty)} + fd(x, Ty) + gd(x, y)$$

for all $x, y \in X$ for which $x \neq y$, $d(x, Tx) + d(y, Yy) \neq 0$, $d(x, Tx) + d(x, Ty) \neq 0$ and $a + 2b + c + 2f + e + g < 1$, where $a, b, c, e, f, g \in (0, 1)$, then T has fixed point in X.

Corollary 2 : If T be an itself mapping of the complete metric space X and if

$$d(Tx, Ty) \leqslant \frac{a\,d(x, Tx)\,d(y, Ty)}{d(x, Ty) + d(x, y)} + \frac{b\,d(x, Tx)\,d(x, Ty)}{d(y, Tx) + d(x, y)}$$
$$+ \frac{c\,d(y, Ty)\,d(x, Ty)}{d(x, Tx) + d(y, Ty)} + \frac{e\,d(y, Ty)\,d(x, y)}{d(x, Tx) + d(x, Ty)}$$

for all $x, y \in X$, for which $x \neq y$, $d(x, Tx) + d(y, Ty) \neq 0$, $d(x, Tx) + d(x, Ty) \neq 0$ and $a + 2b + c + e < 1$, where $a, b, c, e, \in (0, 1)$, then T has a fixed point in X.

Corollary 3 If T be an itself mapping of the complete metric space X and if

$$d(Tx, Ty) \leqslant \frac{a\,d(x, Tx)\,d(y, Ty)}{d(x, Tx) + d(x, y)} + \frac{b\,d(x, Tx)\,d(x, Ty)}{d(y, Tx) + d(x, y)} + f\,d(x, Ty) + g\,d(x, y)$$

for all $x, y \in X$, for which $x \neq y$ and $a + 2b + 2f + g < 1$, where $a, b, f, g \in (0, 1)$, then T has a fixed point in X.

Corollary 4 : If T be an itself mapping of the complete metric space X and if $d(Tx, Ty) \leqslant f\,d(x, Ty) + g\,d(x, y)$, $x, y \in X$ and $2f + g < 1$, $f, g \in (0, 1)$, then T has a fixed point in X.

Corollary 5 : If T be an itself mapping of the complete metric space X satisfying the inequality

$$d(Tx, Ty) \leqslant \frac{b\,d(x, Tx)\,d(x, Ty)}{d(y, Tx) + d(x, y)} + f\,d(x, Ty)$$

for all $x, y \in X$ for which $x \neq y$ and $b + f < \frac{1}{2}$ where $b, f \in (0, 1)$, then T has a fixed point in X.

Corollary 6 : If T be an itself mapping of the complete metric space X satisfying

$$d(Tx, Ty) \leqslant \frac{a\,d(x, Tx)\,d(y, Ty)}{d(x, Ty) + d(x, y)} + \frac{c\,d(y, Ty)\,d(x, Ty)}{d(x, Tx) + d(y; Ty)}$$
$$+ \frac{e\,d(y, Ty)\,d(x, y)}{d(x, Tx) + d(x, Ty)} + f\,d(x, Ty) + g\,d(x, y)$$

for all $x, y \in X$ for which $x \neq y$, $d(x, Tx) + d(y, Ty) \neq 0$, $d(x, Tx) + d(x, Ty) \neq 0$ and $a + c + e + 2f + g < 1$, where $a, c, e, f, g \in (0, 1)$, then T has a fixed point in X.

Corollary 7 : If T be an itself mapping of the complete metric space X satisfying the rational inequality

$$d(Tx, Ty) \leqslant a \frac{d(x, Ty)\, d(y, Ty)}{d(x, Ty)+d(x, y)} + \frac{d(x, Tx)\, d(x, Ty)}{d(y, Tx)+d(x, y)} + \frac{d(y, Ty)\, d(x, Ty)}{d(x, Tx)+d(y, Ty)}$$

$$+ \frac{ed(y, Ty)\, d(x, y)}{d(x, Tx)+d(x, Ty)} + d(x, Ty) + d(x, y)$$

for all $x, y \in X$ for which $x \neq y$, $d(x, Tx) + d(y, Ty) \neq 0$, $d(x, Tx) + d(x, Ty) \neq 0$ and $a < \frac{1}{8}$, $a \in (0, 1)$, then T has a fixed point in X.

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REFERENCE ;

[1] Fisher, Brian : On fixed point mappings and constant mappings, Rivista Matematica della Universita di Parma, (4) .6 (1980) 1-5.

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Dept. of Pure Math.
Calcutta University.

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OPERATIONAL DERIVATION OF SOME GENERATING FUNCTIONS FOR THE LAGUERRE POLYNOMIALS

BANDANA GHOSH

1. Introduction : In a recent paper O. V. Viskov [5] has derived an operational representation of $L_n^{\alpha}(x)$ in the form :

$$(1.1) \quad L_n^{\alpha}(x) = \frac{(-1)^n}{n!} e^{x} (xD^2+\alpha D+D)^n e^{-x}; \quad D \equiv d/dx.$$

An analogous representation has been given by T. D. Banerjee [1]. Subsequently the authoress [4] has made some comments on Viskov's work. Also S. K. Chatterjea [2, 3] has studied Viskov's operator in details. The object of the present paper is to point out a method of using Viskov's operator in the derivation of some generating functions for the Laguerre polynomials including the Hardy-Hille formula.

The results proved in this paper are the following :

$$(1.2) \quad \sum_{n=0}^{\infty} L_n^{\alpha}(x)\, t^n = (1-t)^{-\alpha-1} \exp\left(\frac{-xt}{1-t}\right)$$

$$(1.3) \quad \sum_{n=0}^{\infty} \frac{(n+k)!}{n!\,k!} L_{n+k}^{\alpha}(x)\, t^n = (1-t)^{-\alpha-k-1} \exp\left(\frac{xt}{t-1}\right) L_k^{\alpha}(x/(1-t)).$$

$$(1.4) \quad \sum_{n=0}^{\infty} \frac{n!}{(1+\alpha)_n} L_n^{\alpha}(x)\, L_n^{\alpha}(y)\, t^n = (1-t)^{-\alpha-1} \exp\left(\frac{-(x+1)t}{1-t}\right) {}_0F_1\left(-;\, 1+\alpha;\, \frac{xyt}{(1-t)^2}\right).$$

In order to prove our results we require the following operational formula :

$$(1.5) \quad e^{-t(xD^2+\alpha D+D)}\, e^{-kx} = (1-kt)^{-\alpha-1}\, e^{\frac{kx}{kt-1}},$$

2. Derivation of the operational formula :

Let $u = e^{-t(xD^2+\alpha D+D)}\, e^{-kx}$,

then $\dfrac{\partial u}{\partial t} = e^{-t(xD^2+\alpha D+D)} \left[-\left(k^2 x e^{-kx} - (\alpha+1) k e^{-kx} \right) \right]$

and $$\frac{\partial u}{\partial t} = - e^{-t(xD^2+\alpha D+D)} xe^{-kx}$$

Thus $$\frac{\partial u}{\partial t} = k^2 \frac{\partial u}{\partial k} + (\alpha+1) ku ,$$

which has the general solution in the form

$$ku^{\frac{1}{\alpha+1}} = \Phi\left(t-\frac{1}{k}\right)$$

To determine Φ we notice that $u = e^{-kx}$ when $t = 0$.

$$\therefore\ ke^{\frac{-kx}{\alpha+1}} = \Phi\left(-\frac{1}{k}\right)$$

$$\text{or,}\quad \Phi(-k) = \frac{1}{k} e^{\frac{-x}{(\alpha+1)k}}$$

$$\therefore\quad \Phi(k) = -\frac{1}{k} e^{\frac{x}{(\alpha+1)k}}$$

Thus $$ku^{\frac{1}{\alpha+1}} = -\frac{1}{t-1/k}\, e^{\frac{x}{\alpha+1}\,\frac{k}{kt-1}}$$

$$\therefore\quad u = (1-kt)^{-(\alpha+1)}\, e^{\frac{kx}{kt-1}} .$$

3. Derivation of generating functions ;

From (1.1) and (1.5) we easily obtain

$$\sum_{n=0}^{\infty} L_n^{\alpha}(x)\, t^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} e^{x} (xD^2+\alpha D+D)^n\, e^{-x} t^n$$

$$= e^{x} e^{-t(xD^2+\alpha D+D)} e^{-x}$$

$$= (1-t)^{-\alpha-1} e^{\frac{xt}{t-1}},$$

which is (1.2).

Employing the same technique we get

$$\sum_{n=0}^{\infty} \frac{(n+k)!}{n!\,k!} L_{n+k}^{\alpha}(x)\, t^n$$

$$= \sum_{n=0}^{\infty} \frac{(n+k)!}{n!\,k!} \frac{(-1)^{n+k}}{(n+k)!} e^{x} (xD^2+\alpha D+D)^{n+k} e^{-x} t^n$$

$$= e^{x} \frac{(-1)^k}{k!} (xD^2+\alpha D+D)^k \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} (xD^2+\alpha D+D)^n e^{-x}$$

$$= e^{x} \frac{(-1)^k}{k!} (xD^2+\alpha D+D)^k e^{-t(xD^2+\alpha D+D)} e^{-x}$$

$$= (1-t)^{-\alpha-1} e^{x} \frac{(-1)^k}{k!} (xD^2+\alpha D+D)^k e^{\frac{x}{t-1}}.$$

On putting $\frac{x}{t-1} = -y$, the right hand member becomes

$$(1-t)^{-\alpha-k-1} e^{-ty} \frac{(-1)^k}{k!} e^{y} [yD_y^2+\alpha D_y+D_y]^k e^{-y}$$

$$= (1-t)^{-\alpha-k-1} e^{-ty} L_k^{\alpha}(y)$$

$$= (1-t)^{-\alpha-k-1} e^{\frac{xt}{t-1}} L_k^{\alpha}\left(\frac{x}{1-t}\right),$$

which is (1.3).

Lastly using the same operational technique we shall prove Hardy-Hille formula.

We have

$$\sum_{n=0}^{\infty} \frac{n!}{(1+\alpha)_n} L_n^{\alpha}(x)\, L_n^{\alpha}(y)\, t^n$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{n!}{(1+\alpha)_n} \frac{(-1)^k (1+\alpha)_n y^k t^n}{k!\,(n-k)!\,(1+\alpha)_k} \frac{(-1)^n}{n!} e^{x} (xD^2+\alpha D+D)^n e^{-x}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{y^k t^{n+k}}{k!\, n!\, (1+\alpha)_k} (-1)^{n+k} e^{x} (xD^2+\alpha D+D)^{n+k} e^{-x}$$

$$= \sum_{k=0}^{\infty} e^{x} (-1)^{2k} \frac{(yt)^k}{k!\,(1+\alpha)_k} (xD^2+\alpha D+D)^{k} \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} (xD^2+\alpha D+D)^{n} e^{-x},$$

$$= \sum_{k=0}^{\infty} e^{x} \frac{(-1)^{2k} (yt)^k}{k!\,(1+\alpha)_k} (xD+\alpha D+D)^k (1-t)^{-\alpha-1} e^{\frac{x}{t-1}}$$

On putting $\frac{x}{t-1} = -z$ the right hand member becomes

$$(1-t)^{-\alpha-1} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{-yt}{1-t}\right)^k}{k!\,(1+\alpha)_k} e^{z} e^{-zt} [zD_z^2+\alpha D_z+D_z]^k e^{-z}$$

$$= (1-t)^{-\alpha-1} e^{-zt} \sum_{k=0}^{\infty} \frac{\left(\frac{-yt}{1-t}\right)^k}{(1+\alpha)_k} \frac{(-1)^k}{k!} e^{z} [zD_z^2+\alpha D_z+D_z]^k e^{-z}$$

$$= (1-t)^{-\alpha-1} e^{-zt} \sum_{k=0}^{\infty} \frac{\left(\frac{-t}{1-t}\right)^k}{(1+\alpha)_k} L_k^{\alpha}(z)$$

$$= (1-t)^{-\alpha-1} e^{-zt} \sum_{k=0}^{\infty} \frac{L_k^{\alpha}(z)}{(1+\alpha)_k} \left(\frac{-yt}{1-t}\right)^k$$

Now we know that

$$\sum_{n=0}^{\infty} \frac{L_n^{\alpha}(x)\, t^n}{(1+\alpha)_n} = e^{t} {}_0F_1(-;\, 1+\alpha;\, -xt)$$

Thus

$$\sum_{n=0}^{\infty} \frac{n!}{(1+\alpha)_n} L_n^{\alpha}(x)\, L_n^{\alpha}(y)\, t^n$$

$$= (1-t)^{-\alpha-1} e^{-zt} e^{-yt/(1-t)} {}_0F_1\left(-;\, 1+\alpha;\, \frac{yzt}{(1-t)}\right)$$

$$= (1-t)^{-\alpha-1} e^{-(x+y)t/(1-t)} {}_0F_1\left(-,\, 1+\alpha,\, \frac{yzt}{(1-t)^2}\right),$$

which is (1.4).

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REFERENCES :

[1] Banerjee, T. D. Extension of an identity of Redei for Laguerre polynomials — To appear in Bull. Cal. Math. Soc.

[2] Chatterjea, S. K. : On the identity of Viskov for Laguerre polynomiaal (in Russian), to appear in Math. Balkanica.

[3] — — — — — — : On a result of Viskov fot Laguerre polynomials, Pure Math' Manuscript 1 (1982), 41-44.

[4] Ghosh, B. : A note on Laguerre polynomials, Pure Math. Manuscript 1 (1982), 37–40'

[5] Viskov, O. V. : On the identity of L. B. Redei for Laguerre polynomials (in Russian), Acta Sci Math. (Szeged) 39 (1977), 27-28.

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Dakshin Barasat Sri Sri
Saradamani Balika Vidyalaya
24 Parganas

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ON NORMAL VARIATION OF THE TANGENT MANIFOLD OF A HYPERSURFACE OF A RIEMANNIAN MANIFOLD

BANDANA BARUA (nee GUPTA)
and
BIJAN KUMAR SEN

Introduction :

The normal variations of a submanifold of a Riemannian manifold have been discussed by Chen and Yano [1]. They have obtained the following theorems :

Theorem A. A submanifold is totally geodesic iff every normal variation of the submanifold is isometric.

Theorem B. A submanifold is totally umbilical iff every normal variation of the submanifold is conformal.

Theorem C. A submanifold is minimal iff every normal variation of the submanifold is volume-preserving.

In this paper we have considered the tangent manifold of a hypersurface of a Riemannian manifold which is a submanifold of codimension 2 of the tangent manifold of the ambient manifold. In the first part of the present paper we have studied the nature of the immersion of the tangent manifold of the hypersurface in the tangent manifold of the ambient manifold, corresponding to different types of immersion of the hypersurface. In the second part of the paper we have studied the relations between different types of normal variations of the tangent submanifold and the corresponding normal variations of the hypersurface.

1. Nature of immersion of T(M^n).

Let M^n be a hypersurface of a Riemannin manifold M^{n+1} isometrically immersed in it and let the immersion be given by the equations

(1.1) $y^{\alpha} = y^{\alpha}(x^i)$, $\alpha = 1, \dots, n+1$, $i = 1, \dots, n$, where M^n is covered by coordinate neighbourhoods $(U; x^h)$ and M^{n+1} is covered by coordinate neighbourhoods $(V; y^{\alpha})$. Since the immersion is isometric, we have

(1.2) $g_{jk} = a_{\alpha\beta} \; B^{\alpha}_{j} \; B^{\beta}_{k}, \; B^{\alpha}_{j} = \dfrac{\partial y^{\alpha}}{\partial x^{j}}$

where h, i, j, k, ... , run over 1, 2 ... , n and $\alpha, \beta, \gamma, \delta$ ... run over 1, 2, ... , n, n+1.

If $T(M^n)$ and $T(M^{n+1})$ denote the tangent manifolds of M^n and M^{n+1} respectively, then $(U \times E^n, \bar{x}^h)$ and $(V \times E^{n+1}, \bar{y}^A)$ are the coordinate neighbourhoods covering $T(M)$ and $T(M^{n+1})$ [2] respectively, where H, I, J, K, ... run over 1.2 ... , n, n+1, ... , 2n and A, C, D, ... run over 1, 2, ... , n+1, n+2, ··· , 2n+2. Also, $\bar{x}^h = x^h$ and $\bar{x}^{n+h}$ are tangent vectors to M^n and $\bar{y}^{\alpha} = y^{\alpha}$ and $\bar{y}^{(n+1)+\alpha}$ are tangent vectors to M^{n+1}

We shall denote n+h by $\bar{h}$ and (n+1) + α by $\bar{\alpha}$. Then, $\bar{y}^{\bar{\alpha}} = \bar{x}^{\bar{h}} \; B^{\alpha}_{h}$.

The immersion of $T(M^n)$ in $T(M^{n+1})$ is given by the equaions

(1.3) $\bar{y}^A = \bar{y}^A (x^1, \dots , x^{2n})$

If $\bar{g}_{JK}$ and $\bar{a}_{AC}$ denote the fundamental metric tensors in $T(M^n)$ and $T(M^{n+1})$, then

(1.4) $\bar{g}_{JK} = \bar{a}_{AC} \; B^A_J \; B^C_K, \; B^A_J = \dfrac{\partial \bar{y}^A}{\partial \bar{x}^J}$

Let (N^{α}) be the unit vector normal to M^n in M^{n+1};

then

$$a_{\alpha\beta} \; B^{\alpha}_{i} \, N^{\beta} = 0 \text{ and } a_{\alpha\beta} \, N^{\alpha} \, N^{\beta} = 1 .$$

Let us now define

$$\left.\begin{aligned} &(1.5) \quad (\bar{N}^A_1) = (N^{\alpha}, - \left\{ {\alpha \atop \beta\sigma} \right\} N^{\beta} \, \bar{y}^{\bar{\sigma}}) \\ &\text{and } (\bar{N}^A_2) = (0, N^{\alpha}) \end{aligned}\right\}$$

Straight forward calculations show that

(1.6) $\bar{a}_{AC} \; B^A_J \; \bar{N}^C_x = 0 , \; \bar{a}_{AC} \; \bar{N}^A_x \bar{N}^C_y = \delta_{xy} , \; x, y = 1, 2 .$

Thus $(\bar{N}_1^A)$ and $(\bar{N}_2^A)$ are mutually orthogonal unit vectors to $T(M^n)$.

Theorem 1.1 If N is a unit vector normal to M^n in M^{n+1}, then $\bar{N}_1$ and $\bar{N}_2$ given by (1.5) form an orthonormal basis of the normal bundle of $T(M^n)$ in $T(M^{n+1})$,

If b_{jk} are the components of the second fundamental tensor of M^n, then [3]

$$b_{jk} = a_{\alpha\beta} \left(\nabla_k B_j^\alpha + \left\{ {\alpha \atop \beta\gamma} \right\} B_j^\beta B_k^\gamma \right)$$

where ∇ denotes covariant differentiation with respect to g_{jk}. If $\bar{b}_{J,Kx}$ denote the components of the second fundamental tensor of $T(M^n)$ with respect to the normal vector $\bar{N}_x$, then straight forward calculations show that

(1.7) $\bar{b}_{jk1} = b_{jk}$, $\bar{b}_{\bar{j}k1} = \frac{1}{2} \left(\nabla_m b_{kj} - \nabla_j b_{km} \right) \bar{x}^m$, $\bar{b}_{\bar{j}\bar{k}1} = 0$,

and

$$\bar{b}_{jk2} = a_{\alpha\beta} N^\beta \left[\bar{\nabla}_{jk} \bar{y}^\alpha + \left\{ {\alpha \atop \beta\gamma} \right\} \left(\bar{\nabla}_j \bar{y}^\gamma B_k^\delta + \bar{\nabla}_k \bar{y}^\delta B_j^\gamma \right) \right.$$
$$\left. + \frac{1}{2} \left(R^\alpha_{\gamma\delta\sigma} + R^\alpha_{\delta\gamma\sigma} + 2 \frac{\partial}{\partial \bar{y}^\sigma} \left\{ {\alpha \atop \gamma\delta} \right\} \right) \bar{y}^\sigma B_j^\gamma B_k^\delta \right]$$
$$+ a_{\alpha\beta} N^\beta \left(\bar{\nabla}_{jk} \bar{y}^\mu + \left\{ {\mu \atop \gamma\delta} \right\} B_j^\gamma B_k^\delta \right) \bar{y}^\sigma \left\{ {\alpha \atop \mu\sigma} \right\}$$

(1.8) $\bar{b}_{\bar{j}k2} = b_{jk}$, $\bar{b}_{\bar{j}\bar{k}2} = 0$

where $\bar{\nabla}$ denotes covariant differentiation with respect to $\bar{g}_{JK}$.

From (1.7) and (1.8) we get the followings

Theorem 1.2 $T(M^n)$ is totally umbilical in $T(M^{n+1})$ iff it is totally geodesic.

Theorem 1.3 If (TM^n) is totally geodesic in $T(M^{n+1})$ then M^n is totally geodesic in M^{n+1}.

The converse of Th. 1.3 is not true. We can only say that if M^n is totally geodesic in M^{n+1}, then $\bar{N}_1$ is a geodesic section of $T(M^n)$.

If $\mu = \frac{1}{n} g^{jk} b_{jk}$ denote the mean curvature of M^n, we find from (1.7)

$$\bar{\mu}_1 = \frac{1}{2n} \bar{g}^{JK} \bar{b}_{JK1}$$

$$(1.9) \quad = \tfrac{1}{2}\mu - \tfrac{1}{4n} g^{tj} \left\{ {i \atop hj} \right\} (\nabla_k b_{1t} - \nabla_1 b_{kt}) \bar{x}^k \bar{x}^h$$

Theorem 1.4 If $\bar{N}_1$ is a minimal section of $T(M^n)$, then M^n is minimal iff M^{n+1} is of constant curvature.

2. Isometric and conformal normal variations

An infinitesimal normal variation of $T(M^n)$ in $T(M^{n+1})$ is given by an equation of the form

$$(2.1) \quad {}'\bar{y}^A = \bar{y}^A + \bar{x}^A \, dt.$$

where

$$(2.2) \quad \bar{x}^A = X^x \bar{N}^A_x \qquad x = 1, 2$$

is a vector normal to $T(M^n)$ and dt is an infinitesimal.

$$(2.3) \qquad \text{Then , } {}'\bar{B}^A_J = \frac{\partial {}'\bar{y}^A}{\partial \bar{x}^J} = \frac{\partial \bar{y}^A}{\partial \bar{x}^J} + \frac{\partial \bar{x}^A}{\partial \bar{x}^J} \, dt.$$

Displacing the vector $\bar{B}^A_J = \dfrac{\partial \bar{y}^A}{\partial \bar{x}^J}$ parallelly from $(\bar{y}^A)$ to $({}'\bar{y}^A)$ along $\bar{x}^A$, we obtain [1]

$$(2.4) \quad \widetilde{B}^A_J = \bar{B}^A_J - \left\{ {A \atop DC} \right\} \bar{B}^C_J \, \bar{X}^D \, dt$$

Putting $\delta(\bar{B}^A_J) = \bar{B}^A_J - \widetilde{B}^A_J$, weget,

$$\delta(\bar{B}^A_J) = \bar{\nabla}_J \bar{X}^A \, dt$$

where

$$(2.5) \quad \bar{\nabla}_J \bar{X}^A = \frac{\partial \bar{X}^A}{\partial \bar{x}^J} + \left\{ {A \atop DC} \right\} \bar{B}^C_J \, \bar{X}^D$$

If ${}^{/}\bar{g}_{JK}$ denote the fundamental metric tensor of the deformed submanifold, then [1]

(2.6) $\delta(\bar{g}_{JK}) = {}^{/}\bar{g}_{JK} - \bar{g}_{JK} = -2\, \bar{b}_{JKx} X^{x}\ dt$

By virtue of Theorem A and Theorem 1.3, we now state

Theorem 2.1 If an infinitesimal normal variation of $T(M^n)$ in $T(M^{n+1})$ is an isometry, then the corresponding normal variation of M^n in M^{n+1} is also an isometry.

The converse of the above theorem is not true. Also by virtue of Theorem B and Theorem 1.2 we can state

Theorem 2.2 An infinitesimal normal variation of $T(M^n)$ in $T(M^{n+1})$ cannot be a non-trivial conformal motion.

By virtue of Theorem C and Theorem 1.4 we state

Theorem 2.3 If a nornal variation of $T(M^n)$ in the direction of N_1 is volume preserving, then the normal variation of M^n is also volume preserving iff M^{n+1} is of constant curvature.

3. Affine and Projective normal variations

If has been shown by Chen and Yano [1] that the variation in the affine connection due to the infinitesimal normal variation (2.1) is given by

(3.1) $\delta\left(\left\{ {H \atop JK} \right\}\right) = -\bar{g}^{HT}\left\{\bar{\nabla}_J (\bar{b}_{KTx}\, X^{x}) + \bar{\nabla}_K (\bar{b}_{JTx}\, X^{x}) - \bar{\nabla}_T (\bar{b}_{JKx}\, X^{x})\right\} dt$

An affine motion and a projective motion are characterized by

(3.2) $\delta\left(\left\{ {H \atop JK} \right\}\right) = 0$

(3.3). $\delta\left(\left\{ {H \atop JK} \right\}\right) = \left(\delta^{H}_{J}\, \bar{p}_K + \delta^{H}_{K}\, \bar{p}_J\right) dt$

where $\bar{p}_J$ is a covariant vector field in $T(M^n)$.

From (3.1) and (3.3) we find

(3.4) $2\, \nabla_T (\bar{b}_{JKx}\, X^{x}) = -(2\bar{g}_{JK}\, \bar{p}_T + \bar{g}_{JT}\, \bar{p}_K + \bar{g}_{KT}\, \bar{p}_J).$

Conversely, if (3.4) holds, then substituting in the right hand side of (3.1) we get (3.3). Hence

Theorem 3.1 An infinitesimal normal variation of $T(M^n)$ in $T(M^{n+1})$ given by (2.1) is a projective motion iff there exists a covariant vector field $\bar{p}_K$ in $T(M^n)$ satisfying (3.4).

Let the corresponding normal variation of M^n in M^{n+1} be a projective motion ; then the equation (3.4) takes the form

$$2\ \nabla_t\ b_{jk} = -\ (2g_{jk}\, p_t + g_{jt}\ p_k + g_{kt}\ p_j)$$

where p_k is a covariant vector in M^n. Transvecting by g^{jk} we get after simplification

$$(3.5) \qquad p_k = -\ \frac{n}{n+1}\mu_k, \quad \mu_k = \nabla_k\, \mu$$

where μ is the mean curvature of M^n in M^{n+1}. If M^n is a minimal hypersurface in M^{n+1}, then $p_k = 0$ and the infinitesimal normal variation of M^n reduce to an affine motion.

Theorem 3.2 If the infinitesimal normal variation of a hypersurface of a Riemannian manifold is a projective motion, then the vector of the projective motion is given by (3.5).

Corollary A minimal hypersurface of a Riemannian manifold cannot admit a non-trivial infinitesimal projective normal variation.

Next let us consider the infinitesimal normal variation of $T(M^n)$ in $T(M^{n+1})$ in the direction of $\bar{N}_1$ be projective and the corresponding variation of M^n in M^{n+1} be affine. Then writting $\bar{b}_{JK}$ for $\bar{b}_{JK1}$, we find

$$2\bar{\nabla}_T\ \bar{b}_{JK} = -\ (2\bar{g}_{JK}\ \bar{p}_T + \bar{g}_{JT}\ \bar{p}_K + \bar{g}_{KT}\ \bar{p}_J)$$

$$\text{and} \quad \nabla_t\, b_{jk} = 0$$

Choosing $J = \bar{j}$, $K = \bar{k}$, $T = \bar{t}$, we get, from (1.7)

$$2\, g_{jk}\, \bar{p}_{\bar{t}} + g_{jt}\, \bar{p}_{\bar{k}} + \bar{g}_{kt}\, \bar{p}_{\bar{j}} = 0$$

Transvecting by g^{jk} we find $\bar{p}_{\bar{t}} = 0$.

Again, choosing $J = \bar{j}$, $K = \bar{k}$ and $T = t$, we find $\bar{p}_t = 0$. Thus

Theorem 3.3 If the infinitesimal normal variation of M^n in M^{n+1} is affine, then the infinitesimal normal variation of $T(M^n)$ in $T(M^{n+1})$ in the direction of $\bar{N}_1$ is also affine.

Finally, let us consider the infinitesimal normal variation of $T(M^n)$ in the direction of $\bar{N}_2$ to be projective, the infinitesimal normal variation of M^n being affine. Proceeding as above, we find that the vector of the projective motion is

(3.6) $\bar{p} = \left(-\frac{1}{2n}\, g_{ij}\, b_{tj}\, R^t{}_{kir}\, v^r,\ 0 \right)$

where $R^t{}_{kir}$ are components of the Riemann curvature tensor in M^n.

Theorem 3.4 If the infinitesimal normal variation of M^n in M^{n+1} is affine and the infinitesimal normal variation of $T(M^n)$ in $T(M^{n+1})$ in the directon of $\bar{N}_2$ is projective then the vector of the projective motion is given by (3.6)

REFERENCES

[1] Chen and Yano : On the theory of normal variations. J. Diff. Geom. 13(1978) 1-10.

[2] Eisenhart L. P. : Riemannian Geometry Princeton University Press, (1949.)

[3] Sasaki, S. : On the differential Geometry of tangent bundles of Riemannian manifolds, Tohoku Math. Jour. 10(1958), 238-354.

Depart. of Pure Math.
Calcutta University
and
Dept. of Mathematics,
Nabadwip Vidyasagar College
Nabadwip, Nadia.

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PAIRWISE CONFLUENT AND PAIRWISE PSEUDO CONFLUENT MAPPING IN BITOPOLOGICAL SPACES

DIPTI SINHA

1. **Infroduction.** The concept of bitopological spaces was introduced by J. C. Kelly [2]. In [3] Pervin defined connectedness and continuity in bitopological spaces. We have defined pairwise confluent, pairwise pseudo confluent and pairwise weakly confluent mapping in section 1. We have studied some properties of the mappings of the hereditarily normal bitopological spaces in section 2. In section 3, we have investigated some properties of the mappings onto pairwise locally arcwise connected spaces. In [2] Kelly defined quasi pseudo metric space. Some properties of quasi pseudo metric space have been discussed in section 4.

In this paper we take the definitions of connectedness and continuity given by Pervin in [3].

Now we shall define the connectedness between two sets in a bitopological space as follows :

Definition 1.1 Let A, B be two subsets of (X, P, L). Then X is said to be connected between A and B if and only if $X \neq M \cup N$ where $A \subset M$, $B \subset N$ and $(M \cap P\,cl\,N) \cup (L\,cl\,M \cap N) = \phi$.

Definition 1.2 Let A be a subset of (X, P, L). A is said to be P regularly open w.r.t. L if $A = P\,Int\,(L\,cl\,A)$.

Definition 1.3 Let A be a subset of (X, P, L). A is said to be P regularly closed w. r. t. L if $A = Pcl\,(L\,Int\,A)$.

Definition 1.4 Let $f : (X, P, L) \to (Y, P', L')$ be a continuous mapping of X onto Y. The mapping f is said to be PP′ confluent w. r. t. L, PP′ pseudo confluent w. r. t. L or PP′ weakly confluent w. r. t. L provided for each connected, non empty P′ regularly closed set C w. r. t. L′ of Y, the following conditions are satisfied respectively :

(c) for each pair of points $x \in f^{-1}(C)$ and $y \in C$ the set $f^{-1}(C)$ is connected between $\{x\}$ and $\{f^{-1}(y)\}$.

(p) for each pair of points $y, y' \in C$, the set $f^{-1}(C)$ is connected between $f^{-1}(y)$ and $f^{-}(y')$.

(w) there exists a point $x_0 \in f^{-1}(C)$ such that, for each point $y \in C$, the set $f^{-1}(C)$ is connected between $\{x_0\}$ and $f^{-1}(y)$.

We say that a continuous surjective mapping is pairwise confluent if it is both PP′ confluent w. r. t. L and LL′ confluent w. r. t. P. Similarly pairwise pseudo confluent and pairwise weakly confluent mappings are defined.

We take the definition of pairwise compact space given by Fletcher, Hoyle & Patty in [1].

Definition 1.5 The quasi component of the point $p \in X$ is the set of all points $x \in X$ such that the bitopological space (X, P, L) is connected between p and x.

Proposition 1.1 If A, B, C are subsets of a bitopological space (X, P, L), x_0 is a point of it and the set C is connected between A and $\{x_0\}$ as well as between B and $\{x_0\}$, then C is connected between A and B.

Proof : It easily follows from definition 1.1.

Remark 1.1 Each confluent mapping is both pseudo confluent and weakly confluent. By proposition 1.1, each weakly confluent mapping is pseudo confluent.

Proposition 1.2 Let $f : (X, P, L) \rightarrow (Y, P', L')$ be a continuous mapping of X onto Y such that $f^{-1}(y)$ is pairwise compact for $y \in Y$. Then f is PP′ confluent w. r. t. L, PP′ pseudo confluent w. r. t. L of PP′ weakly confluent w. r. t. L if and only if for each connected, non empty P′ regularly closed set C w. r. t. L′ of Y, the following conditions are satisfied respectively :

(C′) for each quasi component Q of $f^{-1}(C)$, we have $C = f(Q)$

(p) for each pair of points $y, y' \in C$, there exists a quasi component Q of $f^{-1}(C)$ such that $y, y' \in f(Q)$.

(w′) there exists a quasi component Q of $f^{-1}(C)$ such that $C = f(Q)$

2. Hereditarily Pairwise normal-bitopological spaces

We take the definition of pairwise normal space given by J. C. Kelly in [2]. From the definition of pairwise normal space, it follows that

Tneorem 2.1 Let A be a L closed and B be a P closed set in a pairwise normal space (X, P, L) and $A \cap B = \phi$, then there exists a P open set G such that $A \subset G$, $B \cap L \text{ cl } G = \phi$

Theorem 2.2 If A and B are two separated sets in a hereditarily normal bitopological space (X, P, L), then there is a P open set G such that $A \subset G$ and $L \text{ cl } G \cap B = \phi$.

Proof : It readily follows from Theorem 2.1.

Theorem 2.3 Let $f : (X, P, L) \rightarrow (Y, P', L')$ be a PP' closed mapping of a hereditarily normal bitopological space (X, P, L) onto a bitopological space (Y, P', L') and $y, y' \in Y$ be points. Then the following two conditions are equivalent :

(i) if $U \subset Y$ is a P' open set connected between $\{y\}$ and $\{y'\}$ then the set $f^{-1}(U)$ is connected between $f^{-1}(y)$ and $f^{-1}(y')$

(ii) if Z is a set connected between $\{y\}$ and $\{y'\}$ then the set $f^{-1}(Z)$ is connected between $f^{-1}(y)$ and $f^{-1}(y')$

Proof : obviously (ii) implies (i).

Let us suppose (ii) is violated, which means that there exists a set $Z \subset Y$ connected that between $\{y\}$ and $\{y'\}$ such that $f^{-1}(Z) = M \cup N$, $f^{-1}(y) \subset M$, $f^{-1}(y') \subset N$ and $(M \cap P \text{ cl } N) \cup (L \text{ cl } M \cup N) = \phi$.

It follows by theorem 2.2, that there is a P open set G in X such that $M \subset G$ and $L \text{ cl } G \cap N = \phi$

$\therefore \; N \subset X - L \text{ cl } G$.

$\therefore \; f^{-1}(Z) = M \cup N \subset G \cup (X - L \text{ cl } G)$; whence $(L\text{cl } G - G) \cap f^{-1}(Z) = \phi$ and $G \cup (X - L \text{ cl } G)$ is not connected between $f^{-1}(y)$ and $f^{-1}(y')$. Now $(L \text{ cl } G - G)$ is a P closed set. Since f is PP' closed, $f(L \text{ cl } G - G)$ is P' closed in Y. But we have $f(L \text{ cl } G - G) \cap Z = \phi$, whence $Z \subset U$. Therefore U is also connected between $\{y\}$ and $\{y'\}$.

Because $f_{1}^{-1}(U) \subset X - (L \text{ cl } G - G) = G \cup (X - L \text{ cl } G)$, we conclde that $f^{-1}(U)$ is not connected between $f^{-1}(y)$ and $f^{-1}(y')$, so that condition (i) is violated. This completed the proof.

Theorem 2.4 Let $f : (X, P, L) \to (Y, P', L')$ be a PP' closed mapping of a hereditarily normal space (X, P, L) onto a bitopological space (Y, P', L') and let $x \in X$, $y \in Y$ be points. Then the following two conditions are equivalent.

(i) if $U \subset Y$ be a P' open set connected between $\{f(x)\}$ and $\{y\}$, then the set $f^{-1}(U)$ is connected between $\{x\}$ and $\{f^{-1}(y)\}$

(ii) if $Z \subset Y$ is a set connected between $\{f(x)\}$ and $\{y\}$ then the set $f^{-1}(Z)\}$ is connected between $\{x\}$ and $\{f^{-1}(y)\}$

Proof : Obviously (ii) implies (i). To prove (i) implies (ii) one can use exactly the same argument as in the proof of Theorem 2.3 with $f^{-1}(y')$ replaced by $\{x\}$ and y' replaced by $f(x)$.

Theorem 2.5 Let $f : (X, P, L) \to (Y, P', L')$ be a PP' closed mapping of a hereditarily normal bitopological space onto a bitopological space Y. Let $y_0 \in Y$ be a point such that $f^{-1}(y_0)$ is P compact. Then the following conditions are equivalent :

(i) for each P' open set $U \subset Y$ there exists a point $x_0 \in f^{-1}(y_0)$ such that if $y \in Y$ and the set U is connected between $\{y_0\}$ and $\{y\}$, the set $f^{-1}(U)$ is connected between $\{x_0\}$ and $\{f^{-1}(y)\}$

(ii) for each set $Z \subset Y$, there exists a point $x_0 \in f^{-1}(y_0)$ such that if $y \in Y$ and the set Z is connected between $\{y_0\}$ and $\{y\}$, then the set $f^{-1}(Z)$ is connected between $\{x_0\}$ and $\{f^{-1}(y)\}$.

Proof : Obviously (ii) implies (i). Suppose on the contrary that (i) holds and (ii) does not, which means that there exists a set $Z \subset Y$ with the following property : for each $x \in f^{-1}(y_0)$ there exists a point $y(x) \in Y$ such that Z is connected between $\{y_0\}$ and $\{y'x)\}$ but $f^{-1}(Z)$ is not connected between $\{x\}$ and $f^{-1}[y(x)]$.

As in the proof of theorem 2.3, we obtain a P open set $G(x) \subset X$ such that

(1) .... $f^{-1}(Z) \subset G(x) \cup [X - L \text{ cl } G(x)]$, $x \in G(x)$

and $f^{-1}[y(x)] \subset X - L \text{ cl } G(x)$

Thus the P compact set $f^{-1}(y_0)$ is covered by the P open sets $G(x)$, where $x \in f^{-1}(y_0)$ and there exists a finite sequence of points $x_1, x_2, \ldots \ldots \ldots x_n$, of $f^{-1}(y_0)$ such that

(2) ... ... $f^{-1}(y_0) \subset G(x_1) \cup \ldots \ldots \ldots \cup G(x_n)$

Now L cl $G(x_i) - G(x_i)$ is a P closed set. Since f is a PP′ closed mapping the set,

(3) ... ... $U = Y - \bigcup_{i=1}^{n} f[L \text{ cl } G(x_i) - G(x_i)]$ is a P′ open set in Y and

$Z \subset U$ by (1)

Hence the set U is connected between $\{y_o\}$ and $\{y(x_i)\}$ for $i = 1, 2, \ldots \ldots n$.

Let $x_0 \in f^{-1}(y_0)$ be a point whose existence is guaranteed by (i). Consequently the set $f^{-1}(U)$ is connected between $\{x_0\}$ and $f^{-1}[y(x_i)]$ for $i = 1, 2, \ldots \ldots \ldots \ldots n$.

By (2) there exsits an integer $k = 1, 2, \ldots \ldots \ldots, n$, such that $x_0 \in G(x_k)$ and it follows from (3) that $[L \text{ cl } G(x_k) - G(x_k)] \cap f^{-1}(U) = \phi$

whence $f^{-1}(U) \subset G(x_k) \cup [X - L \text{ cl } G(x_k)]$

By (1) we get, $f^{-1}[y(x_k)] \subset X - L \text{ cl } G(x_k)$

Contradicting the fact that the set $f^{-1}(U)$ is connected between $\{x_0\}$ and $f^{-1}[y(x_K)]$ This completes the proof.

3. Mapping onto a pairwise locally arcwise connected space.

Definition 3.1 A bitobological space (X, P, L) is said to be P locally connected w. r. t. L at a point $x \in X$ if for every P regularly open neighbourhood G w. r. t. L containing x there exists a connected P neighbourhood V of x such thal $x \in V \subset G$.

Definition 3.2 (X, P, L) is said to be P Locally connected w. r. t. L if and only if it is P locally connected w. r. t. L at each of its points.

We say that (X, P, L) is pairwise locally connected if and only if it is both P locally connected w. r. t. L and L locally connected w. r. t. P.

Definition 3.3 A pairwise connected and P compact set in (X, P, L) is called a P continuum.

A set which is both P continuum and L continuum is called a continuum.

Definition 3.4 Let (X, P, L) be a bitopological space. A regularly closed continuum $T \subset X$ will be called an arc from a to b where $a, b \in T$, if x is any point of T distinct from a and b, Then $T - x$ is connected between a and b.

The points a, b are called the end points of the arc T and T is said to join a and b.

Definition 3.5 (X, P, L) is said to be arcwise connected if every pair of its points can be joined by an arc.

Definition 3.6 (X, P, L) is said to be P locally arcwise connected (l. a. c) w. r. t. L at a point $x \in X$ if in every P regularly open neighbourhood w. r. t. L of x there exists an arcwise connected P neighbourhood of x.

(X, P, L) is said to be pairwise locally arcwise connected if and only if it is both P l. a. c. w. r. t. L and L l. a. c. w. r. t. P.

Definition 3.7 A mapping $f : (X, P, L) \to (Y, P', L')$ is called pairwise perfect provided f is pairwise closed and $f^{-1}(y)$ is pairwise compact for each $y \in Y$.

Definition 3.8 The maximal connected subset C(x) in (X, P. L), containing a point $x \in X$ is called a component of x in X.

Propostition 3.1 If A, B are pairwise compact sets and a set C is connected between A and B, then there exists a quasi component Q of C such that $A \cap Q \neq \phi \neq B \cap Q$

Theorem 3.1 Let $f : (X, P, L) \to (Y, P', L')$ be a pairwise continuous perfect mapping hereditarily pairwise normal space X onto pairwise locally arcwise connected space Y. Then the following four conditions are equivalent .

(i) f is pairwise confluent

(ii) for each arc $A \subset Y$ and each quasi component Q of $f^{-1}(A)$ we have $A = f(Q)$.

(iii) for each connected set $C \subset Y$ and each quasi component Q of $f^{-1}(C)$ we have $C = f(Q)$.

(iv) for each $Z \subset Y$, each point $z \in Z$ and each point $x \in f^{-1}(Z)$ we have $Q(Z, z) = f [Q (f^{-1}(Z), x)]$. [$Q(Z, z)$ = the set of all points $x \in Z$ such that Z is connected between z and x]

Proof : By proposition 1.2 (i) implies (ii) and (iii) implies (i). Clearly (iv) implies (iii).

To prove that (ii) impl es (iv), let us consider a set $Z \subset Y$ and a point $x \in X$ such that $z = f(x) \in Z$.

Let $y \in Q\ (Z, z)$, $y \neq z$ and let $U \subset Y$ be an arbitrary pairwise regularly open set connected between $\{z\}$ and $\{y\}$. Donated by U_0 the component of U which contains z. Since U is a regularly open set in Y, U is locally connected. Therefore U_0 is a quasi component of U and so $y \in U_0$. Moreover U_0 is pairwise open, so there exists an arc $A \subset U_0$ joining y and z. It follows from (ii) and from the inclusion $A \subset U$ that $A = f\ [\ Q\ (\ f^{-1}(A), x\ )\ ] \subset f[\ Q\ (\ f^{-1}(U), x\ )\ ]$

whence $y \in f\ [\ Q\ (\ f^{-1}(U), x\ )\ ]$ i.e., $f^{-1}(y) \cap Q\ (\ f^{-1}(U),\ x\ ) \neq \phi$.

we conclude that $f^{-1}(u)$ is connected between $\{x\}$ and $f^{-1}(y)$. By Theorem 2.4, the set $f^{-1}(z)$ is connected between $\{x\}$ and $f^{-1}(y)$ because the set Z is connected between $\{z\}$ and $\{y\}$. Since $f^{-1}(y)$ is pairwise compact, there exists, by proposition 3.1, a quasi component of $f^{-1}(z)$ which meets $\{x\}$ and $f^{-1}(y)$. This quasi component is $Q(\ f^{-1}(z), x\ )$ and we get $y \in f\ [\ Q(f^{-1}(z), x)\ ]$. As a result the inclnsion $Q(\ Z, z) \subset f\ [Q(f^{-1}(z), x)\ ]$ holds.

To prove rhe reverse inclusion, for $x \in f^{-1}(z)$, let $x' \in Q(\ f^{-1}(z), x\ )$ i. e., $f^{-1}(z)$ is connected between x and x'. Since f is continuous, Z is connected between f(x) and f(x'). $\therefore$ $f(x') \in Q\ (\ Z, z\ )$

$\therefore$ $f\ [\ Q\{\ f^{-1}(z), x\ \}\ ] \subset Q\ (\ Z, z\ )$

So (iv) is proved. This completes the proof.

Lemma 3.1 Let f be a pairwise continuous mapping of a bitopological space (X, P, L) onto an arc A with end points a_1, a_2 such that $f^{-1}(a)$ is pairwise compact for $a \in A$ and let Q be a quasi component of X.

Then $A = f(Q)$ if and only If $a_1, a_2 \in f(Q)$.

Proof : It readily follows from definitlon 1.1.

Theorem 3.2 Let $f : (\ X, P, L\ ) \rightarrow (\ Y, P', L'\ )$ be a pairwise continuous and perfect mapping of a hereditarily normal space onto a pairwise locally arcwise connected space Y. Then the following four conditions are equivalent ;

(i) f is pairwise pseudo confluent

(ii) for each arc $A \subset Y$, there exists a quasi component Q of $f^{-1}(A)$ such that $A=f(Q)$:

(iii) for each connected set $C \subset Y$ and each pair of points $y, y' \in C$ there exists a quasi component Q of $f^{-1}(C)$ such that $y, y' \in f(Q)$

(iv) for each set $Z \subset Y$ and each $z \in Z$, we have

$$Q(Z, z) = \bigcup_{x \in f^{-1}(z)} f[Q\{f^{-1}(Z), x\}]$$

Proof : (iii) $\Longrightarrow$ (i) by proposition 1.2 and (i) $\Longrightarrow$ (ii) by proposition 1.2 and Lemma 3.1 Clearly (iv) $\Longrightarrow$ (ii). To prove that (ii) $\Longrightarrow$ (iv), let us consider a set $Z \subset Y$ and a point $z \in Z$. Let $y \in Q(Z, z)$, $y \neq z$ and let $U \subset Y$ be an arbitrary pairwise regularly open set conneeted between $\{y\}$ and $\{z\}$. Then as in the proof of Theorem 3.1, there exists an arc $A \subset U$ joining y and z. By (ii) the set $f^{-1}(A)$ is connected between $f^{-1}(y)$ and $f^{-1}(z)$ and so is the set $f^{-1}(U)$. It follows from Theorem 2.3 (for $y' = z$) that the set $f^{-1}(Z)$ too, is connected between the compact sets $f^{-1}(y)$ and $f^{-1}(z)$. Now by Proposition 3.1, a quasi component of $f^{-1}(Z)$ meets both $f^{-1}(y)$ and $f^{-1}(z)$ whence $y \in f[Q\{f^{-1}(Z), x\}]$ for at least one point $x \in f^{-1}(z)$.

Thus $Q(Z, z) \subset \bigcup_{x \in f^{-1}(z)} f[Q\{f^{-1}(Z), x\}]$ and as in the proof of theorem 3.1, the reverse inclusion always holds. This proves (iv). Hence this completes the proof.

4. Quasi pseudo metric space and pseudo confluent mapping

We take the definition of quasi pseudo metric space given by J. C. Kelly in [2]

Theorem 4.1 Let (X, P, L) be a quasi pseudo metric space and $C \subset X$ Then if C is not connected, there exists a P open set G such that $C \subset G \neq \phi \neq C - L \text{ cl } G$ and $C \cap L\,Fr(G) = \phi$ [Here P and L are topologies induced on X by the conjugate quasi pseudo metrics p and l respectively].

Proof : It can be easily proved.

Theorem 4.2 In a pairwise compact quasi pseudo metric space (X, P, L). the quasi components are connected and coincide therefore with the components.

Proof : It follows from Theorem 4.1.

Theorem 4.3 For subsets of a hereditarily locally connected quasi pseudo metric continua their quasi components coincide with the components.

Proof : It follows from Theorem 4.2.

So from Theorem 3.2 we get the proposition 4.1

Proposition 4.1 Let $f : (X, P, L) \rightarrow (Y, P', L')$ be a pairwise pseudo confluent and perfect mapping of a hereditarily normal and hereditarily locally connected quasi pseudo metric continuum (X, P, L) onto a locally arcwise quasi pseudo metric continuum (Y, P', L'). Then for each connected non-empty set $C \subset Y$ and each pair of points $y, y' \in C$, there exists a component K of $f^{-1}(C)$ such that $y, y' \in f(K)$

Definition 4.1 A quasi pseudo metric space (X, P, L) is said to be finitely connected provided each collection of pairwise disjoint connected subsets of P having diameters greater than a positive number is finite.

Theorem 4.4 Let $f : (X, P, L) \rightarrow (Y, P', L')$ be a pairwise pseudo confluent and perfect mapping of a hereditarily locally connected, hereditarily normal, finitely connected quasi pseudo metric continuum onto a pairwise locally arcwise connected quasi pseudo metric continuum Y. Then (Y, P', L') is finitely connected.

Proof : Suppose Y is not finitely connected i. e., there exists a number $\epsilon_0 > 0$ and an infinite sequence of pairwise disjoint connected subsets $C_1, C_2 \ldots \ldots \ldots$ of Y such that P diam $C_i > \epsilon_0$ for i. $= 1, 2, \ldots \ldots \ldots$ Consequently, there are points $y_i, y_i' \in C_i$ with P dist $(y_i, y_i') > \epsilon_0$ for $i = 1, 2, \ldots \ldots \ldots$ By proposition 4.1, we get connected sets $K_i \subset f^{-1}(C_i)$ such that $y_i, y_i' \in f(K_i)$. The sets K_i are pairwise disjoint. By the continuity of f, their diameters must be all greater than some positive number δ_0. But this is a contradiction.

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REFERENCES

[1] Fletcher, Hoyle Patty : The comparison of topologies—Duke Math. Jour, 36 (1969) 325

[2] Kelly, J. C. : Bitopological spaces, Proc. London Math. Soc. (3) 13(1963). 71-89

[3] Fervin, W. J. : Connectedness in bitopological spaces" Indagationes Mathematical 29 (1967) 369 – 372

Dept. of Pure Math.
Calcutta University
and
Dept. of Mathematics,
Nabadwip Vidyasagar College
Nabadwip, Nadia.

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A CHARACTERISATION OF Γ-RINGS

T. K. DUTTA

1. Introduction

Nobusawa [8] in the year 1964 introduced the notion of a Γ-ring M as follows :

Let M and Γ be two additive abelian groups. If for all a, b, c $\in$ M and $\alpha, \beta, \gamma \in \Gamma$ the conditions

N_1 ; $a \alpha b \in M$, $\alpha a \beta \in \Gamma$

N_2 : $(a+b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha+\beta)b = a\alpha b + a\beta b$, $a\alpha(b+c) = a\alpha b + a\alpha c$.

N_3 ; $(a\alpha b)\beta c = a\alpha(b\beta c) = a(\alpha b\beta)c$.

N_4 : $a\alpha b = 0$ for all a, b $\in$ M implies $\alpha=0$

are satisfied then M is called a Γ-ring.

Barnes [1] weakened slightly the defining conditions for Nobusawa's Γ-ring. Following Barnes we say that an additive abelian group M is a Γ-ring if the following conditions are satisfied for all a, b, c $\in$ M and $\alpha, \beta \in \Gamma$

B_1 : $a \alpha b \in M$

B_2 : $(a+b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha+\beta)c = a\alpha c + a\beta c$, $a\alpha(b+c) = a\alpha b + a\alpha c$.

B_3 : $(a\alpha b)\beta c = a\alpha(b\beta c)$.

It is known [2, p 42] that every Γ-ring M is a Γ'-ring in the sense of Nobusawa for some additive abelian group Γ'. Many fundamental results for Γ-rings were obtained by Nobusawa, Barnes, Luh, Coppage, Kyono, Ravisankar and Shukla. Nobusawa [8] proved analogue of the Wedderburn—Artin theorem for simple Γ-rings and for semi-simple Γ-rings. Barnes [1] obtained analogues of the classical Noether—Laskar theorems concerning primary representation of ideals for Γ-rings. Luh [6, 7] gave a generalisation of the Jacobson structure theorem for primitive Γ-rings having minimum one-sided ideals ; Coppage Luh [2] introduced the notions of Jacobson radicals, Levitzki nil radical nil radical and strongly nilpotent radical for Γ-rings. S. Kyono [3, 4, 5] extended

the notion of semiprime ideals to Γ-rings, studied the structure of Γ-ring with minimum conditions, also studied the structure of Γ-rings with left and right unities. Ravisankar and Shukla [9] studied Γ-rings in the setting of modules. Now in this paper we have characterised Γ-ring (in the sense of Barnes) and R Γ-module,

2. Preliminaries :

Definition ; If M_i is a Γ_i-ring for $i = 1, 2$ then an ordered pair (θ_1, θ_2) of mappings is called a homomorphism of M_1 into M_2 if it satisfies the following properties

(1) θ_1 is a group homomorphism from M_1 into M_2,

(2) θ_2 is a group homomorphism from Γ_1 into Γ_2,

(3) for every $x, y \in M_1$ $\gamma \in \Gamma_1$, $(x \gamma y)\theta_1 = (x\theta_1)(\gamma\theta_2)(y\theta_1)$.

Definition : Let R be a Γ-ring. An additive abelian group M is called a right $R\Gamma$-module if there exists a map $\phi : M \times \Gamma \times R \to M$ satisfying $\phi(m, \alpha, x)$ will be denoted by $m \alpha x$ (in short)

(1) $(m+n)\alpha x = m\alpha x + n\alpha x$,

(2) $m\alpha(x+y) = m\alpha x + m\alpha y$,

(3) $m\beta(x\alpha y) = (m\beta x)\alpha y$ for all x, y in R, α, β in Γ and m, n in M.

3. Let A and B be two additive abelian groups. $M = \text{Hom}(A, B)$ denotes the set of all homomorphisms of A into B, $\Gamma = \text{Hom}(B, A)$ denotes the set of all homomorphisms of B into A. Let $x, y \in M$ and $\alpha, \beta \in \Gamma$. If $x\alpha y$ and $\alpha x \beta$ be the usual composite maps, then it can be shown easily that M is a Γ-ring. Define the mappings $(x, a) \to xa$ from $M \times A \to B$ and $(b, \alpha) \to b\alpha$ from $B \times \Gamma \to A$ for all $a \in A$, $b \in B$, $x \in M$ and $\alpha \in \Gamma$ where xa and $b\alpha$ denote the images of a and b under x and α respectively. It can be shown that the above two mappings will satisfy the following conditions

i) $(x+y)a = xa + ya$, $x(a_1 + a_2) = xa_1 + xa_2$,

ii) $b(\alpha + \beta) = b\alpha + b\beta$, $(b_1 + b_2)\alpha = b_1\alpha + b_2\alpha$

ii) $y(xa)\alpha = (y\alpha x)a$ for all $x, y \in M$, $\alpha, \beta \in \Gamma$, $a, a_1, a_2 \in A$, $b, b_1 b_2 \in B$.

Also we note that for any $0 \neq x$ in M, $xA \neq 0$ and for any $0 \neq \alpha$ in Γ, $B\alpha \neq 0$.
Following the above discussion, let us write the following definition.

Definition : Let (A, B) be two additive abelien groups and M be a Γ-ring. (M, Γ) is said to act on (A, B) if there are mappings $M \times A \to B$ and $B \times \Gamma \to A$ satisfying a) $(x+y)a = xa+ya$, $x(a_1+a_2)=xa_1+xa_2$, b) $b(\alpha+\beta)=b\alpha+b\beta$, $(b_1+b_2)\alpha=b_1\alpha+b_2\alpha$ c) $y(xa)\alpha=(y\alpha x)a$ for all $a, a_1, a_2 \in A$, $b, b_1, b_2 \in B$; $x, y \in M$, $\alpha, \beta, \in \Gamma$.
Moreover (M, Γ) is said to act faithfully on (A, B) if for any $0 \neq x$ in M, $xA \neq 0$ and for any $0 \neq \alpha$ in Γ, $B\alpha \neq 0$.

Thus we have the following theorem.

Theorem 1. Given a pair (A, B) of additive abelian groups, there exists a Γ-ring M such that (M, Γ) acts faithfully on (A, B).

Theorem 2. Let M be a Γ-ring and A, B be two additive abelian groups, If (M, Γ) acts faithfully on (A, B) then the Γ-ring M is isomorphic to a Γ_1-ring M_1 such that each element of M_1 is a homomorphism of A into B and each element of Γ_1 is a homomorphism of B into A.

Proof : Let $x \in M$. Difine $l_x : A \to B$ by $l_x(a)=xa$. Let $\alpha \in \Gamma$

Define $r_\alpha : B \to A$ by $r_\alpha(b) = b\alpha$. Suppose $a_1, a_2 \in A$,

$l_x(a_1+a_2)=x(a_1+a_2)=xa_1+xa_2$ {by (a) of Definition 1}

$=l_x(a_1)+l_x(a_2)$. Hence l_x is a homomorphism of A into B.

Similarly we can show that r_α is a homomophism of B into A.

Let $S = \{l_x : x \in M\}$ and $T = \left\{r_\alpha : \alpha \in \Gamma\right\}$. Let $l_{x_1}, l_{x_2} \in S$.

Define $\left(l_{x_1}+l_{x_2}\right)a = l_{x_1}(a) + l_{x_2}(a)$. Then $\left(l_{x_1}+l_{x_2}\right)a = l_{x_1}(a) + l_{x_2}(a)$

$= x_1a + x_2a = \left(x_1 + x_2\right)a = l_{x_1+x_2}(a)$.

So $l_{x_1} + l_{x_2} = l_{x_1+x_2} \in S$. Define $\left(r_\alpha + r_\beta\right)b = r_\alpha(b) + r_\beta(b)$

We can show that S and T are both additive abelian groups.

Define $l_{x_1} r_\alpha l_{x_2}$ by the usual mapping product. Now $\left(l_{x_1} r_\alpha l_{x_2}\right) a =$

$x_1(x_2 a)\alpha = (x_1 \alpha x_2)a, \ \forall a \in A$. Hence $l_{x_1} r_\alpha l_{x_2} \in S$.

We can show that S is a T-ring. Define $\theta = (\theta_1, \theta_2)$ from the Γ-ring M into the T-ring S by $\theta_1(x) = l_x$, $\theta_2(\alpha) = r_\alpha$ Then $\theta_1 :$ M$\rightarrow$S and $\theta_2 : \Gamma \rightarrow$T are group homomorphisms. Also $\theta_1(x\alpha y) = l_{x\alpha y} = l_x r_\alpha l_y = \theta_1(x)\theta_2(\alpha)\theta_1(y)$. Hence θ is a homomorphism of the Γ-ring M into the T-ring S. Suppose $\theta_1(x) = 0$. Then $l_x = 0$. Hence $l_x(a) = 0, \ \forall a \in A$. So $xA = 0$. Now since (M, Γ) acts faithfully on (A, B), $xA = 0$ implies $x = 0$. Consequently θ_1 is injective. Similarly we can show that θ_2 is also injective. Hence the Γ-ring M is isomorphic to the T-ring S. Hence the theorem.

Theorem 3. Given an additive abelian group A there exists a Γ-ring R such that A is a right $R\Gamma$-module.

Proof. Let B be another additive abelian group. Also let Γ=Hom(A, B) and R = Hom (B, A). Then R is a Γ-ring. Now we define a mapping $\phi : A \times \Gamma \times R \rightarrow A$ (written $\phi(a, \gamma, x) = a \gamma x$) by $a\gamma x = (\gamma a)x$. It can be easily verified that the above mapping ϕ satisfies the following conditions i) $(a_1+a_2)\gamma x = a_1\gamma x + a_2\gamma x$ ii) $a\gamma(x_1+x_2) = a\gamma x_1 + a\gamma x_2$ iii) $a\gamma_1(x_1\gamma_2 x_2) = (a\gamma_1 x_1)\gamma_2 x_2$, for all a, a_1, $a_2 \in A$; γ, γ_1, $\gamma_2 \in \Gamma$ and x. x_1, $x_2 \in R$. Consequently A is a right R Γ-module.

Theorem 4. Let A be an additive abelian group and (θ_1, θ_2) be a homomorphism from a Γ_1-ring R_1 into a Γ-ring R where Γ = Hom (A, B) and R = Hom (B, A), B is an additive abelian group ; then A is also a right $R_1\Gamma_1$-module.

Proof. We define a mapping $\phi : A \times \Gamma_1 \times R_1 \rightarrow A$ by $\phi(a, \gamma_1, x_1) = a\gamma_1 x_1 = ((\gamma_1\theta_2)a)(x_1\theta_1)$. It can be verified that the mapping ϕ satisfies the following conditions i) $(a_1+a_2)\gamma x = a_1\gamma x + a_2\gamma x$ ii) $a\gamma(x_1+x_2) = a\gamma x_1 + a\gamma x_2$ iii) $a\gamma_1(x_1\gamma_2 x_2) = (a\gamma_1 x_1)\gamma_2 x_2$ for all a, a_1, a_2 in A : γ, γ_1 γ_2 in Γ and x, x_1, x_2 in R. Hence A is a right $R_1\Gamma_1$-module.

Theorem 5. Let (A, B) be a pair of additive abelian groups, and R_1 be a Γ-ring such that (R_1, Γ_1) acts on (B, A) then A is a right $R_1\Gamma_1$-module.

Proof : Since (R_1, Γ_1) acts on (B, A) it follows by Theorem 2 that there exists a homomorphism (θ_1, θ_2) from the Γ_1-ring R_1 into the Γ-ring R where R = Hom (B, A). and Γ = Hom (A, B). Consequently by Theorem 4 it follows that A is a right $R_1\Gamma_1$-module.

Lastly combining Theorem 4 and Theorem 5 we have the following theorem ;

Theorem 6. Let (A, B) be a pair of additive abelian groups and R_1 be a Γ_1-ring ; then A will be a right $R_1\Gamma_1$-module and (R_1, Γ_1) acts on (B, A) if and only if there exists a homomorphism (θ_1, θ_2) from the Γ_1-ring R_1 into the Γ-ring R where Γ = Hom (A, B) and R = Hom (B, A)

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REFERENCES

[1] Barnes, W. E. —On the Γ-rings of Nobusawa. Pacific J. Math. 18(1966), 411—422.

[2] Coppage, W. E. and Luh, J. —Radicals of Γ-rings. J. Math. Soc. Japan 23 (1971), 40—52.

[3] Kyuno, S. —On the radicals of Γ-rings. Osaka J. Math. 12(1975), 639—645.

[4] " —A Γ-ring with right and left Unities. Math. Japonica 24 2(1978), 191—193.

[5] , —Nobusawa's Γ-rings with right and left unities. Math Japonica 25, 2(1980), 179—190.

[6] Luh, J. —On primitive Γ-rings with minimal one sided ideals. Osaka J. Math 5(1968), 165—175.

[7] " —On theory of simple Γ-rings. Michigan Math. J. 16(1969) 65—75.

[8] Nobusawa, N. —On a generatisation of the ring theory. Osaka J. Math. 1(1964), 81—89.

[9] Ravisanka, I. S. and Shukla, U. S. —Structure of Γ-rings. Pacific J. Math. Vol. 80, 2(1979).

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1. 8. 83

Dept. of Pure Math.
Calcutta University

Jour. Pure Math.
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APPLICATION OF LAGUERRE PARTIAL DIFFERENTIAL OPERATORS TO PARTIAL DIFFERENTIAL EQUATIONS

SARAMA DAS

1. Introduction :

In a recent paper [1] Isaac. I. H. Chen and T. W. Barrett have used Bessel's ordinary differential operators which raise and lower the index of Bessel's function of the first kind to solve some second-order linear ordinary differential equations. It may be pointed out that the operators used by Chen and Barrett are not proper Lie elements in order to generate a Lie algebra. So we [2] have recently used the partial differential operators of B. Kaufman [3 in connection with the Bessel function of the first kind, which are regarded as generators of Lie algebra, in the derivation of some operational results and finally in the solution of those partial differential equations which can be factorized by means of the generators of the Lie algebra for the Bessel function of the first kind. In the present paper we similarly use the partial differential operators of E. B.McBride [4] in connection with the Laguerre polynomial in the derivation of some operational results and finally in the solution of those partial differential equations which can be factorized by means of the generators of the Lie algebra for the Laguerre polynomials.

In fact from [4 ; p. 28], we have

1.1 $$[\, xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + (\alpha + 1 - x)\, y \,]\, X_n = (n+1)\, X_{n+1}$$

$$[\, xy^{-1} \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \,]\, X_n = -(\alpha + n)\, X_{n-1}$$

where $X_n = L^{\alpha}_{n}(x)\, y^n$, $L^{\alpha}_{n}(x)$ is the Laguerre polynomials.

If we put

(1.2) $$R = xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + (\alpha + 1 - x)\, y$$

$$L = xy^{-1} \frac{\partial}{\partial x} - \frac{\partial}{\partial y}$$

then

(1.3) $\begin{pmatrix} 0 & R \\ -L & 0 \end{pmatrix} \begin{pmatrix} X_{n+1} \\ X_n \end{pmatrix} = \begin{pmatrix} (n+1)X_{n+1} \\ (\alpha+n+1)X_n \end{pmatrix}$

where $X_n = L_n^{\alpha}(x)\, y^n$.

Also we notice that

(1.4) $LR\,[\,L_n^{\alpha}(x)\, y^n\,] = -(\alpha+n+1)(n+1)\, L_n^{\alpha}(x)\, y^n,$

which yields the well-known relation

$$[\, x^2 \frac{\partial^2}{\partial x^2} + (\alpha+1-x)\, x \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}\,][\,L_n^{\alpha}(x)\, y^n\,] = 0.$$

Again $[\,R, L\,] = 2y\frac{\partial}{\partial y} + \alpha+1,$

where $[\,R, L\,] = RL - LR.$

Also we have

(1.5) $x\frac{\partial}{\partial x} + \frac{\alpha+1-x}{2} = \frac{y^{-1}R+yL}{2}$

$y\frac{\partial}{\partial y} + \frac{\alpha+1-x}{2} = \frac{y^{-1}R-yL}{2}$

2. Derivation of operational formulas from the raising and lowering operators for Laguerre polynomials :

Consider the partial differential equation

(2.1) $xy\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + (\alpha+1-x)\, y\, u = F(x, y),$

which is equivalent to

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + (\alpha+1-x)\, u = y^{-1} F(x, y).$$

Hence the corresponding system of ordinary differential equations is

(2.2) $\frac{dx}{x} = \frac{dy}{y} = \frac{du}{y^{-1}F(x, y) - (\alpha+1-x)\, u}$

Solving (2.2) we get $y = C_1 x$ and

$$u\ x^{\alpha+1} e^{-x} = C_1^{-1} \int x^{\alpha-1}\, e^{-x} F(x, C_1 x)\, dx + C_2$$

(2.3) $\qquad = \Psi_1(x, C_1) + C_2,$ (say).

Hence

(2.4) $\frac{1}{R}[F(x, y)] = x^{-\alpha-1} e^{x}\, \Psi_1(x, C_1)\Big|_{C_1 = yx^{-1}} + x^{-\alpha-1} e^{x} \phi_1\left(\frac{y}{x}\right)$

where Ψ_1 is given in (2.3) and ϕ_1 is arbitrary.

Corollaries :

(i) If $y^{-1}F(x, y)$ = a function of x only = $P(x)$, (say), then

(2.5) $$\frac{1}{R}[F(x,y)] = x^{-\alpha-1}e^{x}\left[\int e^{-x}x^{\alpha}P(x)\,dx + \phi_1\left(\frac{y}{x}\right)\right]$$

where ϕ_1 is arbitrary.

(ii) If $e^{-x}F(x, y)$ = a function of y only = $Q(y)$, (say) then

(2.6) $$\frac{1}{R}[F(x,y)] = e^{x}y^{-\alpha-1}\left[\int y^{\alpha-1}Q(y)\,dy + \phi_1(yx^{-1})\right],$$

where ϕ_1 is arbitrary.

Next Consider the partial differential equation

(2.7) $$xy^{-1}\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = f(x, y),$$

which is equivalent to the system of ordinary differantial equations

(2.8) $$\frac{dx}{xy^{-1}} = \frac{dy}{-1} = \frac{du}{f(x,y)}$$

solving (2.8) we get $xy = C_1$ and

$$u = C_1 \int x^{-2} f(x, C_1x^{-1})\,dx + C_2$$

(2.9) $$= \Psi_2(x, C_1) + C_2, \text{ (say).}$$

Hence

(2.10) $$\frac{1}{L}[f(x,y)] = \Psi_2(x, C_1)\Big|_{C_1 = xy} + \phi_2(xy),$$

where Ψ_2 is given in (2.9) and ϕ_2 is arbitrary.

Corollaries :

(i) If $yf(x, y)$ = a function of x only = $P(x)$, (say) then

(2.11) $$\frac{1}{L}[f(x,y)] = \int x^{-1}P(x)\,dx + \phi_2(xy),$$

where ϕ_2 is arbitrary.

(ii) If $f(x, y)$ = a function of y only = $Q(y)$, (say), then

(2.12) $$\frac{1}{L}[f(x,y)] = -\int Q(y)\,dy + \phi_2(xy),$$

where ϕ_2 is arbitrary.

3. Application of operational formulas to second-order partial differential equations :

Consider the differential equation

(3.1) $$x^2\frac{\partial^2 u}{\partial x^2} - y^2\frac{\partial^2 u}{\partial y^2} + (\alpha+1-x)\left(x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y}\right) - 2y\frac{\partial u}{\partial y} - (\alpha+1)u = f(x,y).$$

Since

$$\left[xy^{-1}\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right]\left[xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + (\alpha+1-x)y\right]$$
$$= x^2\frac{\partial^2}{\partial x_2} - y^2\frac{\partial^2}{\partial y^2} + (\alpha+1-x)\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right) - 2y\frac{\partial}{\partial y} - (\alpha+1).$$

The above equation becomes

$LRu = f(x,y).$

It follows therefore from (2. 10) that

(3.2) $$Ru = \Psi_2(x, C_1)\Big|_{C_1 = xy} + \phi_2(xy) = F(x,y), \text{ (say)}$$

where Ψ_2 is given in (2 9) and ϕ_2 is arbitrary.

Again it follows from (2.4) that

(3.3) $$u = e^x x^{-\alpha-1}\,\Psi_1(x, C_1)\Big|_{C_1 = yx^{-1}} + e^x x^{-\alpha-1}\phi_1(y/x),$$

where Ψ_1 is given in (2.3) and ϕ_1 is arbitrary.

On the otherhand if we consider the equation

$RLu = F(x,y),$

which is equivalent to

(3.4) $$x^2\frac{\partial^2 u}{\partial x^2} - y^2\frac{\partial^2 u}{\partial y^2} + (\alpha+1-x)\left(x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y}\right) = F(x,y),$$

Then from (2.4) we get

(3.5) $$Lu = e^x x^{-\alpha-1}\Psi_1(x, C_1)\Big|_{C_1 = yx^{-1}} + e^x x^{-\alpha-1}\phi_1(y/x) = f(x\,y,), \text{ (say)},$$

where Ψ_1 is given in (2.3) and ϕ_1 is arbitrary.

Again it follows from (2.10) that

$$u = \Psi_2 (x, C_1) \Big|_{C_1 = xy} + \phi_2 (xy), \tag{3.6}$$

where Ψ_2 is given in (2.9) and ϕ_2 is arbitrary.

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REFERENCES

[1] Chen, Isaac I.H. and Barret, T.W. — Bessel's differential operators and application to linear drfferential equafions, Internat. J. Math. Educ. Sci. Tech. 13 (1982), No. 2, 149—153.

[2] Das, Sarama — Applications of Bessel's partial differential operators to partial differential equations — to appear in Pure Math. Manuscript Vol.2 (1983)

[3[Kaufman, B — Special functions of mathematical physics from the view point of Lie algebra, J. Math. Physics. 7 (1966), 447-457.

[4] Mc Bride, E.B. — Obtaining Generating functions, Springer-Verlag, Newyork (1971).

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Dept. of Pure Math.
University of Calcutta

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APPLICATION OF PARTIAL DIFFERENTIAL OPERATORS FOR SIMPLE BESSEL POLYNOMIALS TO PARTIAL DIFFERENTIAL EQUATIONS

SARAMA DAS

1. Introduction :

In a recent paper [1] Isaac. I. H. Chen and T. W. Barrett have used Bessel's ordinary differential operators which raise and lower the index of Bessel's function of the first kind to solve some second-order linear ordinary differential equations. It may be pointed out that the operators used by Chen and Barrett are not proper Lie elements in order to generate a Lie algebra.. So we [2] have recently used the partial differential operators of B. Kaufman [3] in connection with the Bessel's function of the first kind in the derivation of some operational results and finally in the solution of those partial differential equations which can be factorized by means of the generators of the Lie algebra for the Bessel function of the first kind. In the present paper we similarly use the partial differential operators of E. B. McBride [4] in connection with the simple Bessel polynomials, in the derivation of some operational formulas and finally in the solution of those partial differential equations which can be factorized by means of the generators of the Lie algebra for simple Bessel polynomials.

Now simple Bessel polynomials is defined by

$$f_n(x) = {}_2F_0\left(-n, n+1 ; \frac{x}{2}\right), \; n > 0, \text{ where } f_{-n}(x) = f_{n-1}(x) \text{ and } f_{-1}(x) = f_0(x) = 1.$$

From [4 : 47] we have

$$\text{(1.1)} \quad \left.\begin{aligned} &\left[x^2y \frac{\partial}{\partial x} + xy^2 \frac{\partial}{\partial y} + xy + y\right] X_n = X_{n+1} \\ &\left[x^2y^{-1} \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} + y^{-1}\right] X_6 = X_{6-1}, \end{aligned}\right\}$$

where $X_n = f_n(x)\, y^n$, $f_n(x)$ is the simple Bessel polynomial.

Let

$$(1.2)\quad \left.\begin{aligned} R &= x^2y\frac{\partial}{\partial x} + xy^2\frac{\partial}{\partial y} + xy + y \\ L &= x^2y^{-1}\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + y^{-1} \end{aligned}\right\}$$

$$(1.3)\quad \begin{pmatrix} O & R \\ L & O \end{pmatrix}\begin{pmatrix} X_{n+1} \\ X_n \end{pmatrix} = \begin{pmatrix} X_{n+1} \\ X_n \end{pmatrix},$$

where $X_n = f_n(x)\, y^n$.

Also we notice that

$$(1.4)\quad LR\,[\, f_n(x)\, y^n \,] = f_n(x)\, y^n,$$

which yields the well-known relation

$$(1.5)\quad x^2\frac{\partial^2 u}{\partial x^2} - y^2\frac{\partial^2 u}{\partial y^2} + (2x+2)\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = 0.$$

Again $[\, R, L \,] = 0$,

where $[\, R, L \,] = RL - LR$.

Also we have

$$(1.6)\quad \begin{cases} \dfrac{y^{-1}R + yL}{2} = x^2\dfrac{\partial}{\partial x} + \dfrac{x}{2} + 1 \\ \dfrac{y^{-1}R - yL}{2} = xy\dfrac{\partial}{\partial y} + \dfrac{x}{2}. \end{cases}$$

2. Derivation of operational formulas from the raising and lowering operators for simple bessel polynomials :

Consider the partial differential equation

$$(2.1)\quad x^2y\frac{\partial u}{\partial x} + xy^2\frac{\partial u}{\partial y} + (x+1)\, yu = f(x, y),$$

which is equivalent to

$$x^2\frac{\partial u}{\partial x} + xy\frac{\partial u}{\partial y} + (x+1)\, u = y^{-1}f(x, y).$$

Hence the corresponding system of ordinary differential equations is

$$(2.2)\quad \frac{dx}{x^2} = \frac{dy}{xy} = \frac{du}{y^{-1}f(x, y) - (x+1)\, u}$$

Solving (2.2) we get $y = C_1 x$ and

$$u\ x\, e^{-1/x} = \frac{1}{C_1}\int x^{-2} e^{-1/x} f(x, C_1x)\, dx + C_2$$

$$(2.3)\quad = \Psi_1(x, C_1) + C_2, \text{ (say)}.$$

Hence

$$(2.4)\quad \frac{1}{R}\left[f(x,y)\right] = x^{-1} e^{-1/x} \Psi_1(x, C_1)\Big|_{C_1} = yx^{-1} + x^{-1} e^{1/x} \phi_1(yx^{-1})$$

where Ψ_1 is given in (2.3) and ϕ_1 is arbitrary.

Corollaries :

(i) If $y^{-1}f(x,y)$ = a function of x only = $P(x)$, (say) then

$$(2.5)\quad \frac{1}{R}\left[f(x,y)\right] = x^{-1} e^{1/x} \int x^{-1} e^{-1/x} p(x)\,dx + x^{-1} e^{1/x} \phi_1(yx^{-1}),$$

where ϕ_1 is arbitrary.

(ii) If $x^{-1} e^{-1/x} f(x,y)$ = a function of y only = $Q(y)$, (say), then

$$(2.6)\quad \frac{1}{R}\left[f(x,y)\right] = y^{-1} e^{1/x} \int y^{-1} Q(y)\,dy + y^{-1} e^{1/x} \phi_1(yx^{-1}),$$

where ϕ_1 is arbitrary.

Next Consider the partial differential equation

$$(2.7)\quad x^2y^{-1} \frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} + y^{-1}u = F(x,y),$$

which is equivalent to

$$x^3 \frac{\partial u}{\partial x} - xy\frac{\partial u}{\partial y} + u = y\,F(x,y).$$

Hence the correspondsng system of ordinary differential equatins is

$$(2.8)\quad \frac{dx}{x^2} = \frac{dy}{-xy} = \frac{du}{y\,F(x.\,y) - u}$$

Solving (2.8) we get $xy = C_1$ and

$$u\,e^{-1/x} = C_1 \int e^{-1/q} x^{-3} F(x, C_1x^{-1})\,dx + C_2.$$

$$(2.9)\quad = \Psi_2(x.\,C_1) + C_2,\ \text{(say)}$$

Hence

$$(2.10)\quad \frac{1}{L}\left[F(x,y)\right] = e^{1/x}\, \Psi_2(x, C_1)\Big|_{C_1=xy} + e^{1/x} \phi_2(xy).$$

where Ψ_2 is given in (2.9) and ϕ_2 is arbitrary.

Corollaries :

(i) If $y\,F(x, y)$ = a function of x only = $P(x)$, (say)

then

$$(2.11)\quad \frac{1}{L}\,[F(x,y)] = e^{-1/x}\int e^{1/x}\, x^{-2}\, P(x)\, dx + e^{1/x}\,\phi_2(xy).$$

where ϕ_2 is arbitrary.

(ii) If $x^{-1}\, e^{-1/x}\, F(x,y)$ = a function of y only = $Q(y)$, say)

then

$$(2.12)\quad \frac{1}{L}\,[F(x,y)] = -e^{1/x}\int Q(y)\, dy + e^{1/x}\,\phi_{\cdot}(xy),$$

where ϕ_2 is arbifrauy.

3. Application of operational formulas to second-order partial differential equations :

Consider the partial differential equation

$$(3.1)\quad x^2\frac{\partial^2 u}{\partial x^2} - y^2\frac{\partial^2 u}{\partial u^2} + 2(x+1)\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} + x^{-2}\, u = g(x,y).$$

Since $\left[x^2 y\frac{\partial}{\partial x} + xy^2\frac{\partial}{\partial y} + xy + y\right]\left[x^2 y^{-1}\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + y^{-1}\right]$

$$= x^4\frac{\partial^2}{\partial x^2} - x^2y^2\frac{\partial^2}{\partial y^2} + 2x^2(x+1)\frac{\partial}{\partial x} - 2x^2 y\frac{\partial}{\partial y} + 1$$

The above equation (3.1) becomes

$RLu = x^2 g(x,y) = f(x, y)$, (say)

It follows therefore from (2. 4) that

$$(3.2)\quad Lu = x^{-1}e^{1/x}\,\Psi_1(x, C_1)\Big|_{C_1 = yx^{-1}} + x^{-1}\, e^{1/x}\,\phi_1(yx^{-1})$$

(3.3) $= F(x,y)$. (say). where Ψ_1 is given in (2.3) and ϕ_1 is arbitrary.

Again it follows from (2.10) that

$$(3.4)\quad u = e^{1/x}\,\Psi_2(X, C_1)\Big|_{C_1 = xy} + e^{1/x}\, x\,\phi_2(xy),$$

where Ψ_1 is given in (2.9) and ϕ_1 is arbitrary.

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REFERENCES

[1] Chen, Isaac I. H. and Barrett, T. W. — Bessel's differential operators and application to linear differential equation, Int. J. Math. Educ. Sci. Tech. 13 (1982). No. 2, 149-153.

[2] Das, Sarama — Application of Bessel's partial differential operators to partial differential equations, — To appear in Pure Matn. Manuscript Vol-2 (1983).

[3] Kaufmann, B. — Special functions of Mathematical Physics from tha view point of Lie algebra, J. Math. Physics, 7 (1966), 447-457.

[4] McBride, E. B. — Obtaining Generatlng functions, Springer-Verlag, Berlin Heidelberg, NewYork (1971).

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University of Calcutta

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METHODS OF OBTAINING GENERATING FUNCTIONS FOR CERTAIN SPECIAL FUNCTIONS*

S. K. CHATTERJEA

Generating functions play a fundamental role in the investigation of various special functions of mathematical physics. In the earlier treatises on special functions, generating functions are obtained mainly by means of series manipulation method, although sometimes contour integral method and Laplace transform method are also mentioned in books of complex analysis and integral transforms. In recent works on generating functions two methods are frequently adopted, viz. differential operator method and group-theoretic method, besides the series manipulation method. Differential operator method originates from the Rodrigues' formula for the classical orthogonal polynomials as well as for the Bessel polynomials introduced by H. L. Krall and O. Frink. For example, if for a particular polynomial $p_n(x)$ we have

$$(1)\quad p_n(x) = [K_n\ \omega(x)]^{-1} D^n [\omega(x) X^n],$$

then we have.

$$(2)\quad K_n p_n(x) = \prod_{j=1}^{n} \left\{ X\left(D + \frac{\omega'(x)}{\omega(x)}\right) + j X' \right\}. 1$$

where $D \equiv d/dx$ and the prime indicates the derivative with respect to x.

It may be noted that for classical orthogonal polynomials and Bessel polynomials, K_n is a constant, X is a polynomial in x (of degree at most two) whose coefficients are independent of n and $\omega(x)$ is a non-negative weight function in a suitably chosen region. The formula like (1) is known as Rodrigues' formula and (2) is known as an operational representation for $p_n(x)$.

Now when X is a constant (say $X = a$) one can find by means of (2) that

$$(3)\quad \sum_{n=0}^{\infty} K_n\ p_n(x) \frac{t^n}{n!} = \frac{\omega(x+at)}{\omega(x)},$$

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which is the case of the Hermite polynomials as well as the case of Gould-Hopper's function $H_n^r(x, a, p)$ defined by

$$(4) \quad H_n^r(x, a, p) = (-1)^n x^{-a} e^{px^r} D^n (x^a e^{-px^r}).$$

Next when $X = ax+b$ one can find by means of (2) that

$$(5) \quad \sum_{n=0}^{\infty} K_n p_n \left(\frac{x-b}{a}\right) \frac{t^n}{n!} = (1-at)^{-1} \left\{\omega\left(\frac{x-b}{a}\right)\right\}^{-1} \omega\left(\frac{x-b(1-at)}{a(1-at)}\right),$$

which is the case of the Laguerre polynomials as well as the case of the present aurhor's function $T_{kn}^{\alpha}(x, p)$ defined by

$$(6) \quad T_{kn}^{\alpha}(x, p) = \frac{1}{n!} x^{-\alpha} e^{px^k} D^n \left(x^{\alpha+n} e^{-px^k} \right).$$

On the otherhand, when $X = ax^2 + bx + c$, one can find by using Lagrange's expansion, viz.

$$(7) \quad \frac{F(z)}{1-t\phi'(z)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} D^n \left[\left\{\phi(x)\right\}^n F(x) \right],$$

where $z = x + t\phi(z)$,

the following generating relation

$$(8) \quad \sum_{n=0}^{\infty} K_n \; p_n(x) \frac{t^n}{n!} = \frac{\omega(z)}{\omega(x) \{1-bt-2atz\}},$$

where $atz^2 + (bt-1)z + (ct+x) = 0$ and that branch of z is to be taken which tends to x as t tends to zero, which is the case of the Jacobi polynomials as well as the case of the Bessel polynomials $y_n(x, a, b)$ defined by

$$(9) \quad y_n(x, a, b) = b^{-n} x^{2-a} e^{b/x} D^n (x^{2n+a-2} e^{-b/x}).$$

Recently some authors [1, 15, 16] have given representations of some orthogonal polynomials in terms of differential operators operating on generating functions. One such representation is as follows

$$(10) \quad y L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{\alpha+n}{n-k} \frac{t^{n-k}(1-t)^{2k}}{k!} \frac{d^k y}{dt^k},$$

where $y \equiv (1-t)^{1-\alpha} \exp(-xt(1-t)^{-1}) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\, t^n$.

We like to point out that if there exists a function ϕ (x, t, T) such that G (x, t)-G (x, T) / G (x, t + ϕ (x, t, T)) is a function of t and T only, where G (x, t) denotes a generating function of the polynomial in question, then the desired representation is possible. In fact, for the Laguerre polynomials $L_n^{(\alpha)}(x)$ generated by

$$G(x,t) = (1-t)^{1-\alpha} \exp[-xt/(1-t)],$$

there exists function $\phi(x,t,T) = \frac{T(1-t)^2}{1-tT}$ such that G (x, t) G (x, T) / G (x, t. + ϕ (x, t, T)) is a function of t and T only and by virtue of this existence of the relation

$$G(x,t)\, G(x,T) = (1-tT)^{-1-\alpha}\; G\left(x, t + \frac{T(1-t)^2}{1-tT}\right)$$

implies the representation (10). For the Hermite polynomials $H_n(x)$ generated by $G(x,t) = \exp(xt - t^2/2)$, $\phi(x,t,T) = T$ and therefore $G(x,t)\, G(x,T) = e^{tT}\, G(x, t+T)$ implies

$$G(x,t)\, H_n(x) = \sum_{k=0}^{n} \binom{n}{k} t^k \frac{d^{n-k} G(x,t)}{dt^{n-k}}.$$

For the ultraspherical polynomials $P_n^{\lambda}(x)$ generated by $G(x.t) = (1-2xt+t^2)^{-\lambda}$,

$$\phi(x.t,T) = \frac{T(1-2xt+t^2)}{1-tT} \text{ and therefore}$$

$$G(x,t)\, G(x,T) = (1-tT)^{-2}\; G\left(x.t + \frac{T(1-2xt+t^2)}{1-tT}\right) \text{ implies}$$

$$G(x,t)\, P_n^{\lambda}(x) = \sum_{k=0}^{n} \binom{n+2\lambda-1}{k} \frac{t^k}{G^{(n-k)/\lambda}(x,t)\,(n-k)!} \; \frac{d^{n-k} G(x,t)}{dt^{n-k}}.$$

Another operational representation for the Laguerre polynomials is

$$(11) \quad L_n^{(\alpha)}(x) = \frac{1}{n!}\, e^x \prod_{j=1}^{n} (\delta+\alpha+j)\, e^{-x},$$

which is frequently useful for deriving linear and bilinear generating relations for the Laguerre polynomials. An equivalent representation of (10) is due to L. Carlitz (1960).

The following operational representation for the Laguerre polynomials is due to O V. Viskov (1977)

$$(12) \quad L_n^{(\alpha)}(x) = \frac{(-1)^n}{n!}\, e^x (xD^2 + \alpha D + D)^n\, (e^{-x})$$

It may be noted that it follows from (11)

(13) $$L_n^{(\alpha)}(x) = \frac{(-1)^n}{n!} e^x (xD + \alpha + 1)_n D^n e^{-x}.$$

Thus (12) follows at once from (13) owing to the fact

(14) $$(xD + \alpha + 1)_n D^n = (xD^2 + \alpha D + D)^n.$$

Furthermore the result of Viskov can be extended in the form :

(15) $$n! \sum_{r=0}^{n} \frac{x^r}{r!} L_{n-r}^{(\alpha+r)}(x) D^r y$$
$$=(-1)^n e^x (xD^2 + \alpha D + D)^n e^{-x} y$$
$$+(-1)^{n+1} e^x (xD+\alpha+1)_n \sum_{r=1}^{n} (-1)^{n-r} \binom{n}{r} e^{-x} D^r y,$$

which can be expressed in the following elegant form ;

(15′) $$\frac{(-1)^n}{n!} e^x (xD^2+\alpha D+D)^n e^{-x} y$$

$$= \sum_{r=0}^{n} \frac{x^r}{r!} L_{n-r}^{(\alpha+r)}(x) D^r \left(1+ \sum_{m=1}^{n} (-1)^m \binom{n}{m} D^m \right) y.$$

It may be of interest to point out that the result of Viskov may well be applied to derive the following extension of Hardy-Hille formula,

(16) $$\sum_{n=0}^{\infty} \frac{(m+n)!}{m!\,(1+\alpha)_n} L_{m+n}^{(\alpha)}(x) L_n^{(\alpha)}(y) t^n$$

$$=(1-t)^{-\alpha-m-1} \exp\left(-(x+y)t/(1-t)\right) {}_0F_1\left(-; 1+\alpha; \frac{xy\,t}{(1-t)^2}\right) L_n^{(\alpha)}\left(\frac{x}{1-t}\right)$$

Similarly the following operational represntation of the Hermite polynomials

(17) $$H_n(\sqrt{x}) = x^{-n/2} e^x (-2\delta)_n e^{-x},$$

where $\delta \equiv x \dfrac{d}{dx}$,

is also useful for deriving linear and bilinear generating relations for the Hermite polynomials. The following equivalent representation of (17) is due to J. L. Burchnal (1941)

(18) $$H_n(x) = (-1)^n (D-2x)^n. 1$$

Next we shall consider a class of bilateral generating functions for the said polynomials, because in any treatise of special functions only some particular bilinear and

bilateral generating functions are mentioned. Indeed, C. A. Truesdell raised the question whether there was any unified way of deriving bilateral generating functions. To answer this question it may be of interest to mention that differential operator method helps us in the straightforward derivation of a class of bilateral generating functions whenever one is able to find a differentiation formula equivalant to Rodrigues' formula for a particular set of polynomials by the trasformation of variable so that the function under the differential operator is independenf of n. This method is discussed in details by the present author in the work [2] under the head 'classicai Method (A)'. Another method of derivation of a class of bilateral generating relations for certain special functions originates from the method adopted by E. D. Rainville [special Functions (1960)] in deriving particular bilinear or bilateral generating functions for certain special functions. To be precise, this method consists in finding a linear generating relation of the form

$$(19)\qquad \sum_{n=0}^{\infty} A_{m,n} \quad p_{n+m}(x)\, t^n = \frac{f(x.\,t)}{[g(x,t)]^m}\, p_m(h(x,\,t))$$

for suitable coefficient $A_{m,n}$. This method is also discussed in details by the present author in the work [2] under the head 'Classical Method (B)'. Various references of this method may be found in the works of the present author [3, 4, 5, 6 pp. 486-510], the present author and N. B. Ash [7] and J. P. Singhal and H. M. Srivastava [17].

Finally we shall mention that the theory of one-parameter contiunous transformations group enables one to transform a class of generatjng relations for a particular special function into a class of multilateral generating relations. In the case of a class of bilateal generating relations for certain special functions, the present author has discussed in details in the work [2] under the head 'Group-theoretic Method'. Various references in this connection may be found in the works of the present author [6 pp 393-400, 413-418, 449-465, 466-478], the present author and T. D Banejee [8], the present author and A. K. Chongdar [9, 10]. Besides bilateral generating functions, group-theorctic method enabies one to obtain multilatetal generating functions. For example, if we suppose that

$$(20)\quad G(x,\,w,\,u) = \sum_{v=0}^{\infty} a_n w^n P_n^{\lambda}(x)\, p_n(u),$$

where $P_n^{\lambda}(x)$ denotes the ulraspherical polynomial and $p_n(u)$ an arbitrary classical polynonial or function,

and consider the operator $\quad R = (x^2-1)\, y\frac{\partial}{\partial x} + xy^2\frac{\partial}{\partial y} + 2\lambda xy,$

so that

(21) $\exp(\omega R) f(x, y)$

$$= (\omega^2 y^2 - 2\omega xy + 1)^{-\lambda} \; f\left(\frac{x-\omega y}{\sqrt{(\omega^2 y^2-2\omega xy+1)}}, \frac{y}{\sqrt{(\omega^2 y^2-2\omega xy+1)}}\right)$$

(22) $R\,[y^n P_n^{\lambda}(x)] = (n+1)\; y^{n+1}\, P_{n+1}^{\lambda}(x)$,

then we oblain

$$(23) \quad \rho^{-2\lambda} G\left(\frac{x-\omega}{\rho}, \frac{\omega z}{\rho}, u\right) = \sum_{n=0}^{\infty} \omega^n f_n(z, u)\, P_n^{\lambda}(x),$$

where

$$\rho = (1-2wx + \omega^2)^{1/2}, \quad f_n(z, u) = \sum_{m=0}^{n} \binom{n}{m} a_m z^m p_m(u).$$

We remark that this method can be applied to any other classfcal polynomial only when a suitable continuons transformations group given by

$$R \equiv \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} + \zeta(x, y)$$

can be found such that R raises the index of the polynomial by one. By applying such method one can show that.

$$(24) \quad \text{If } G(x.w,u) = \sum_{n=0}^{\infty} a_n w^n L_n^{(\alpha)}(x)\, p_n(u).$$

then

$$(25) \quad (1-w)^{-\alpha-1} \exp\left\{\frac{-wx}{1-w}\right\} G\left(\frac{x}{1-w}, \frac{wz}{1-w}, u\right)$$

$$= \sum_{m=0}^{\infty} L_m^{(\alpha)}(x)\, f_m(z\; u)\, w^m,$$

$$\text{where } f_m(z,u) = \sum_{n=0}^{m} a_n \binom{m}{n} z^n p_n(u).$$

Various applications of the abore results may be found in the works of the present author and B. B. Saha [11. 12]. A nice method of obtaining multilateral generating functions involving Tchebycheff polynomials was discussed by the present author in [13,14].

REFERENCES

[1] Allaway, Wm. R. — The representation of orthogonal polynomials in terms of a differential operator, Jour. Math. Anal. Appl. 56 (1976), 288-293

[2] Chatterjea, S. K. — Recent researches on bilateral generating functions for certain special functions, Ganit (J. Bangladesh Math. Soc.) 2 (1982), 114—137

[3] Chatterjea, S. K. — On a Class of bilateral generating relations for certains special fumctions, Ranchi Univ. Math. J. 6 (1975), 1—5

[4] Chatterjea, S. K. — Unification of a class of bilateral generating relatlans for certaim special functions. Bull. Cal. Math. Soc. 67 (1975), 115—127

[5] Chatterjea, S. K. — An extension of a class of bilateral generating functions for certain special functions, Bull, Inst. Math, Acad, Sinica 5 (1977), 323—331,

[6] Chatterjea, S. K — Classical and group-theoretic analysis of some problems of special functions, D. Sc. thesis, University of Calcutta (1974)

[7] Chatterjea, S, K, and Ash, N. B. — Some extensions of bilateral generating functions of certain special functions, Math. Seminar Notes 10 (1982), 661—666

[8] Chatterjea, S. K. and Banerjee, T, D. — Two parameter Lie groups and special functions, Bull. Cal, Math. Soc. 75 (1983), 239—243

[9] Chatterjea, S. K. and Chongdar, A. K. — On a Class of trilateral generatlng relations with Tchebychev polynomials from the vfew—point of one parameter group of continuous transformations, Bull. Cal. Math. Soc. 73 (1981), 127—140

[10] Chatterjea, S. K. and Chongder, A. K, — on a bilateral generating relation involving Laguerre polynomials, to appear in Math. Balkanica

[11] Chatterjee, S. K. and Saha, B. B. — On generating functions of Gegenbauer polynomials from the view-yoint of Lie groups C. R. Acad. Sc, Paris 284 (1977), 1057—1060

[12] Chatterjea, S. K. and Saha B. B. — On generating functions of Laguerre polyomials from the view-point of Lie groups, C. R. Acad, Sc. Paris 284 (1977), 1121 —1124

[13] Chatterjea, S. K. — Some multilateral generating functions, Bull. Inst. Math. Acad, Sinica 9 (1981), 179—191

[14] Chatterjea, S. K. — Some multilateral generating relations involving Hermite and Tchebycheff polynomials, Jour. Pure Math. 1 (1981), 9—13

[15] Haradze, A. K. — On the representation of ultraspherical polynomials by a differential operator containing the generating function of the polynomials, Soviet. Math. Dokl. 5 (1964), 1191—1193

[16] Poli, L. — Hermite polynomials of differential equation, Mathesis 63 (1954), 319—325

[17] Siinghal, J. P. and Srivastava, H. M. — A class of bilateral generating functions for certain classical polynomiials, Pacific J. Math. 42 (1972), 755—762

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Dept. of Pure Math.
Calcutta University

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A NOTE ON A CLASS OF STEADILY INCREASING CONTINUOUS FUNCTIONS WHICH ARE NOT ABSOLUTELY CONTINUOUS

D. K. GANGULY

1. **Introduction.** In some text books of classical analysis there are some functions which are continuous and monotone increasing but not absolutely continuous in $0 \leqslant x \leqslant 1$. The classical Cantor function falls into this category. E. Hill and T. D. Tamarkin [1] studied the behaviour of Cantor function and established some interesting properties of the function.

Starting with a convergent series $\sum_{n=1}^{\infty} a_n$ of positive terms satisfying the conditions

(i) $\sum_{n=1}^{\infty} a_n=1$ (i) $a_n \geqslant R_n=a_{n+1}+a_{n+2}+\ldots\ldots\ldots$ and (iii) $0 \leqslant a_n-R_n < \frac{b_n}{2^{n-1}}$, where $\sum_{n=1}^{\infty} b_n$ is an arbitrary convergent series of positive terms such that $\Sigma b_n=b<1$, Sengupta and Ganguly [3] constructed a class of steadily increasing functions which are continuous but not absolutely continuos in $0 \leqslant x \leqslant 1$.

In this note we have constructed some functions with similar properties, which include all the above mentioned functions.

2. Construction of the Linear Set C_k in $[0, 1]$.

In a study of theory of sets Cantor middle third set C occupies an important place. We now construct a set C_k in $0 \leqslant x \leqslant 1$ which includes the classical Cantor set as a particular case and whose construction and properties are similar to those of C. The construction of C_k can be done as follows.

We divide the unit interval [0, 1] into $(2k+1)$ equal parts and remove k open intervals in the second, fourth, sixth, ...(2k) th position and thus C_k is now contained in

$(k+1)$ closed intervals each of length $\frac{1}{2k+1}$. We again divide each of the remaining $(k+1)$ closed intervals into $(2k+1)$ equal parts and remove the open intervals in the even positions and hence C_k is now contained in $(k+1)^2$ closed intervals each of length $\frac{1}{(2k+1)^2}$. We continue this process indefinitely. Total length removed in the above process is $\sum_{n=1}^{\infty} \frac{k(k+1)^{n-1}}{(2k+1)^n} = 1$. Hence the set C_k which remains after the removal of the above open intervals being the complement of an everywhere dense set of openi ntervals which do not overlap nor about is a non-dense perfect set of Lebesgue measure zero [2, p. 117]. It should be noted that classical Cantor set C is a particular case of C_k when $k = 1$.

Therefore C_k is the set of all real numbers x, $0 \leqslant x \leqslant 1$, such that $x = \sum_{n=1}^{\infty} \frac{x_n}{(2k+1)^n}$, where $x_n = (0,2,4,\ldots\ldots 2r,\ldots\ldots 2n)$ for all n.

3. Constrution of a Class of Function f (x).

We take an arbitrary convergent series of positve terms $\sum_{n=1}^{\infty} b_n = b(<1)$ and thenc onstruct another series of positive terms such that (i) $\sum_{n=1}^{\infty} a_n = 1$ (ii) $a_n \geqslant k R_n$, $R_n = a_{n+1} + a_{n+2} + \ldots\ldots\ldots\ldots\ldots\ldots$ and k being a fixed positive integer and (iii) $0 \leqslant a_n - k R_n < \frac{b_n}{(k+1)^{n-1}}$,

An actual construction of a series of the above type can be ensured as follows.

We consider a sequence of positive numbers $\left\{ a_n \right\}_{n=1}^{\infty}$ chosen successively according to the following way :

$$\frac{k}{k+1} \leqslant a_1 < \frac{k+b_1}{(k+1)},$$

$$\frac{k}{k+1}(1-a_1) \leqslant a_2 < \min\left\{\frac{k}{k+1}(1-a_1) + \frac{b_2}{(k+1)^2},\ 1-a_1\right\}$$

$$\ldots\ldots\ldots \qquad \ldots\ldots\ldots \qquad \ldots\ldots\ldots \qquad \ldots\ldots\ldots$$

$$\frac{k}{k+1}(1-a_1 \ldots\ldots\ldots - a_{n-1}) \leqslant a_n < \min \left\{ \frac{k}{k+1}(1-a_1- \ldots\ldots - a_{n-1}) + \frac{b_n}{(k+1)^n}, \right.$$

and so on. $\left. (1-a_1-a_2- \ldots\ldots\ldots\ldots\ldots - a_{n-1}) \right\}$

This ensures that (i) $\Sigma a_n = 1$, (ii) $a_n \geqslant k R_n$, and

(iii) $0 \leqslant a_n - k R_n < \dfrac{b_n}{(k+1)^{n-1}}$.

Verification :

Since $a_n < 1-a_1-a_2- \ldots\ldots\ldots - a_{n-1}$, then $a_1 + a_2 + \ldots\ldots + a_n < 1$.

Thus Σa_n is convergent and $\Sigma a_n \leqslant 1$ ... .(3.1)

Hence $a_n \to 0$ as $n \to \infty$.

Again, $(k+1)\, a_n > k\,(1-a_1-a_2- \ldots\ldots -a_{n-1})$

or, $a_1 + a_2 + \ldots\ldots\ldots\ldots + a_n > 1 - \dfrac{a_n}{k}$

Thus $\underset{n\to\infty}{\text{Lt}} \sum_{i=1}^{n} a_i \geqslant 1$ i. e. $\sum_{n=1}^{\infty} a_n \geqslant 1$ ....(3.2)

From (3.1) and (3.2) it follows that $\Sigma a_n = 1$.

Again $\dfrac{a_n}{k} \geqslant 1-(a_1+a_2- \ldots\ldots\ldots +a_n)$ or $a_n \geqslant k R_n$.

Next $a_n < \dfrac{k}{k+1}(1-a_1-a_2- \ldots\ldots\ldots -a_{n-1}) + \dfrac{b_n}{(k+1)^n}$

or, $a_n < k\,(1-a_1-a_2- \ldots\ldots\ldots - a_{n-1} - a_n) + \dfrac{b_n}{(k+1)^{n-1}}$

or, $a_n < k\, R_n + \dfrac{b_n}{(k+1)^{n-1}}$

or. $a_n - k\, R_n < \dfrac{b_n}{(k+1)^{n-1}}$.

We can choose the series of the above type in an infinite number of ways and the cardinal number of the aggregate of the series of the above type is c.

After construction of the series we now define a function f (x) in $0 \leqslant x \leqslant 1$ as follows :

We first define our function for $z \in C_k$ according to the following rule.

If $x \in C_k$, then x can be uniquely represented as

$x = \sum_{n=1}^{\infty} \dfrac{d_n}{(2k+1)^n}$ where $d_n = (0, 2, 4, \ldots \ldots 2k)$ for every n.

For such an x, we set $f(x) = \sum_{n=1}^{\infty} a'_n$ where

$a'_n = \left(o, \frac{a_n}{k}, \frac{2a_n}{k}, \ldots\ldots \frac{ra_n}{k}, \ldots\ldots \frac{ka_n}{k}\right)$ according as

$d_n = (o, 2, 4, \ldots\ldots\ldots\ldots 2r \ldots\ldots\ldots\ldots 2k)$.

Next we define f (x) linearly on the complemeutary set of C_k as follows.

The complementary set of C_k consists of an enumerably infinite number of open intervals with their end points belonging to C_k. If $\xi < x < \eta$ be such an interval ($\xi \in C_k, \eta \in C_k$) we define f (x) in its interior linearly i. e. by $f(x) = f(\xi) + (x-\xi)\frac{f(\eta)-f(\xi)}{\eta-\xi}$, it being obvious that $f(\xi) < f(\eta)$.

4. Propsrties of f (x).

We shall derive some of the more important properties of f (x).

Property 1.

f(x) is a non-decreasing function of x in [0, 1].

Improving this it is sufficient to consider p ints x of C_k. since f (x) is obviously increasing on the complementary set of C_k. We now have to show that if x' and x'' are any two points of C_k and $x' < x''$, then $f(x') \leqslant f(x'')$.

Let $x' = \sum_{n=1}^{\infty} \frac{d'_n}{(2k+1)^n}$ and $x'' = \sum_{n=1}^{\infty} \frac{d_n''}{(2k+1)^n}$

where $d'_n, d_n'' = (0, 2, 4, \ldots\ldots 2n)$ for every n, be the representations of x' and x'' so that there exists a suffix n such that $d'_m = d_m''$ (for all $m < n$) and $d_n' < d_n''$. Hence $f(x') \leqslant f(x'')$.

Property 2.

f(x) is continuous in [0, 1].

As f(x) is linear on complementary intervals of C_k, f (x) is continuous on each of them. Hence it is sufficient to consider points x, x'' of C_k and show that $f(x) \to f(x')$ as $x \to x'$.

Let $x = \sum_{n=1}^{\infty} \frac{d_n}{(2k+1)^n}$ and $x' = \sum_{n=1}^{\infty} \frac{d'_n}{(2k+1)^n}$

Where $d_n, d_n' = (0, 2. 4, \ldots \ldots 2n)$ for all n.

In that case, let $d_i = d_i'$ for $1 \leqslant i < j$.

Hence

$$|f(x)-f(x')| = |(a'_j+a'_{j+1}+\ldots)-(a''_j+a''_{j+1}+\ldots\ldots)|$$
$$\leqslant |(a_j+a_{j+1}+\ldots\ldots)-(0+0+0+\ldots\ldots)|$$
$$\to 0 \text{ if } j\to\infty \text{ and thus if } x\to x'.$$

Property 3.

$f(x)$ is not absolutely continuous in $[0, 1]$.

Now from the construction of $f(x)$ on the complementary intervals constituting the complementary set of C_k, it follows on each of the k—intervals of the first stage, $f'(x) = \frac{(2k+1)}{k}(a_1-kR_1)$ on each of $k(k+1)$ intervals of the second stage, we have $f'(x) = \frac{(2k+1)^2}{k}(a_2-kR_2)$. In general on each of the $k(k+1)^{n-1}$ intervals of the nth stage, we have $f'(x) = \frac{(2k+1)^{n-1}}{k}(a_n-kR_n)$. Since $f(x)$ is continuous and of bounded variation in $[0, 1]$ hence $f'(x)$ is summable over $[0, 1]$, [2, p. 590],.

Thus $\int_0^1 f'(x)\,dx = \int\limits_{[0,1]-C_k} f'(x)\,dx$

$$=\frac{(2k+1)}{k}(a_1-kR_1)\frac{k}{2k+1}+\frac{(2k+1)^2}{k}(a_2-kR_2)\frac{k(k+1)}{(2k+1)^2}$$
$$+\frac{(2k+1)^3}{k}(a_3-kR_3)\frac{k(k+1)^2}{(2k+1)^3}+\ldots\ldots\ldots\ldots$$
$$=(a_1-kR_1)+(k+1)(a_2-kR_2)+(k+1)^2(a_3-kR_3)+\ldots\ldots\ldots\ldots$$
$$<b_1+b_2+b_3+\ldots\ldots\ldots\ldots = b<1.$$

This completes the proof.

Remarks

The result of Sengupta and Ganguly follows when we take $k=1$ (incidentally the result of Hille and Tamankin also follows as shown by Sengupta and Ganguly).

Note 1. We have seen that C_k is closed and of Lebesgue measure zero. We examine measure of the sets $f(C_k)$. As the function $f(x)$ is continuous in $0 \leqslant x \leqslant 1$, the set $f(C_k)$ is closed and hence measurable. It is easy to see that each of the $k(k+1)^{n-1}$ contiguous intervals of the nth stage is mapped into an interval of length $\frac{1}{k}(a_n-kR_n)$

Thus $mf(C_k) = 1 - \left[\frac{k(a_1 - kR_1)}{k} + k(k+1)\frac{(a_2 - kR_2)}{k} + k(k+1)^2\frac{(a_3 - kR_3)}{k} + \ldots\ldots\right]$.

But $0 \leqslant a_n - k\,R_n < \dfrac{b_n}{(k+1)^{n-1}}$

Hence $mf(C_k) > 1 - (b_1 + b_2 + b_3 + \ldots\ldots\ldots\ldots) = 1 - b > 0$

Thus it is interesting to note that the measure of the set $f(C_k)$ is positive although the measure of the set C_k is zero.

Note 2. It follows that corresponding to each of the series Σa_n constructed above we get a non-absolutely continuous and steadily increasing function in $0 \leqslant x \leqslant 1$. The aggregate of such a class of functions has the cardinal number c.

Theorem 1.

For any series $\Sigma a_n = 1$ with $a_n > 0$ and $a_n \geqslant kR_n$ and

$0 \leqslant a_n - kR_n < \dfrac{b_n}{(k+1)^{n-1}}$ where $\Sigma b_n = b\ (<1)$, $\underset{n\to\infty}{\mathrm{Lt}}\ (k+1)^n \dfrac{a_n}{k}$

is positive and equals the measure of the set $f(C_k)$.

Proof : We have seen that each of $k(k+1)^{n-1}$ open intervals of nth order is mapped by $f(x)$ into an open interval of length $\dfrac{a_n - k\,R_n}{k}$. But $\dfrac{a_n - k\,R_n}{k} = \dfrac{R_{n-1} - (k+1)R_n}{k}$.

Also for a given n, the number of such intervals is $k(k+1)^{n-1}$ and therefore rhe total length of such intervals is

$$\frac{k(k+1)^{n-1}[R_{n-1} - (k+1)R_n]}{k} = (k+1)^{n-1}R_{n-1} - (k+1)^n R_n.$$

$$\text{Therefore} \quad \sum_{n=1}^{\infty}\left[(k+1)^{n-1}R_{n-1} - (k+1)^n R_n\right]$$

$$= 1 - \underset{n\to\infty}{\mathrm{Lt}}\ (k+1)^n R_n.$$

Thus we get the measure of the complementary set of the set $f(C_k)$ from which it follows that the measure of $f(C_k)$ is $\underset{n\to\infty}{\mathrm{Lt}}\ (k+1)^n R_n$.

Thus $mf(C_k) = \underset{n\to\infty}{\mathrm{Lt}}\ (k+1)^n R_n$.

Again $0 \leqslant a_n - k\,R_n < \dfrac{b_n}{(k+1)^{n-1}}$.

Therefore, $\underset{n\to\infty}{\mathrm{Lt}} (k+1)^n \frac{a_n}{k} = \underset{n\to\infty}{\mathrm{Lt}} (k+1)^n R_n.$

Thus $m f(C_k) = \underset{n\to\infty}{\mathrm{Lt}} (k+1)^n \frac{a_n}{k}.$

Theorem 2. (R) $\int_0^1 f(x)\,dx = \frac{1}{2}$

Proof.

Let $x \in C_k$. Then $x = \sum_{n=1}^{\infty} \frac{x_n}{(2k+1)^n}$,

where $x_n = (0,2,4,\ldots\ldots\ldots\ldots 2k)$ for every n.

Then $f(x) = \sum_{n=1}^{\infty} d_n$ where $d_n = \left(0, \frac{a_n}{k}, \frac{2a_n}{k} \ldots\ldots\ldots\ldots \frac{ka_n}{k}\right)$

according as $x_n = (0, 2, 4, \ldots\ldots\ldots\ldots 2k)$.

Now $1-x = \sum_{n=1}^{\infty} \frac{2k-x_n}{(2k+1)^n} = \sum_{n=1}^{\infty} \frac{x'_n}{(2k+1)^n}$, where $x'_n = 2k - x_n = 2k, 2k-2, \ldots\ldots\ldots 2, 0$

according as

$x_n = (0, 2, 4, \ldots\ldots\ldots 2k)$ for every n. Hence $1-x \in C_k$.

Therefore. $f(1-x) = \sum_{n=1}^{\infty} e_n$ where $e_n = \left(\frac{ka_n}{k}, \frac{(k-1)\,a_n}{k}, \ldots\ldots\ldots\ldots \frac{a_n}{k}, 0\right)$

according as $x'_n = (2k, 2k-2, \ldots\ldots\ldots 4, 2, 0)$ i. e.

according as $x_n = (0, 2, 4, \ldots\ldots\ldots\ldots\ldots 2k)$.

We get $d_n + e_n = a_n$ for every n.

Hence $f(x) + f(1-x) = \sum_{n=1}^{\infty} (d_n + e_n) = \sum_{n=1}^{\infty} a_n = 1$, for every n.

Next let x be a point in the complementary set of C_k. Then we can find an open interval (ξ, η) such that $\xi < x < \eta$ where $\xi \in C_k$ and $\eta \in C_k$.

In the intetior of (ξ, η), $f(x) = f(\xi) + \frac{x-\xi}{\eta-\xi}\left\{ f(\eta) - f(\xi) \right\}.$

It is easy to verify that $1-x$ also belongs to the complementary set of C_k. Thus there exists an open interval $(\xi' \eta')$ containing the point $(1-x)$ where $\xi'=1-\eta$ and $\eta'=1-\xi$ with $\xi'<\eta'$.

Therefore

$$f(1-x) = f(\xi') + \frac{1-x-\xi'}{\eta'-\xi'}\left\{ f(\eta') - f(\xi') \right\}$$

$$= 1-f(\eta) + \frac{\eta-x}{\eta-\xi}\left\{ 1-f(\xi)-1+f(\eta) \right\}$$

$$= 1-f(\eta) + \frac{\eta-x}{\eta-\xi}\left\{ f(\eta)-f(\xi) \right\}$$

Thus

$$f(x) + f(1-x) = f(\xi) + 1 - f(\eta) + \frac{\eta-\xi}{\eta-\xi}\left\{ f(\eta) - f(\xi) \right\}$$

$$=1.$$

Hence $f(x)$ satisfies the functional equation $f(x) + f(1-x) = 1$ everywhere in $0 \leqslant x \leqslant 1$. Now taking μ to be Lebesgue measure we have $\mu(C_k)=0$ and $\mu\{[0,1]-C_k\}=1$.

If we put $y = 1-x$ in $(R)\int_0^1 f(1-x)\,dx$ we get $(R)\int_0^1 f(1-x)\,dx = (R)\int_0^1 f(y)\,dy = (R)\int_0^1 f(x)\,dx$.

Therefore $(R)\int_0^1 f(x)\,dx = (L)\int_0^1 f(x)\,d\mu = (L)\int_{[0,1]-C_k} f(x)\,d\mu$.

If follows that

$$(L)\int_{[0,1]-C_k} f(x)\,d\mu + (L)\int_{[0,1]-C_k} f(1-x)\,d\mu = (L)\int_{[0,1]} 1.\,d\mu = 1$$

Thus $(R)\int_0^1 f(x)\,dx + (R)\int_0^1 f(1-x)\,dx = 1$.

Therefore $(R)\int_0^1 f(x)\,dx = \frac{1}{2}$

Note. It can similarly be shown that $\frac{1}{2\delta}\int_{\frac{1}{2}-\delta}^{\frac{1}{2}+\delta} f(x)\,dx = \frac{1}{2}$.

for every δ satisfying the condition $0 < \delta \leqslant 1/2$.

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REFERENCES

[1] Hill. E and Tamarkin, T. D. (1929) : Remarks on a known example of a monotone continuous function. Amer. Math. Monthly. Vol. 36, p 255.

[2] Hobson, E. W. (1929) : Theory of functions of a real variable, Vol. I, Dover Publication, N. Y.. p. 117, 590.

[3] Sengupta, S. and Ganguly. P. L. (1959) : A class of steadily increasing coninuous functions which are not absolutely continuous. Bull. Cal. Math. Soc., Vol. 51, P. 53.

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Calcutta University

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RODRIGUES' FORMULISTIC ORIGINS OF CERTAIN GENERATING FUNCTIONS FOR HERMITE POLYNOMIALS.

TAHA IBRAHIM SULTAN
(an Egyptian Student)

The Rodrigues' formula for the Hermite polynomials $H_n(x)$ is

$$H_n(x) = (-1)^n e^{x^2} D^n (e^{-x^2}), \quad D = d/dx.$$

Since in the above Rodrigues' formula the function under the sign of differentiation does not depend upon n, we require no transformation of variabie to make the said function independent of n and we can at once use the operator e^{-tD} on any m—lateral generating relation involving Hermite polynomials in order to derive (m+1) lateral generating relation by means of the operational formula

$$e^{-tD} f(x) = f(x-t),$$

where f (x) is an arbitrarily differentiable function of x. Such Rodrigues' formulistic origins of generating functions involving other special functions exist, provided suitable transformation of variable is found out.

First we consider the generating relation [3]

$$(1) \quad \sum_{n=0}^{\infty} (-t/\alpha)^n L_n^{k-n}(\alpha) H_n(x)$$
$$= (t/\alpha)^k \exp(2xt - t^2) H_k\left(\frac{2t^2 - 2xt + \alpha}{2t}\right).$$

Replacing t by ty, and multiplying both sides by e^{-x^2} and then operating with e^{-tD} we obtain

$$e^{-tD} e^{-x^2} \sum_{n=0}^{\infty} (-ty/\alpha)^n L_n^{-n}(\alpha) H_n(x)$$
$$= e^{-tD} e^{-x^2} (ty/\alpha)^k \exp(2xyt - t y) H_k\left(\frac{2t^2y^2 - 2xyt + \alpha}{2ty}\right).$$

Since $e^{-tD} f(x) = f(x-t)$, then the above right member is equal to

$$(ty/\alpha)^k \exp.[2xt(y+1) - t^2(y+1)^2 - x^2]\ H_k\left(\frac{2t^2y^2-2(x-t)ty+\alpha}{2ty}\right).$$

By virtue of Rodrigues' formula the left member is equal to

$$\sum_{m=0}^{\infty}\frac{(-tD)^m}{m!} e^{-x^2} \sum_{n=0}^{\infty}(-ty/\alpha)^n L_n^{k-n}(\alpha)\, e^{-x^2}(-1)^n D^n e^{-x^2}$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-y/\alpha)^n L_n^{k-n}(\alpha)(-1)^{n+m}\frac{t^{n+m}}{m!} D^{n+m} e^{-x^2}$$

$$= e^{-x^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-y/\alpha)^n L_n^{k-n}(\alpha)\frac{t^{n+m}}{m!} H_{n+m}(x)$$

$$= e^{-x^2}\sum_{n=0}^{\infty}(-ty/\alpha)^n H_n(x)\sum_{m=0}^{n} L_{n-m}^{k-n+m}(\alpha)\frac{(-\alpha/y)^m}{m!}.$$

Equating the two members, we get

$$(2)\quad \sum_{n=0}^{\infty}(-ty/\alpha)^n H_n(x)\sum_{n=0}^{n} L_{n-m}^{k-n+m}(\alpha)\frac{(-\alpha/y)^m}{m!}$$

$$= (ty/\alpha)^k \exp[2xt(y+1)-t^2(y+1)^2]\ H_k\left(\frac{2t^2y^2-2(x-t)ty+\alpha}{2ty}\right),$$

which does not seem to appear before. It my be of interest to note that the generating relation (2) can otherwise be derived by using (1) and the well known generating relation

$$\sum_{n=0}^{\infty} H_{m+n}(x)\frac{t^n}{n!} = \exp(2xt-t^2) H_m(x-t).$$

Secondly we consider the generating relation (4, pp 248)

$$(3)\quad \sum_{n=0}^{\infty}\frac{(ye^{i\alpha})^n}{n!} L_m^{n-m}(2y^2) H_n(x)$$

$$= \frac{(ye^{i\alpha})^m}{m!}\exp(2xye^{i\alpha} - y^2e^{2i\alpha}) H_m(x-2y\cos\alpha).$$

Replacing y by ty, and multiplying both sides by e^{-x^2} and operating by e^{-tD},

we obtain

$$(4)\quad e^{-tD}\, e^{-x^2} \sum_{n=0}^{\infty} \frac{(yte^{i\alpha})^n}{n!} L_m^{n-m}(2y^2t^2)\, H_n(x)$$

$$= e^{-tD} e^{-x^2} \frac{(te^{i\alpha})^m}{m!} \exp\left(2xyte^{i\alpha} - t^2y^2 e^{2i\alpha}\right) H_m(x-2yt\cos\alpha).$$

Since $e^{-tD} f(x) = f(x-t)$, the left member of (4) is equal to :

$$\sum_{r=0}^{\infty} \frac{-t^r}{r!} D^r e^{-x^2} \sum_{n=0}^{\infty} \frac{(yte^{i\alpha})^n}{n!} L_m^{n-m}(2y^2t^2)\, e^{x^2} (-1)^n \quad D^n e^{-x^2}$$

$$= \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{(ye^{i\alpha})^n}{n!} L_m^{n-m}(2y^2t^2) \frac{t^{r+n}}{r!} (-1)^{r+n} \quad D^{r+n} \quad e^{-x^2}$$

$$= e^{-x^2} \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{(ye^{i\alpha})^n}{n!} L_m^{n-m} (2y^2t^2) \frac{t^{r+n}}{r!} \quad H_{r+n}(x)$$

$$= e^{-x^2} \sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!} \sum_{r=0}^{n} \binom{n}{r} (ye^{i\alpha})^{n-r} L_m^{n-m-r}(2y^2t^2)$$

The right member of (4) is equal to :

$$e^{-(x-t)^2} \frac{(yte^{i\alpha})^m}{m!} \exp\left(2yte^{i\alpha}(x-t) - t^2y^2e^{2i\alpha}\right) H_m(x-t-2yt\cos\alpha).$$

Equating the two members, we have

$$(5)\quad \sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!} \sum_{r=0}^{\infty} \binom{n}{r} (ye^{i\alpha})^{n-r} L_m^{n-m-r} (2t^2y^2)$$

$$= \frac{(yte^{i\alpha})^m}{m!} \exp\left(2yte^{i\alpha}(x-t) + 2xt - t^2 - t^2y^2e^{2i\alpha}\right).$$

$$\cdot H_m(x-t-2yt\cos\alpha),$$

which also does not seem to appear before.

Thirdly we consider the generating relation due to G. Doetsch [2]

$$(6)\quad \sum_{n=0}^{\infty} \frac{H_n(x)t^n}{[n/2]!} = (1+4t^2)^{-3/2}(1+2xt+4t^2)\exp\left(\frac{4x^2t^2}{1+4t^2}\right).$$

Replacing t by ty, and multiplying both sides by e^{-x^2} and operating by e^{-tD}, we obtain

$$(7)\quad e^{-tD}\, e^{-x^2} \sum_{n=0}^{\infty} \frac{H_n(x)\,(ty)^n}{[n/2]!}$$

$$= e^{-tD}\, e^{-x^2}\, (1+4t^2y^2)^{-3/2}\, (1+2xty+4t^2y^2)\, \exp\left(\frac{4x^2t^2y^2}{1+1t^2y^2}\right).$$

Since $e^{-tD} f(x) = f(x-t)$, the right member of equation (7) is equal to

$$e^{-(x-t)^2}.\ (1+4t^2y^2)^{-3/2}.\ (1+2yt\,(x-t)+4y^2t^2).\ \exp\left(\frac{4y^2t^2(x-t)^2}{1+4y^2t^2}\right).$$

The left member of equation (7) is equal to

$$\sum_{r=0}^{\infty} \frac{(-t)^r}{r!} D^r e^{-x^2} \sum_{n=0}^{\infty} \frac{H_n(x)\,(ty)^n}{[n/2]!}$$

$$= \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{r+n}\, D^{r+n}}{r!\left[\frac{n}{2}\right]!} e^{-x^2}.\quad (t)^{r+n}\, y^n$$

$$= e^{-x^2} \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{H_{r+n}(x)\, t^{r+n}}{r!\left[\frac{n}{2}\right]!}\, y^n$$

$$= e^{-x^2} \sum_{n=0}^{\infty} H_n(x)\, t^n \sum_{r=0}^{n} \frac{y^{n-r}}{r!\left[\frac{n-r}{2}\right]!}.$$

Equating the to members, we have

$$(8)\quad \sum_{n=0}^{\infty} H_n(x)\,(yt)^n \sum_{r=0}^{n} \frac{y^{-r}}{r!\left[\frac{n-r}{2}\right]!}$$

$$=(1+4y^2t^2)^{-3/2}.\ (1+2yt(x-t)+4y^2t^2)\, \exp\left(\frac{4x^2y^2t^2+2xt-t^2}{1+4y^2t^2}\right)$$

Fourthly we consider the well-known Mehler's formula ;

$$\sum_{k=0}^{\infty} \frac{t^k}{2^k\, k!} H_k(x)\, H_k(y)$$

$$=(1-t^2)^{-\frac{1}{2}}\exp\left(\frac{2xyt-t^2(x^2+y^2)}{(1-t^2)}\right).$$

Replacing t by tz, and multipling both sides by e^{-x^2} and operating by e^{-tD}, we obtain

$$(10)\quad e^{-tD}\,e^{-x^2}\sum_{k=0}^{\infty}\frac{(tz)^k}{2^k\,k!}H_k(x)\,H_k(y)$$

$$=e^{-tD}e^{-x^2}(1-t^2z^2)^{-\frac{1}{2}}\exp\left(\frac{2xyzt-t^2z^2(x^2+y^2)}{(1-t^2z^2)}\right).$$

Since $e^{-tD}f(x)=f(x-t)$, the right member of (10) is equal to

$$e^{-(x-t)^2}(1-t^2z^2)^{-1/2}\exp\left(\frac{2yzt\,(x-t)-t^2z^2\,(\,(x-t)^2+y^2)}{(1+t^2z^2)}\right).$$

The left member of equation (10) is equal to

$$\sum_{n=0}^{\infty}\frac{(-1)^n t^n D^n}{n!}e^{-x^2}\sum_{k=0}^{\infty}\frac{(tz)^k}{2^k\,k!}e^{x^2}(-1)^k D^k e^{-x^2}H_k(y)$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{z^k}{2^k\,n!\,k!}H_k(y)\,t^{n+k}(-1)^{n+k}D^{n+k}e^{-x^2}$$

$$=e^{-x^2}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{z^k}{2^k\,n!\,k!}H_k(y)\,t^{n+k}H_{n+k}(x)$$

$$=e^{-x^2}\sum_{n=0}^{\infty}\frac{H_n(x)\,t^n}{n!}\sum_{k=0}^{n}\binom{n}{k}\frac{z^k}{2^k}H_k(y).$$

By equating the two members, we obtain

$$(11)\quad\sum_{n=0}^{\infty}\frac{H_n(x)\,t^n}{n!}\sum_{k=0}^{\infty}\binom{n}{k}(z/2)^k H_k(y)$$

$$=(1-t^2z^2)^{-1/2}\exp\left(\frac{2yzt(x-t)-t^2z^2(x^2+y^2)+2xt-t^2}{(1-t^2z^2)}\right).$$

Finally we use extended Mehler's formula due to L. Carlitz [1]

$$(12)\quad\sum_{k=0}^{\infty}\frac{t^k}{2^k\,k!}H_{k+m}(x)\,H_{k+n}(y)$$

$$=(1-t^2)^{-\frac{1}{2}(m+n+1)} \cdot \exp\left(\frac{2xyt-t^2(x^2+y^2)}{1-t^2}\right).$$

$$\sum_{r=0}^{\min(m,n)} 2^r r! \binom{m}{r}\binom{n}{r} t^r H_{m-r}\left[\frac{x-yt}{(1+t^2)^{1/2}}\right] H_{n-r}\left[\frac{y-xt}{(1-t^2)^{1/2}}\right].$$

Replacing t by tz, and multipling both sides by e^{-x^2}, and then operating with e^{-tD}, we obtain

$$(13)\quad e^{-tD} e^{-x^2} \sum_{k=0}^{\infty} \frac{(tz)^k}{2^k k!} H_{k+m}(x) H_{k+n}(y)$$

$$= e^{-tD} e^{-x^2} (1-t^2z^2)^{-\frac{1}{2}(m+n+1)} \cdot \exp\left(\frac{2xyzt-t^2z^2(x^2+y^2)}{1-t^2z^2}\right).$$

$$\sum_{r=0}^{\min(m,n)} 2^r r! \binom{m}{r}\binom{n}{r} (tz)^r H_{m-r}\left[\frac{x-tyz}{(1-t^2z^2)^{1/2}}\right] H_{n-r}\left[\frac{y-xtz}{(1-t^2z^2)^{1/2}}\right]$$

Since $e^{-tD} f(x) = f(x-t)$, the right member of (13) is equal to

$$e^{-(x-t)^2} (1-t^2z^2)^{-\frac{1}{2}(m+n+1)} \cdot \exp\left(\frac{2yzt(x-t)-t^2z^2(y^2+(x-t)^2)}{(1-t^2z^2)}\right).$$

$$\sum_{r=0}^{\min(m,n)} 2^r r! \binom{m}{r}\binom{n}{r} (tz)^r H_{m-r}\left[\frac{x-t-yzt}{(1-t^2z^2)^{1/2}}\right] H_{n-r}\left[\frac{y-zt(x-t)}{(1-t^2z^2)^{1/2}}\right].$$

The left member of (13) is equal to

$$\sum_{p=0}^{\infty} \frac{(-1)^p t^p}{p!} D^p e^{-x^2} \sum_{k=0}^{\infty} \frac{(tz)^k}{2^k k!} H_{k+m}(x) H_{k+n}(y)$$

$$= e^{-x^2} \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \frac{z^k}{2^k k!\, p!} t^{p+k} H_{k+n}(y) H_{k+m+p}(x)$$

$$= e^{-x^2} \sum_{p=0}^{\infty} \frac{H_{p+m}(x) t^p}{p!} \sum_{k=0}^{p} \binom{p}{k} (z/2)^k H_{k+n}(y).$$

Equating the two members, we obtain

$$(14)\quad \sum_{p=0}^{\infty} \frac{H_{p+m}(x) t^p}{p!} \sum_{k=0}^{p} \binom{p}{k} (z/2)^k H_{k+n}(y)$$

$$= (1-t^2z^2)^{-\frac{1}{2}(m+n+1)} \cdot \exp\left(\frac{2yzt(x-t)-t^2z^2(x^2+y^2)+2xt-t^2}{(1-t^2z^2)}\right)$$

$$\sum_{r=0}^{\min(m,n)} 2^r\, r! \binom{m}{r}\binom{n}{r} (tz)^r\, H_{m-r}\left[\frac{x-t-yzt}{(1-t^2z^2)^{1/2}}\right] H_{n-r}\left[\frac{y-zt(x-t)}{(1-t^2z^2)^{1/2}}\right]$$

Now if we put $m = 0$, $n = 0$, we have

$$\sum_{p=0}^{\infty} \frac{H_p(x)t_p}{p!} \sum_{k=0}^{p} \binom{p}{k} (z/2)^k H_k(y)$$

$$=(1-t^2z^2)^{-1/2}.\ \exp\left(\frac{2yzt(x-t)-t^2z^2(x^2+y^2)+2xt-t^2}{(1-t^2z^2)}\right)$$

which is our result (11).

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REFERENCES

[1] Carlitz, L. — Boll. Un. Mat. Ital. 1 (1970), 43—46.

[2] Doetsch, G. — Math. Zeitschr. 32 (1930), 587 — 599.

[3] Sultan, Taha Ibrahim — Pure Math. Manuscript, 2 (1983), 45 — 48.

[4] Talman, J D. — Special Functions, W. A. Benjamin Inc. (1968).

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Dept. of Pure Math.
Calcutta University

Jour. Pure Math.
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SOME THEOREMS ON GENERATING FUNCTIONS FOR CHARLIER POLYNOMIALS.

N. BARIK

1. Introduction :

Noticing the existence of the following type of summation formula involving Charlier polynomials

$$(1.1)\quad (1-t)^{\alpha}\ c_{\alpha}\ (a\ ;x-xt) = \sum_{n=0}^{\alpha} \frac{(-\alpha)_n}{n!} t^n\ c_{\alpha-n}\ (a\ ;\ x),$$

we are led to investigate more general generating relation by group-theoretic method. It may be pointed out that the theorems on Charlier polynomirls can be easily stated in terms of modified Laguerre polynomials by means of the relation

$$f_n^{-a}\ (x) = \frac{x^n}{n!}\ c_n\ (a\ ;\ x).$$

Here we like to consider the operator

$$L = xy^{-1}\frac{\partial}{\partial x}+\frac{\partial}{\partial y}-xy^{-1}.$$

Since the above operator is independent of α, n and a, we have considered the action of the operator on different functions viz.,

$$(1\ 2)\quad L\ (c_{\alpha+n}\ (a\ ;\ x)\ y^a) = -x\ c_{\alpha+n+1}\ (a\ ;\ x)\ y^{a-1},$$

$$(1\ 3)\quad L\ (c_{\alpha-n}\ (a\ ;\ x)\ y^a) = -x\ c_{\alpha-n+1}\ (a\ ;\ x)\ y^{a-1},$$

which help us in deriving different theorems on generating functions. The main theorems of the present work are as follows :

Theorem 1. If there exists the following linear generating relation

$$G(x,\ t) = \sum_{n=0}^{\infty}\ a_n\ c_{\alpha+n}\ (a\ ;\ x)\ t^n$$

then the following bilateral generating relation holds

$$\exp(-xt)(1+t)^a G(x+xt, t)$$

$$= \sum_{k=0}^{\infty} \frac{(-xt)^k}{k!} c_{\alpha+k}(a;x) \sum_{n=0}^{k} a_n (-k)_n x^{-n}$$

Theorem 2. If

$$G(x,t) = \sum_{n=0}^{\alpha} a_n c_{\alpha-n}(a;x)\, t^n$$

then the following generating relation holds

$$\exp(-xt)(1+t)^a G(x+xt, t)$$

$$= \sum_{k=0}^{\infty} \frac{(-xt)^k}{k!} \sum_{n=0}^{\alpha} a_n c_{\alpha+k-n}(a;x)\, t^n$$

Applications of the above theorems are pointed out.

2. Derivation of generating functions :

Let us consider the generating function

$$\text{(2.1)}\quad G(x,t) = \sum_{n=0}^{\infty} a_n c_{\alpha+n}(a;x)\, t^n \quad (\alpha \text{ being a non-negative integer})$$

Now multiplying (2.1) by y^a, we have

$$\text{(2.2)}\quad y^a G(x,t) = \sum_{n=0}^{\infty} a_n c_{\alpha+n}(a;x)\, y^a t^n.$$

We consider the operator

$$L = xy^{-1}\frac{\partial}{\partial x} + \frac{\partial}{\partial y} - xy^{-1},$$

such that

$$\text{(2.3)}\quad L\left[c_{\alpha+n}(a;x)\, y^a\right] = -x\, c_{\alpha+n+1}(a;x)\, y^{a-1}$$

The extended form of the group generated by L is given by

$$\text{(2.4)}\quad \exp(tL) f(x,y) = \exp(-txy^{-1}) f(x+txy^{-1}, y+t).$$

Operating both members of (2.2) by exp (tL) we have

$$\text{(2.5)}\quad \exp(tL)\left[y^a G(x,t)\right] = \exp(tL)\left[\sum_{n=0}^{\infty} a_n c_{\alpha+n}(a;x)\, y^a t^n\right].$$

The left membar of (2.5) becomes by means of (2.4)

$$\exp\ (tL)\ \left[y^a\ G(x,t)\ \right] = \exp\ (-txy^{-1})\ (y+t)^a . G(x+txy^{-1},\ t).$$

On the otherhand the right member of (2.5) becomes by means of (2.3)

$$\exp\ (tL) \left[\sum_{n=0}^{\infty} a_n\ c_{\alpha+n}\ (a\ ;\ x)\ y^a\ t^n \right]$$

$$= y^a \sum_{k=0}^{\infty} \frac{\left(-\frac{tx}{y}\right)^k}{k\ !} \sum_{n=0}^{\infty} a_n\ c_{\alpha+n+k}\ (a : x)\ t^n.$$

Therefore (2.5) can be expressed as

$$\exp\ (-\ txy^{-1})\ (y+t)^a\ G(x+txy^{-1},\ t)$$

$$= y^a \sum_{k=0}^{\infty} \frac{\left(-\frac{tx}{y}\right)^k}{k\ !} \sum_{n=0}^{\infty} a_n\ c_{\alpha+n+k}\ (a\ ;\ x)\ t^n$$

In otherwords,

$$\exp\ (\ -txy^{-1})\ \ (1+ty)^{-1\ a}\ G\ (x+txy^{-1},\ t)$$

$$= \sum_{k=0}^{\infty} \frac{\left(-\frac{tx}{y}\right)^k}{k\ !}\ c_{\alpha+k}\ (a\ ;\ x) \sum_{n=0}^{k} a_n (-k)_n \left(\frac{y}{x}\right)^n$$

Hence we obtain theorem 1 on using y=1.

Next we consider the another type of generating function

$$(2.6)\quad G\ (x,t) = \sum_{n=0}^{\alpha} a_n\ c_{\alpha-n}\ (a\ ;\ x).t^n$$

(α being a non-negative integer).

Now multiplying (2.6) by y^a we get

$$(2.7)\quad y^a\ G(x,t)\ = \sum_{n=0}^{\alpha} a_n\ c_{\alpha-n}\ \ (a\ ;\ x)\ y^a\ t^n.$$

Operating both members of (2 7) by exp (tL) we have

$$(2.8)\quad \exp\ (tL) \left[y^a\ G(x,t) \right] = \exp\ (tL) \left[\sum_{n=0}^{\alpha} a_n\ \ c_{\alpha-n}\ (a\ ;\ x)\ y^a t^n \right]$$

The left member of (2.8) can be expressed by means of (2.4) as

$$\exp(tL)[y^a G(x,t)] = \exp(-txy^{-1})(y+t)^a G(x+txy^{-1}, t)$$

The right member of (2.8) can be expressed by means of (2.3) as

$$\exp(tL)\left[\sum_{n=0}^{\alpha} a_n c_{\alpha-n}(a;x)\, y^a t^n\right]$$

$$= y^a \sum_{k=0}^{\infty} \frac{(-tx/y)^k}{k!} \sum_{n=0}^{\alpha} a_n c_{\alpha-n+k}(a;x)\, t^n$$

So we can rewrite (2.8) in the following way :

$$\exp(-txy^{-1})(1+ty^{-1})^a G(x+txy^{-1}, t)$$

$$= \sum_{k=0}^{\infty} \frac{(-tx/y)^k}{k!} \sum_{n=0}^{\alpha} a_n c_{\alpha-n+k}(a;x)\, t^n$$

Hence we derive theorem 2 on using $y=1$.

For an application of the the orem 2, we consider

$$a_n = \frac{(-\alpha)_n}{n!}.$$

Then we have [1 ; p. 71]

$$(1-t)^{\alpha} c_{\alpha}(a;x-xt) = \sum_{n=0}^{\alpha} \frac{(-\alpha)_n}{n!} t^n c_{\alpha-n}(a;x).$$

Thus we obtain the following type of generating relation from the above generating relation by means of theorem 2.

$$\exp(-xt)(1+t)^a (1-t)^{\alpha} c_{\alpha}(a;x(1-t^2))$$

$$= \sum_{k=0}^{\infty} \frac{(-tx)^k}{k!} \sum_{n=0}^{\alpha} \frac{(-\alpha)_n}{n!} c_{\alpha+k-n}(a;x)\, t^n.$$

Conclusion : One can at once obtain a large number of generating relations by means of our theorems by attributing suitable values to a_n in the respective theorems.

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REFERENCE

1- Mc Bride, E. B. — Obtaining generating functions ; Springer—Verlag, Berlin. (1971)

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Gurudas College of Commerce
Calcutta-54

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SOME THEOREMS ON GENERATING FUNCTIONS FOR MODIFIED LAGUERRE POLYNOMIALS

MANIK CHANDRA MUKHERJEE

1. Introduction ;

Noticing the existence of the following types of generating relations

$$(1.1)\quad e^{xt}(1-t)^{-(k+\beta)} f_k^{\beta}(x(1-t)) = \sum_{n=0}^{\infty} \frac{(k+1)_n}{n!} f_{k+n}^{\beta}(x)\, t^n$$

$$(1.2)\quad e^{zt}(1-t)^{-\beta} = \sum_{n=0}^{\infty} f_n^{\beta}(x)\, t^n$$

as well as the following finite summation formula

$$1.3)\quad f_k^{\beta}(x-t) = \sum_{n=0}^{k} \frac{1}{n!} f_{k-n}^{\beta}(x)\, t^n,$$

we are led to investigate more general generating relations by group-theoretic method, For this purpose we have considered the operator

$$R = xy\frac{\partial}{\partial x} - y^2\frac{\partial}{\partial y} - (x+k)\, y,$$

such that

$$(1.4)\quad R\left[f_{k+n}^{\beta}(x)\, y^{\beta+n}\right] = -(k+n+1)\, f_{k+n+1}^{\beta}(x)\, y^{\beta+n+1}$$

$$(1.5)\quad R\left[f_{k-n}^{\beta}(x)\, y^{\beta-n}\right] = -(k-n+1)\, f_{k-n+1}^{\beta}(x)\, y^{\beta-n+1},$$

which help us in deriving different theorems on generating functions.

The main theorems of the present work are as follows :

Theorem 1

If there exists the following type of generating function

$$G(x, t) = \sum_{n=0}^{\infty} a_n f_{k+n}^{\beta}(x)\, t^n,$$

then the following bilateral generating relation holds

$$e^{-xy}(1+y)^{-(k+\beta)} G(x(1+y),\ ty(1+y)^{-1})$$

$$= \sum_{n=0}^{\infty} (-y)^m f_{k+m}^{\beta}(x) \sum_{n=0}^{m} a_n \binom{m+k}{n+k} (-t)^n.$$

Theorem 2

If

$$G(x, t) = \sum_{n=0}^{k} a_n f_{k-n}^{\beta}(x)\, t^n,$$

then

$$e^{-xy}(1+y)^{-(k+\beta)} G\left(x(1+y),\ \frac{t}{y}(1+y)\right)$$

$$= \sum_{r=0}^{\infty} (-y)^r \sum_{n=0}^{k} a_n \binom{k-n+r}{r} f_{k-n+r}^{\beta}(x) \left(\frac{t}{y}\right)^n.$$

Applications of the above theorems are pointed out.

2. Derivation of generating functions :

Proof of theorem 1

Let

$$(2.1)\quad G(x,t) = \sum_{n=0}^{\infty} a_n f_{k+n}^{\beta}(x)\, t^n.$$

Multiplying both sides of (2·1) by y^{β} and writing ty for t, we have

$$(2.2)\quad y^{\beta} G(x,\ ty) = \sum_{n=0}^{\infty} a_n f_{k+n}^{\beta}(x)\, y^{\beta+n} t^n.$$

Now we consider the operator

$$R = xy\frac{\partial}{\partial x} - y^2\frac{\partial}{\partial y} - (x+k)y,$$

such that

$$(2.3)\quad R\left[f_{k+n}^{\beta}(x)\, y^{\beta+n}\right]$$

$$= -(k+n+1) f_{k+n+1}^{\beta}(x)\, y^{\beta+n+1}$$

The extended torm of the group generated by R is given by

(2.4) $e^{pr}f(x,y) = e^{-pxy}(1+py)^{-k} f(x(1+py), y(1+py)^{-1})$

Now we shall operate both sides of (2 2) by e^{pR}

First we have by (2.4)

(2-5) $e^{pR}\left[y^{\beta} G(x, ty) \right]$

$$=e^{-pxy}\, y^{\beta} (1+py)^{-(k+\beta)}\, G(x(1+py), ty(1+py)^{-1})$$

On the other hand we have by (2 3)

$$e^{pR} \sum_{n=0}^{\infty} a_n f_{k+n}^{\beta}(x)\, y^{\beta+n} t^n$$

$$= \sum_{m=0}^{\infty} \frac{p^m}{m!} R^m \sum_{n=0}^{\infty} a_n f_{k+n}^{\beta}(x)\, y^{\beta+n} t^n$$

(2.6) $$= y^{\beta} \sum_{m=0}^{\infty} \frac{(-py)^m}{(m-n)!} f_{k+m}^{\beta}(x) \sum_{n=0}^{m} a_n (n+k+1)\ m-n \left(-\frac{t}{p}\right)^n$$

Equating the results (2.5) and (2.6) we have

$$e^{-xy}\,(1+y^{-(k+\beta)}\, G(x(1+y), ty(1+y)^{-1})$$

$$= \sum_{m=0}^{\infty} (-y)^m f_{k+m}^{\beta}(x) \sum_{n=0}^{m} a_n \binom{m+k}{n+k} (-t)^n,$$

on using $p=1$,

which is our therem 1. Now we shall discuss applications of our theorem.

Let

$$a_n=\frac{(k+1)_n}{n!}.$$

Then we have [1, p 45, (8)]

$$e^{xt}(1-t)^{-(k+\beta)} f_k^{\beta}(x(1-t)) = \sum_{n=0}^{\infty} \frac{(k+1)_n}{n!} f_{k+n}^{\beta}(x)\, t^n$$

Thus by means of our theorem 1, we obtain

$$e^{xy(t-1)}\,(1+y-ty)^{-(k+\beta)}\, f_k^{\beta}(x(1+y-ty))$$

$$= \sum_{m=0}^{\infty} (-y)^m f_{k+m}^{\beta}(x) \sum_{n=0}^{m} \frac{(k+1)_n}{n!} \binom{m+k}{n+k} (-t)^n.$$

For k=0, we at once get the following corollary from the above theorem :

Corollary : If

$$G(x, t) = \sum_{n=0}^{\infty} a_n f_n^{\beta}(x) t^n$$

Then

$$e^{-xy} (1+y)^{-\beta} G(x(1+y), ty(1+y)^{-1})$$

$$= \sum_{m=0}^{\infty} (-y)^m f_m^{\beta}(x) \sum_{n=0}^{m} a_n \binom{m}{n} (-t)^n.$$

Proof of theorem 2. Let

$$(2.7) \quad G(x, y) = \sum_{n=0}^{k} a_n f_{k-n}^{\beta}(x) t^n.$$

Multiplying both sides of (2.7) by y^{β} and writing $\frac{t}{y}$ for t, we get

$$(2.8) \quad y^{\beta} G\left(x, \frac{t}{y}\right) = \sum_{n=0}^{k} a_n f_{k-n}^{\beta}(x) y^{\beta-n} t^n.$$

Now we concider the operator

$$R = xy \frac{\partial}{\partial x} - y^2 \frac{\partial}{\partial y} - (x+k) y$$

such that

$$(2.9) \quad R\left[f_{k-n}^{\beta}(x) y^{\beta-n}\right]$$

$$= -(k-n+1) f_{k-n+1}^{\beta}(x) y^{\beta-n+1},$$

The extended form of the group generated by R is given by

$$(2.10) \quad e^{pR} f(x,y) = e^{-pxy} (1+py)^{-k} f(x(1+py), y(1+py)^{-1})$$

Now we shall operate both sides of (2·8) by e^{pR}.

First we have

$$(2.11) \quad e^{pR}\left[y^{\beta} G\left(x, \frac{t}{y}\right)\right]$$

$$= e^{-pxy}\, y^{\beta}\, (1+py)^{-(k+\beta)}\, G\left(x\,(1+py),\ \frac{t}{y}\, 1+py)\right)$$

On the other hand, we get

$$e^{pR} \sum_{n=0}^{k} a_n\, f_{k-n}^{\beta}(x)\, y^{\beta-n}\, t^n$$

$$= \sum_{r=0}^{\infty} \frac{p^r}{r\,!} R^r \sum_{n=0}^{k} a_n\, f_{k-n}^{\beta}(x)\, y^{\beta-n}\, t^n$$

$$(2.12) \quad = y^{\beta} \sum_{r=0}^{\infty} (-py)^r \sum_{n=0}^{k} a_n \binom{k-n+r}{r} f_{k-n+r}^{\beta}(x) \left(\frac{t}{y}\right)^n$$

Equating the results (2.11) and (2.12), we get.

$$e^{-xy}\, (1+y)^{-(k+\beta)}\left(x\,(1+y),\ \frac{t}{y}\,(1+y)\right)$$

$$= \sum_{r=0}^{\infty} (-y)^r \sum_{n=0}^{k} a_n \binom{k-n+r}{r} f_{k-n+r}^{\beta}(x) \left(\frac{t}{y}\right)^n,$$

which is our therem 2, on using $p=1$,

Now we consider an application of the above theorem :

$$\text{Let } a_n = \frac{1}{n\,!},$$

then we know [1, p 45, (7)]

$$f_k^{\beta}(x+t) = \sum_{n=0}^{k} \frac{1}{n\,!} f_{k-n}^{\beta}(x)\, t^n.$$

Now by means of our theorem we at once obtain the following generating relation

$$e^{-xy}\, (1+y)^{-(k+\beta)}\, f_k^{\beta}\left(\frac{(xy+t)\,(1+y)}{y}\right)$$

$$= \sum_{r=0}^{\infty} (-y)^r \sum_{n=0}^{k} \frac{1}{n\,!} \binom{k-r+r}{r} f_{k-n+r}^{\beta}(\chi) \left(\frac{t}{y}\right)^n.$$

One can easily obtain a large number of generating relations in future by means of our theorem by giving suitable values to a_n.

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REFERENCE

[1] McBride, E. B.—Obtaining generating functions, Springer—Verlag, Berlin, Heidelberg, New York (1971).

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FOREWARD

This year the University of Calcutta has generously offered financial assistance to hold a Seminar from September 17—21. 1984, in the department of Pure Mathematics and to publish the present volume which contains papers contributed for the seminar.

Journal of Pure Mathematics vol 1 (1981) first appeared entirely out of the research grant for the teachers of the department of Pure Mathematics, Calcutta University, due to undaunted and untiring mettle of S. K. Chatterjea in order to give top priority of the research works of young aspirants of the department as per resolution of the departmental committee meeting dated 12. 5. 1981. Subsequently the University of Calcutta has generously sanctioned a grant for the publication of this journal instead of using research grant for the teachers of the department.

Thanks are due to the staff of the Elm Press for their ready collaboration in vol 4 (1984).

Chief Editor

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ON SOME TOEPLITZ MATRICES WITH CANTOR SET ELEMENTS

N. C. BOSE MAJUMDER AND KEYA GANGULY

1. **Introduction.** If a Toeplitz matrix is made up of elements, which can be expressed as functions of Cantor set numbers, then we shall call such a matrix a (C)-matrix. In this paper, we propose to deal with some particular types of (C)-matrices, constructed from Rienermann matrix [3].

The Rienermann matrix $T = (t_{m,n})$ is defined as follows :

$$t_{m,o} = \prod_{j=o}^{m} (1 - c_j) \;;$$

$$t_{m,n} = c_{n-1} \prod_{j=n}^{m} (1 - c_j),\ 1 \leq n \leq m \;;$$

$$t_{m,m+1} = c_m \;;$$

$$t_{m,n} = 0.\ n > m + 1.$$

T is a Toeplitz matrix, with a proper choice of $\{c_n\}_{n=o}^{\infty}$. By choosing the numbers $\{c_n\}_{n=o}^{\infty}$ in the above matrix T, as Cantor set elements (i.e., $c_n \in \Gamma$, $n = 0, 1, 2 \ldots$, where Γ is the Cantor set) we shall use this matrix as a (C)-matrix.

In this paper, we shall use another Toeplitz matrix T_o defined as follows :

$$T_o = (a_{n,k}),\ a_{n,k} = \frac{1}{n},\ 1 \leq k \leq n \;;\ a_{n,k} = 0,\ k > n.$$

If a sequence $\{u_n\}_{n=1}^{\infty}$ be transformed into a sequence $\{t_n\}_{n=1}^{\infty}$ by this matrix T_o and if $\lim_{n\to\infty} t_n = t$, then we say that the sequence $\{u_n\}_{n=1}^{\infty}$ is T_o-limitable to t [4].

S. Reich, [2] gave the following theorem :

"Let $x_o \in C$, a closed convex subset of a Banach space $(E, \|\cdot\|)$.

Let $g : C \to C$ satisfy,

$$\| g(x) - g(y) \| \leq \tfrac{1}{2} \{ \| x - g(x) \| + \| y - g(y) \| \}.$$

Let the sequence $S = \{x_n : n \in N\}$ be defined by the matrix $T = (t_{m,n})$:

$$x_{n+1} = t_{n,o}\, x_o + \sum_{i=0}^{n} t_{n,i+1}\, g(x_i),\quad n \in N\ (= \text{the set of positive integers}),\ \text{whence}$$

$x_n \in C$; and c_n, s in $(t_{m,n})$ satisfy, $0 < c_n \leqq 1$, for all $n \in N$, the series $\sum_{n=0}^{\infty} c_n$ being divergent and a subsequence of $\{c_n\}_{n=0}^{\infty}$ converges to a certain s wher $0 < s \leqq 1$.

If S converges to x, then x is the fixed point of g."

In this paper, we introduce a function g (with reference to the cantor middle third set $\Gamma \subset [0, 1]$) given by

$$g : \Gamma \to \Gamma, \text{ where } g(x) = 1 - x,\ x \in \Gamma$$

as we know that $g(x) \in \Gamma$, whenever $x \in \Gamma$, and g is continuous on Γ (relative to Γ).

We have, in this paper, considered the existence of a fixed point of the function g, defined on the Cantor set Γ, which is shown here to satisfy all the conditions of S. Reich's Theorem, with respect to a Rienermann matrix [used here as a (C)-matrix] excepting that Γ is not a convex subset of a Banach space E. We have arrived at the conclusion of our Theorem, given in § 3, through the following five sections 2.1, 2.2, 2.3, 2.4 and 2.5.

2.1 It is to be observed that Γ is not a linear space. We define its norm, as the Euclidean norm and thus

$$\| g(x) - g(y) \| = | g(x) - g(y) | = | 1 - x - 1 + y | = | x - y |$$

If $x, y \leqq \frac{1}{2}$, then

$$\tfrac{1}{2}\{ \| x - g(x) \| + \| y - g(y) \| \} = \tfrac{1}{2}\{ | x - 1 + x | + | y - 1 + y | \}$$
$$= \tfrac{1}{2}\{ | 2x - 1 | + | 2y - 1 | \} = \tfrac{1}{2}\{ 1 - 2x + 1 - 2y \} = 1 - x - y$$

If, for definiteness, $x \geqq y$ and $x - y = d \geqq 0$, and thus $\| g(x) - g(y) \| = d$ and

$$\tfrac{1}{2}\{ \| x - g(x) \| + \| y - g(y) \| \} = 1 - x - y = 1 - 2x + d \geqq d.$$

Hence $\| g(x) - g(y) \| \leqq \frac{1}{2}\{ \| x - g(x) \| + \| y - g(y) \| \}$.

If $x \geqq \frac{1}{2}$ any $y \geqq \frac{1}{2}$, then as above

$$\tfrac{1}{2}\{ \| x - g(x) \| + \| y - g(y) \| \} = x + y - 1 = d + 2y - 1 \geqq d$$
$$= | x - y | = \| g(x) - g(y) \|$$

If $x \leqq \frac{1}{2}$ and $y \geqq \frac{1}{2}$, then as above (since $y \geqq x$),

$$\tfrac{1}{2}\{ \| x - g(x) \| + \| y - g(y) \|) \} = \tfrac{1}{2}\{ 1 - 2x + 2y - 1 \}$$
$$= y - x = | x - y | = g(x) - g(y) \|.$$

Hence in all cases, we get

$$\| g(x) - g(y) \| \leqq \tfrac{1}{2}\{ \| x - g(x) \| + \| y - g(y) \| \}.$$

2.2 We now particularise Rienermann matrix $T = (t_{m,n})$ by choosing the sequence $\{c_n\}_{n=0}^{\infty}$ with

$$c_0 = \tfrac{1}{3} \text{ and } c_n = 1, \text{ for all } n = 1, 2, 3, \ldots.$$

Hence with $\{c_n\}_{n=0}^{\infty} \subset \Gamma$, T is now a (C) – matrix.

We observe that c_n's satisfy $0 < c_n \leqq 1$, for all $n = 0, 1, 2, 3, \ldots$ and that $\sum_{i=0}^{\infty} c_i$ is divergent.

Also a subsequence of $\{c_n\}_{n=0}^{\infty}$, namely, $\{c_n\}_{n=1}^{\infty}$, converges to an $s\ (=1)$ when $0 < s \leqq 1$.

2.3 We then apply the process of iteration, i.e., use the formula

$$x_{n+1} = t_{n,0}\, x_0 + \sum_{i=0}^{n} t_{n,i+1}\, g(x_i),\ n \in N;$$

to find the sequence $S = \{x_n\}_{n=0}^{\infty} \subset \Gamma$.

Starting with $x_0 = 1\ (\in \Gamma)$, we obtain successively, $x_1 = \tfrac{2}{3}, x_2 = \tfrac{1}{3}, x_3 = \tfrac{2}{3}, x_4 = \tfrac{1}{3}$, and so on,

So the sequence $S = \{x_n\}_{n=0}^{\infty} = \{1, \tfrac{2}{3}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{3}, \ldots\}$, which is not convergent.

2.4 Tietze's Extension Theorem [1] is given as follows :

"Let X be a metric space and let $F\ (F \subset X)$ be a closed set.

If $f : E \to E'$, is continuous on F (relative to F) and if f is bounded with, $\sup\{\,|f(x)| \,/\, x \in F\} = M < \infty$, then there is a continuous (on X) function $g : X \to E^1$, such that $x \in F \Rightarrow g(x) = f(x)$ and $|g(x)| \leqq M$, for each $x \in X$."

We now consider the Cantor set Γ. We notice that $\Gamma \subset X\ [= [0, 1]]$ is a closed set and

$g : \Gamma \to E^1$, (as given in the section 2.1) is continuous on Γ (relative to Γ) and g is bounded with

$$\sup\{\,|g(x)| \,/\, x \in F\} = M\ (=1) < \infty.$$

Hence according to Tietze's theorem mentioned above, there exists a continuous (on X) function $G : X \to E^1$, such that $x \in \Gamma \Rightarrow G(x) = g(x)$ and

$$|G(x)| \leqq M\ (=1), \text{ for all } x \in X.$$

If we take $G(x) = 1 - x$ for all x on $X [= [0, 1]]$, it satisfies all the conclusions of Tietje's Theorem.

Moreover $x = \frac{1}{2}$ is the fixed point of the transformation of $G : X \to E^1$.

2.5 In 2.3, we have seen that the sequence $S = \{x_n\}_{n=0}^{\infty}$ is not convergent. We now show that S is T_0-limitable to $\frac{1}{2}$.

Let the sequences $\{x_n\}_{n=0}^{\infty}$ be transformed into the sequence $\{t_n\}_{n=1}^{\infty}$ by the matrix T_0.

$$\text{Then } t_{2n} = \frac{x_0 + x_1 + \ldots\ldots + x_{2n-1}}{2n} = \frac{1 + \frac{2}{3} + \frac{1}{3} + \ldots + \frac{2}{3}}{2n}$$

$$= \frac{1 + \dfrac{2n}{3} + \dfrac{n-1}{3}}{2n+1} \to \tfrac{1}{2}, \text{ as } n \to \infty$$

$$t_{2n-1} = \frac{x_0 + x_1 + \ldots + x_{2n}}{2n+1} = \frac{1 + \frac{2}{3} + \frac{1}{3} + \ldots + \frac{1}{3}}{2n+1}$$

$$= \frac{1 + \dfrac{2n}{3} + \dfrac{n}{3}}{2n+1} \to \tfrac{1}{2}, \text{ as } n \to \infty$$

Hence $\lim\limits_{n \to \infty} t_n = \frac{1}{2}$ as $n \to \infty$ and thus S is T_0-limitable to $\frac{1}{2}$.

3. **Conclusion.** Combining the section 2.1 through 2.5, we conclude that the Cantor set Γ (which is not a convex subset of a Banach space E) has the following property (given below in the form of a theorem), which bears some similarity with the property of the set C as given by S. Reich metioned above.

Theorem. Let $x_0 \ (= \frac{1}{3}) \in \Gamma$, where Γ is the Cantor set. The function $g : \Gamma \to \Gamma$ satisfies

$$\| g(x) - g(y) \| \leqq \tfrac{1}{2} \{ \| x - g(x) \| + \| y - g(y) \| \},$$

on taking $g(x) = 1 - x$, $x \in \Gamma$,

it being known that $1 - x \in \Gamma$, if $x \in \Gamma$ and $\| . \|$ is the Euclidean norm $| . |$, the function g being continuous on Γ (relative to Γ). The sequence $S = \{x_n : n \in N\}$ is constructed by the iteration process, with the help of the formula,

$$x_{n+1} = t_{n,\,0}\, x_0 + \sum_{i=0}^{n} t_{n,\,i+1}\, g(x_i),\ n \in N,\ x_0 \in \Gamma,\ x_n \in \Gamma,$$

by using the (C)-matrix $T = (t_{m,\,n})$, which is a particular case of the Rienermann matrix, with the sequence $\{c_n\}_{n=0}^{\infty}$, having values $c_0 = \frac{1}{3}$ and $c_n = 1, 2, 3, \ldots$, ensuring that

$\{c_n\}_{n=0}^{\infty} \subset \Gamma$ and $0 < c_n \leqq 1$ for all $n \in N$ and also ensuring that the series $\sum_{n=0}^{\infty} c_n$ is divergent and also that a subsequence of $\{c_n\}_{n=0}^{\infty}$ converges to $s\ (=1)$ satisfying $0 < s \leqq 1$.

The sequence $S = \{x_n\}_{n=0}^{\infty} = \{1, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3}, \ldots\}$

though not convergent, is T_0-limitable to $\frac{1}{2}$, which is the fixed point of the function G in $0 \leqq \times \leqq 1$, given by $G(x) = 1 - x$, the existence of G being assured by Tietze's extension Theorem, it being noted that the original function g has no fixed point on Γ.

Concluding remarks : It follows from above, that Reich's condition that C is to be a convex subset of a Banach space E, cannot be dropped without affecting the conclusion of his Theorem.

Moreover, Reich *assumes* that his sequence S is convergent ; but he does not discuss the situation in which S fails to converge.

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REFERENCES

[1] Randolph, J. F.—Basic Real and Abstract Analysis, Academic Press (1968) p. 310

[2] Reich, S.—*Bull. Cal. Math. Soc.*, 65 (1973), 203-207

[3] Reinermann, J.—Studies Math., 32 (1969), 209–227

[4] Salat, T.—*Czechoslovak Math. Journal Praha*, 26 (1976), 613-627.

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Calcutta University

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SOME PROPERTIES OF THE GENERALIZED MODIFIED BESSEL FUNCTION FROM THE VIEW POINT OF LIE-ALGEBRA*

DIPAK KUMAR BASU

1. **Introduction.** W. Miller, Jr. [1] introduced an interesting family of 3 dimensional Lie-Algebras $g_{p,q}$ which forms a natural generalization of $\mathscr{T}_3 \cong g_{1,1}$. The special functions associated with this family form a natural generalization of Bessel functions. Thus the functions $I_l^{p,q}(r)$ are group-theoretic generalization of Bessel functions and may be defined by the following generating relation :

$$\text{(1.1)} \qquad \exp\left[\frac{r}{p+q}\left(z^p + z^{-q}\right)\right] = \sum_{l=-\infty}^{\infty} I_l^{p,q}(r) z^l$$

Indeed, $I_l^{1,1}(r) = (-i)^l J_l(ir)$, is the ordinary modified Bessel function.

The object of this paper is to consider some properties of $I_l^{p,q}(r)$ from a different point of view. We first notice [1, p. 318] the following recursion relations for $I_l^{p,q}(r)$:

$$\text{(1.2)} \qquad \left(q\frac{d}{dr} + \frac{m}{r}\right) I_m^{p,q}(r) = I_{m-p}^{p,q}(r)$$

$$\text{(1.3)} \qquad \left(p\frac{d}{dr} - \frac{m}{r}\right) I_m^{p,q}(r) = I_{m+q}^{p,q}(r)$$

Then by giving suitable interpretation to m we form the Lie elements for Lie Algebra (one is called the raising operator and the other lowering operator) and using Lie's canonical variables we reduce the calculation to a minimum. The main results of our investigation are the following :

$$\text{(1.4)} \qquad \left(\frac{r+\alpha(p+q)}{r+\beta(p+q)}\right)^{m/(p+q)} I_m^{p,q}\left[\left(r+\alpha(p+q)\right)^q \left(r+\beta(p+q)\right)^p\right]^{1/(p+q)}$$

$$= \sum_{s,t=0}^{\infty} \frac{\alpha^s \beta^t}{s!\,t!} I_{m-sp+tq}^{p,q}(r).$$

---

* This work was included in the Ph. D. Thesis' Calcutta University (1979) of the author.

(1.5) If $\sum_{m=0}^{\infty} \frac{a_m}{m!} I_m^{p,q}(r)\, z^m = F(r, z)$, then

$$\sum_{m=0}^{\infty} (yz)^m I_m^{p,q}(r)\, b_m(yz)$$

$$= F\left[r^q (r + \alpha(p+q)^p)^{1/(p+q)},\ zy \left(\frac{r}{r+\alpha(p+q)}\right)^{1/(p+q)} \right]$$

where $b_m(x) = \frac{a_{m-lq}(\alpha x^{-q})^l}{l!\,(m-lq)}$

(1.6) If $G(r, z, \theta) = \sum_{m=0}^{\infty} \frac{a_m}{m!} I_m^{p,q}(r) z^m p_m(\theta)$

then

$$G\left[(r^q (r + \alpha(p+q))^p)^{1/(p+q)},\ zy\left(\frac{r}{r+\alpha(p+q)}\right)^{1/(p+q)}, \theta \right]$$

$$= \sum_{m=0}^{\infty} F_m(r, \Phi)\,(yz)^m\, b_m(yz, \theta).$$

where

$$b_m(x, \theta) = \sum_{n=0}^{[m/q]} \frac{a_{m-nq}}{(m-nq)!} \frac{(\alpha x^{-q})^n}{n!} p_{m-nq}(\theta).$$

It may be of interest to remark that the result (1.4) does not seem to appear earlier in the form given here. Furthermore (1.4) gives rise to many interesting special cases already derived by Miller. Next in connection with the results (1.5) and (1.6) we remark that the method adopted here in the derivation of class of bilateral generating relation and a class of mixed trilateral generating relations respectively is perfectly general (i. e. it can be applied to any special function for which a Lie algebra can be constructed) and it is suggested by S. K. Chatterjea as one of the methods of obtaining a general class of bilateral and mixed trilateral generating relations from a given unilateral and a given bilateral generating relation respectively. The importance of the result (1.5) lies in the fact that whenever for a particular value of a_m the unilateral generating function $F(r, \Phi)$ is known, then the corresponding bilateral generating relation can at once be obtained

from our result and thus a large number of bilateral generating relations could be obtained by attributing different values to a_m. Silimar remarks could be made in the case of the result (1.6).

2. Lie Algebra for generalized modified Bessel function.—

The recurssion relation for generalized modified Bessel functions are

$$\left(q \frac{d}{dr} + \frac{m}{r} \right) I_m^{p,q}(r) = I_{m-p}^{p,q}(r)$$

$$\left(p \frac{d}{dr} - \frac{m}{r} \right) I_m^{p,q}(r) = I_{m+q}^{p,q}(r)$$

In order to obtain the raising operator R and the lowering operator L in a form which does not refer to the index of Bessel function, we interpret m as the result of operating with $-i\left(\frac{\partial}{\partial \Phi}\right)$ upon the function and changing $\frac{d}{dr}$ into $\frac{\partial}{\partial r}$ we observe that

$$e^{iq\Phi} \left(\frac{i}{r} \frac{\partial}{\partial \Phi} + p \frac{\partial}{\partial r} \right) [e^{im\Phi} I_m^{p,q}(r)] = e^{(m+q)i\Phi} I_{m+q}^{p,q}(r)$$

and $e^{-ip\Phi} \left(\frac{-i}{r} \frac{\partial}{\partial \Phi} + q \frac{\partial}{\partial r} \right) [e^{im\Phi} I_m^{p,q}(r)]$

$$= e^{(m-p)i\Phi} I_{m-p}^{p,q}(r)$$

Our raising and lowering operators are thus

$$R = e^{iq\Phi} \left(\frac{i}{r} \frac{\partial}{\partial \Phi} + p \frac{\partial}{\partial r} \right)$$

$$L = e^{-ip\Phi} \left(\frac{-i}{r} \frac{\partial}{\partial \Phi} + q \frac{\partial}{\partial r} \right).$$

Since $[R, L] = 0$ and $L\, f(r, \Phi) \not\equiv \rho\,(r, \Phi)\, Rf\,(r, \Phi)$, we can reduce the above pair of operators into Lie's cennonical form and we use Lie's cannonical variables $\bar{\Phi}$ and $\bar{r}$ satisfying the following pairs of equations :

$$\begin{cases} e^{iq\Phi} \dfrac{i}{r} \dfrac{\partial \bar{\Phi}}{\partial \Phi} + p e^{iq\Phi} \dfrac{\partial \bar{\Phi}}{\partial r} = 1 \\ e^{-ip\Phi} \dfrac{-i}{r} \dfrac{\partial \bar{\Phi}}{\partial \Phi} + e^{-ip\Phi} q \dfrac{\partial \bar{\Phi}}{\partial r} = 0 \end{cases}$$

and

$$e^{iq\Phi} \frac{i}{r} \frac{\partial \bar{r}}{\partial \Phi} + p e^{iq\Phi} \frac{\partial \bar{r}}{\partial r} = 0$$

$$e^{-ip\Phi} \frac{-i}{r} \frac{\partial \bar{r}}{\partial \Phi} + e^{-ip\Phi} q \frac{\partial \bar{r}}{\partial r} = 1$$

From the above pair of equations we obtain the desired transformation.

(2.1) $$\bar{\Phi} = \frac{r}{p+q} e^{-qi\phi} \quad \text{and} \quad \bar{r} = \frac{r}{p+q} e^{ip\phi}$$

$$\text{so that} \quad e^{i\phi} = \left(\frac{\bar{r}}{\bar{\Phi}}\right)^{1/(p+q)}, \quad r = (p+q)\,(\bar{r}^q \bar{\Phi}^p)^{1/(p+q)}$$

under which the raising and lowering operators become

$$R = \frac{\partial}{\partial \bar{\Phi}} \quad \text{and} \quad L = \frac{\partial}{\partial \bar{r}}.$$

3. Deductions of unilateral generating relations.

First we consider the general operator $\alpha L + \beta R$, In the space of $\bar{r}$ and $\bar{\Phi}$, the finite operators exp (αL) and exp (βR) are translation operators. Exp (αL) sends $\bar{r}$ into $\bar{r} + \alpha$ and leaves $\bar{\Phi}$ invariant while exp (βR) seconds $\bar{\Phi}$ into $\bar{\Phi} + \beta$ leaving $\bar{r}$ invariant. Thus we have

$$\text{Exp}\,(\alpha L + \beta R)\; G_m(\bar{r}, \bar{\Phi}) = G_m(\bar{r} + \alpha, \bar{\Phi} + \beta)$$

where $G_m(\bar{r}, \bar{\Phi})$ is the transform of $F_m(r, \Phi) \equiv e^{im\phi} I^{p,q}(r)$ under the transformation (2.1).

On the other hand

$$\exp\,(\alpha L + \beta R)\; G_m(\bar{r}, \bar{\Phi}) = \sum_{s,t=0}^{\infty} \frac{\alpha^s \beta^t}{s!\,t!} L^s R^t G_m(\bar{r}, \bar{\Phi})$$

$$= \sum_{s,t=0}^{\infty} \frac{\alpha^s \beta^t}{s!\,t!} G_{m-sp+tq}(\bar{r}, \bar{\Phi})$$

Equating the two results, we obtain the addition theorem

$$G_m(\bar{r} + \alpha, \bar{\Phi} + \beta) = \sum_{s,t=2}^{\infty} \frac{\alpha^s \beta^t}{s!\,t!} G_{m-sp+tq}(\bar{r}, \bar{\Phi})$$

Transferring to original co-ordinate (r, Φ) we get the main result :

(3.1) $$\left(\frac{r + \alpha(p+q)}{r + \beta(p+q)}\right)^{m/(p+q)} I_m^{p,q}\,[\,r + \alpha(p+q))^q\,(r + \beta(p+q))^p]^{1/(p+q)}$$

$$= \sum_{s,t=0}^{\infty} \frac{\alpha^{ts} \beta}{s!\,t!} I^{p,q}_{m-sp+tq}(r),$$

which does not seem to appear in the form cited in (3.1).

It may be of interest to remark that this result gives rise to many interesting results of W. Miller as special cases :

Indeed, If $\alpha = 0$, our result becomes

$$\left[1 + \frac{\beta(p+q)}{r}\right]^{-m/(p+q)} I_m^{p,q}\left[r\left(1 + \frac{\beta(p+q)}{r}\right)^{p/(p+q)}\right]$$

$$= \sum_{t=0}^{\infty} \frac{\beta^t}{t!} I_{m+tq}^{p,q}(r),$$

which may be compared with the result of W. Miller [1 ; p, 321].

Again, if we set $\beta = 0$, the above result reduces to

$$\left[1 + \frac{(p+q)\,b}{r}\right]^{m/(p+q)} I_m^{p,q}\left[r\left(1 + \frac{b(p+q)}{r}\right)^{q/(p+q)}\right]$$

$$= \sum_{t=0}^{\infty} \frac{1}{t!} b^t I_{m-tp}^{p,q}(r),$$

where $b = \alpha$;

which may be compared with the result of W. Miller [1, p. 321]

To deduce another interesting result of W. Miller, we consider the combination

$$\frac{\alpha}{p+q}(L+R).$$

Operating this on a function, we have on one hand,

$$\exp\left(\frac{\alpha}{p+q}(L+R)\right) F_m(r, \Phi) = \exp\left(\frac{\alpha}{p+q} L\right) \exp\left(\frac{\alpha}{p+q} R\right) F_m(r, \Phi)$$

$$= \sum_{t=0}^{\infty} \sum_{s=0}^{\infty} \frac{\left(\frac{\alpha}{p+q}\right)^t R^t}{t!} \frac{\left(\frac{\alpha}{p+q}\right)^s}{s!} L^s F_m(r, \Phi)$$

$$= \sum_{s,t=0}^{\infty} \frac{\left(\frac{\alpha}{p+q}\right)^{t+s}}{t!\ s!} F_{m-sp+tq}(r, \Phi)$$

$$= \sum_{tq-sp=n=-\infty}^{\infty} F_{m+n}(r, \Phi) \left[\sum_{t=0}^{\infty} \frac{\left(\frac{\alpha}{p+q}\right)^{\{t(p+q)-n\}/p}}{t!\left(\frac{tq-n}{p}\right)!}\right].$$

On the other hand

$$\exp\left(\frac{\alpha}{p+q}\,(L+R)\right) F_m(r,\Phi) = F_m(r',\Phi')$$

where r' and Φ' are given by the following :

$$\begin{cases} \dfrac{r'\,e^{ip\Phi'}}{p+q} = \dfrac{re^{ip\Phi}}{p+q} + \dfrac{\alpha}{p+q} \\[2ex] \dfrac{r'\,e^{-iq\Phi'}}{p+q} = \dfrac{re^{-iq\Phi}}{p+q} + \dfrac{\alpha}{p+q} \end{cases}$$

Equating the two results, we hrve

$$F_m(r',\Phi') = \sum_{n=-\infty}^{\infty} F_{m+n}(r,\Phi)\left[\sum_{t=0}^{\infty} \frac{\left(\dfrac{\alpha}{p+q}\right)^{\{t(p+q)-n\}/n}}{t!\left(\dfrac{tq-n}{p}\right)!}\right]$$

$$(3.4)\qquad e^{im\phi'} I_m^{p,q}(r') = \sum_{n=-\infty}^{\infty} e^{i(m+n)\phi}\, I_{m+n}^{p,q}(r)\,.\left[\sum_{t=0}^{\infty} \frac{\left(\dfrac{\alpha}{p+q}\right)^{\{t(p+q)-n\}/n}}{t!\left(\dfrac{tq-n}{p}\right)!}\right]$$

In this relation, if we set $r=\Phi=0$, then $r'=\alpha$ and $\Phi'=0$, we obtain

$$I_m^{p,q}(\alpha) = \sum_{n=-\infty}^{\infty} I_{m+n}^{p,q}(o)\left[\sum_{t=0}^{\infty} \frac{\left(\dfrac{\alpha}{p+q}\right)^{\{t(p+q)-n\}/n}}{t!\left(\dfrac{tq-n}{p}\right)!}\right]$$

Now, it is evident from the recursion relation that $I_k^{p,q}(o)=0$ for every k except $k=0$. Hence, if m an integer, we have

$$I_m^{p,q}(\alpha) = \sum_{t=0}^{\infty} \frac{\left(\dfrac{\alpha}{p+q}\right)^{\{t(p+q)+m\}/p}}{t!\left(\dfrac{tq+m}{p}\right)!}$$

Thus from (3.4) we obtain

$$e^{i(\phi'-\phi)m}\, I_m^{p,q}(r') = \sum_{n=-\infty}^{\infty} e^{in\phi}\, I_{m+n}^{p,q}(r)\; I_{-m}^{p,q}(\alpha)$$

or, $$e^{-im\phi}\left(\frac{re^{ip\phi}+\alpha}{re^{-iq\phi}+\alpha}\right)^{m/(p+q)} I_m^{p,q}\left[(re^{ip\phi}+\alpha)^q\,(re^{-iq\phi}+\alpha)^p\right]^{1/(p+q)}$$

$$=\sum_{n=-\infty}^{\infty} (e^{i\phi})^n\, I_{m+n}^{p,q}(r)\; I_{-m}^{p,q}(\alpha).$$

Setting $ei = t$ we finally obtain

(3.5) $$\left(\frac{r+\alpha t^{-p}}{r+\alpha t^{q}}\right)^{m/(p+q)} I_m^{p,q}\left[r\left(1+\frac{\alpha}{rt^{p}}\right)^{q/(p+q)}\cdot\left(1+\frac{\alpha}{rt^{q}}\right)^{q/(p+q)}\right]$$

$$=\sum_{n=-\infty}^{\infty} t^n \; I_{m+n}^{p,q}(r)\; I_{-m}^{p,q}(\alpha),$$

which was alternatively derived by W. Miller.

4. A class of general bilateral generating relations for generalized modified Bessel function.

For generalized modified Bessel function we shall deduce the following theorem :

Theorem.

(4.1) If

$$\sum_{m=0}^{\infty} \frac{a_m}{m!} I_m^{p,q}(r)\, z^m = F(r,z)$$

then there exists a bilateral generating relation of the form :

$$\sum_{m=0}^{\infty} (yz)^m \; I_m^{p,q}(r)\; b_m(yz)$$

$$= F\left[(r^q(r+\alpha(p+q))^p)^{1/(p+q)},\; xy\left(\frac{r}{r+\alpha(p+q)}\right)^{1/(p+q)}\right]$$

where

$$b_m(x) = \sum_{l=0}^{[m/q]} \frac{a_{m-lq}\,(\alpha x^{-q})^l}{l!\,(m-lq)!}$$

Proof. Replacing z with $zy\,e^{i\phi}$, we have

$$\sum_{m=0}^{\infty} \frac{a_m}{m!} e^{im\phi} I_m^{p,q}(r)\, z^m y^m = F(r, zy\,e^{i\phi})$$

operating both sides with $e^{\alpha R}$ where

$$R = e^{iq\phi}\left(\frac{i}{r}\frac{\partial}{\partial\phi}+p\frac{\partial}{\partial r}\right)$$

we get for the left member of the above equalily

$$e^{\alpha R} \sum_{m=0}^{\infty} \frac{a_m}{m\,!}\, e^{im\phi}\, I_m^{p,\,q}(r)\, z^m y^m -$$

$$= \sum_{m=0}^{\infty} \sum_{l=0}^{[m/q]} \frac{\alpha^l}{l\,!}\, \frac{a_{m-lq}}{(m-lq)\,!}\, e^{im\phi}\, I_m^{p,\,q}(r)\, z^{m-lq} y^{m-lq}\,.$$

Next to calculate $e^{\alpha R} F(r, yze^{i\phi})$ we use Lie's cannonical variables as explained in the previous section. Under the cannonical variables we have the transformation to (2.1) and the operator R reduces to the form $\frac{\partial}{\partial \bar{\Phi}}$.

Now, $$e^{\alpha \frac{\partial}{\partial \bar{\Phi}}} F\left((p+q)\, (\bar{r}^q \bar{\Phi}^p)^{1/(p+q)},\ zy \left(\frac{\bar{r}}{\bar{\Phi}}\right)^{1/(p+q)} \right)$$

$$= F\left[(p+q)\, (\bar{r}^q (\bar{\Phi} + \alpha)^p)^{1/(p+q)},\ zy \left(\frac{\bar{r}}{\bar{\Phi}+\alpha}\right)^{1/(p+q)} \right]$$

$$= F\left(r^q (r + \alpha (p+q)\, e^{iq\phi})^p\right)^{1/p+q},\ zy \left(\frac{r\, e^{ip\phi}}{re^{-iq\phi} + \alpha (p+q)} \right)^{1/(p+q)}$$

equating the two results and putting $\Phi = 0$, we obtain (4.2).

The importance of this class of bilateral generating relations lies in the fact that whenever for a particular value of a_m, the generating function $F(r, \Phi)$ is known, then the corresponding bilateral generating relation can at once obtained from our general result (4.2). Such a class of bilateral generating relations for generalized modified Bessel function does not seem to appear in any other earlier investigatoin.

As for application, if we have $a_m = m\,!$, we know that (4.1) becomes

$$\sum_{m=0}^{\infty} I_m^{p,\,q}(r)\, \Phi^m = \exp\left[\frac{r}{p+q}\,(\Phi^p + \bar{\Phi}^{-q})\right]$$

then from our theorem we at once obtain

(4.3) $$\sum_{m=0}^{\infty} (y\Phi)^m\, I_m^{p,\,q}(r)\, b_m(y\Phi)$$

$$= \exp\left[\frac{r^q (r + \alpha (p+q)^p)^{1/(p+q)}}{p+q} \left\{ \Phi^p y^p \left(\frac{r}{r + \alpha (p+q)}\right)^{p/(p+q)} + \bar{\Phi}^{-q} y^{-q} \left(\frac{r}{r + \alpha (p+q)}\right)^{-q/(p+q)} \right\} \right]$$

where $$b_m{}^l(x) = \sum_{l=0}^{[m/q]} \frac{(\alpha x^{-q})^l}{l\,!}$$

5. **A Class of mixed trilateral generating relations for generalized modified Bessel function.**

(5.1) If $$G(r, z, \theta) = \sum_{m=0}^{\infty} \frac{a_m}{m\,!} I_m^{p,q}(r)\, z^m p^m(\theta)$$

where $I_m^{p,q}(r)$ is the generalised Bessel function and $p_m(\theta)$ is an arbitrary classical function of order m, then there exists a mixed trilateral generating relation of the form.

(5.2) $$G\left[(r^q(r+\alpha(p+q))^p)^{1/(p+q)},\ yz\left(\frac{r}{r+\alpha(p+q)}\right)^{1/(p+q)}, \theta\right]$$

$$= \sum_{m=0}^{\infty} F_m(r)\,(yz)^m\, b_m(yz, \theta)$$

where

(5.3) $$b_m(x, \theta) = \frac{a_{m-nq}}{(m-nq)\,!} \frac{(\alpha x^{-q})^n}{n\,!} p_{m-nq}(\theta),$$

proof.

Replacing z by $bze^{i\phi}$ in (5.1) we have

$$G(r, yze^{i\phi}, \theta) = \sum_{m=3}^{\infty} \frac{a_m}{m\,!} y^m z^m F_m(r, \Phi)\, p_m(\theta)$$

where $F_m(r, \Phi) = e^{im\phi} I_m^{p,q}(r)$.

Now operating both sides of the last equation by $\exp \alpha R$ we obtain for the right member

$$\sum_{n=0}^{\infty} \frac{\alpha^n R^n}{n\,!} \sum_{m=0}^{\infty} \frac{a^m}{m\,!} y^m z_m F_m(r, \Phi)\, p_m(\theta$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{[m/q]} \frac{a_{m-nq}}{(m-nq)\,!} \frac{\alpha^n}{n\,!} y^{m-nq} z^{m-nq} F_m(r, \Phi)\, p_{m-nq}(\theta).$$

$$= \sum_{m=0}^{\infty} F_m(r, \Phi)(yz)^m \sum_{n=0}^{[m/q]} \frac{a_{m-nq}}{(m-nq)!\, n!} (\alpha y^{-q} z^{-q})^n \, p_{m-n^1}(\theta)$$

$$= \sum_{m=0}^{\infty} F_m(r, \Phi)(yz)^m \, b_m(yc, \theta)$$

$$\text{where} \quad b_m(x, \theta) = \sum_{n=0}^{[m/q]} \frac{a_{m-nq}}{(m-uq)!\, n!} (\alpha x^{-q})^n \, p_{m-nq}(\theta).$$

On the other hand the left member reduces to

$$\exp \alpha\, R\, G(r, yze^{i\phi}, \theta)$$

$$= e^{\alpha \frac{\partial}{\partial \bar{\Phi}}} G\left((p+q)\,(\bar{r}^q \bar{\Phi}^p)^{1,(p+q)},\; yz \left(\frac{\bar{r}}{\bar{\Phi}}\right)^{1/(p+q)},\; \theta \right)$$

$$= G\left[(p+q)\,(\bar{r}^q (\bar{\Phi}+\alpha)^p)^{1/(p+q)},\; yz \left(\frac{\bar{r}}{\bar{\Phi}+\alpha}\right)^{1/(p-q)},\; \theta \right]$$

$$= G\left[(r^q (r+\alpha\,(p+q)\, e^{iq\phi})^p)^{1/(p+q)},\; zy \left(\frac{re^{ip\phi}}{re^{-iq\phi}+\alpha(p+q)}\right)^{1/(q+p)},\; \theta \right]$$

Equating the two results and putting $\Phi = 0$ we obtain (5.2).

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REFERENCE

[1] W. Miller, Jr. Lie Theory and Special functions. New York : Academic Press (1968)

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Department of Mathematics
Bangabasi Evening College
Calcutta – 700 009

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A CAUCHY TYPE PROBLEM FOR A SECOND ORDER MATRIX DIFFERENTIAL OPERATOR

N. K. Chakraborty and Swapna Roy Paladhi

1. **Introduction.** The problem under consideration is to solve the system

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial x^2} - Q(x)\, U \tag{1.1}$$

with

$$\begin{cases} U(x,t)\,|_{t=0} = f(x) \\ \dfrac{\partial}{\partial t} U(x,t)\,|_{t=0} = h(x) \end{cases} \tag{1.2}$$

where $U \equiv U(x,t) \equiv \begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix}$; $f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$ and $h(x) = \begin{pmatrix} h_1(x) \\ h_2(x) \end{pmatrix}$ are two arbitrary vector valued functions of x, such that $f(x)$ is twice differentiable and $h(x)$ is the integral of an absolutely continuous function ; also $Q(x) \equiv \begin{pmatrix} p(x) & r(x) \\ r(x) & q(x) \end{pmatrix}$, where $p(x)$, $q(x)$, $r(x)$ are real valued functions integrable in every finite interval.

The present problem is an extension of the Cauchy problem to differential systems, where $\frac{\partial}{\partial t} U(x,t)\,|_{t=0}$ has a value other than zero. The method of procedure leading to the solution of the problem is analogous to the one developed by Povzner [3], Levitan and Sargsyan [2], and is essentially an adaptation of the Riemann and Goursat methods for partial differential equations.

2. **Solutions of the problem by the Riemann method.**

The homogeneous equation corresponding to (1.1) is

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial x^2}, \tag{2.1}$$

which is an equation of the hyperbolic type ; we put $x = \frac{1}{2}(\xi + \eta)$, $t = \frac{1}{2}(\eta - \xi)$. Then (2.1) reduces to

$$\frac{\partial^2 U}{\partial \xi\, \partial \eta} = 0. \tag{2.2}$$

Solving (2.2) in ξ, η we obtain the solution of (2.1) in the form

(2.3) $$U = \begin{pmatrix} C_1(x-t) + C_2(x+t) \\ C_3(x-t) + C_4(x+t) \end{pmatrix}$$

where the C_j, $j = 1, 2, 3, 4$, are arbitrary functions.

Then, from (2.3), the solution of (2.1) satisfying the boundary conditions (1.2) is given by

(2.4) $$U_0(x, t) = \begin{pmatrix} u_0(x, t) \\ v_0(x, t) \end{pmatrix} = \tfrac{1}{2}\left[f(x+t) + f(x-t) + g(x+t) - g(x-t)\right]$$

where

$$g(x) = \int_0^x h(y)\, dy .$$

A particular solution $\bar{U}(x, t)$ of (1.1) satisfying the boundary conditions

$$\bar{U}(x, t)\,|_{t=0} = 0, \qquad \left.\frac{\partial \bar{U}(x, t)}{\partial t}\right|_{t=0} = 0, \text{ is}$$

(2.5) $$\bar{U}(x, t) = -\tfrac{1}{2} \int_{\Delta_{x,t}} Q(y)\, U(y, \tau)\, dy$$

where $\Delta_{x,t}$ is the triangle in the y, τ plane with vertices $(x-t, 0)$, (x, t) and $(x+t, 0)$.

The integral equation corresponding to (1.1) and (1.2) is therefore,

(2.6) $$U(x, t) = U_0(x, t) - \tfrac{1}{2}\int_0^t d\tau \int_{x-(t-\tau)}^{x+(t-\tau)} Q(y)\, U(y, \tau)\, dy$$

Let

(2.7) $$U_k(x, t) = \tfrac{1}{2}\int_0^t d\tau \int_{x-(t-\tau)}^{x+(t-\tau)} Q(y)\, U_{k-1}(y, \tau)\, dy \text{ ; for } k \geqslant 1$$

$U_0(x, t)$ being given by (2.4), where $U_k(x, t) = \begin{pmatrix} u_k(x, t) \\ v_k(x, t) \end{pmatrix}$

Then by the well-known method of successive approximations it follows that the equation (2.6) has the solution

(2.8) $$U(x, t) = U_0(x, t) - U_1(x, t) + U_2(x, t) - U_3(x, t) + \ldots\ldots$$

From (2.7) and (2.4)

$$U(x, t) = \tfrac{1}{4}\int_0^t d\tau \int_{x-(t-\tau)}^{x+(t-\tau)} Q(y)\{f(y+\tau) + f(y-\tau) + g(y+\tau) - g(y-\tau)\}\, dy$$

$$= I_{11} + I_{12}, \text{ say.}$$

where

$$I_{11} = \tfrac{1}{4} \int_0^t d\tau \int_{x-(t-\tau)}^{x+(t-\tau)} Q(y) \{ f(y+\tau) + f(y-\tau) \} dy$$

$$= \tfrac{1}{4} \int_0^t d\tau \int_{x-t+2\tau}^{x+t} Q(s-\tau) f(s)\, ds + \tfrac{1}{4} \int_0^t d\tau \int_{x-t}^{x+t-2\tau} Q(s+\tau) f(s)\, ds$$

$$= \tfrac{1}{4} \int_{x-t}^{x+t} \left[\int_{\frac{1}{2}(s+x-t)}^{\frac{1}{2}(s+x+t)} Q(\sigma)\, d\sigma \right] f(s)\, ds$$

On changing the order of integration in the repeated integral and simplification as in Levitan and Sargsyan [2], we have

$$I_{11} = \tfrac{1}{2} \int_{x-t}^{x+t} W_1(x, t, s) f(s)\, ds$$

where,

$$W_1(x, t, s) = \tfrac{1}{2} \int_{\frac{1}{2}(s+x-t)}^{\frac{1}{2}(s+x+t)} Q(\sigma)\, d\sigma \tag{2.9}$$

Again, on simplification and change in the order of repeated integration as before, we have

$$I_{12} = \tfrac{1}{4} \int_0^t d\tau \int_{x-(t-\tau)}^{x+(t-\tau)} Q(y) \{ g(y-\tau) - g(y+\tau) \} dy$$

$$= -\tfrac{1}{4} \int_{x-t}^{x+t} \left[\left\{ \int_s^{\frac{1}{2}(s+x+t)} + \int_s^{\frac{1}{2}(s+x-t)} \right\} Q(\sigma)\, d\sigma \right] g(s)\, ds$$

$$= -\tfrac{1}{2} \int_{x-t}^{x+t} T_1(x, t, s)\, g(s)\, ds, \text{ say,}$$

where

$$T_1(x, t, s) = \tfrac{1}{2} \left[\int_s^{\frac{1}{2}(s+x+t)} + \int_s^{\frac{1}{2}(s+x-t)} \right] Q(\sigma)\, d\sigma \tag{2.10}$$

Since the matrix Q is symmetric, the two matrices $W_1(x, t, s)$ and $T_1(x, t, s)$ are also symmetric. Altogteher, we have

$$U_1(x, t) = \tfrac{1}{2} \int_{x-t}^{x+t} \{ W_1(x, t, s) f(s) - T_1(x, t, s)\, g(s) \}\, ds \tag{2.11}$$

To obtain a formula analogous to (2.11) for each $U_k(x, t)$ occurring in (2.8), viz.,

$$U_k(x, t) = \tfrac{1}{2} \int_{x-t}^{x+t} \{ W_k(x, t, s)\, f(s) - T_k(x, t, s)\, g(s) \}\, ds \tag{2.12}$$

we assume that the result is true for $k = n-1$. Then from (2.12) and (2.7) we have

$$U_n(x, t) = \tfrac{1}{2} \int_0^t d\tau \int_{x-(t-\tau)}^{x+(t-\tau)} Q(y)\, dy \left[\tfrac{1}{2} \int_{y-\tau}^{y+\tau} \{ W_{n-1}(y, \tau, s)\, f(s) - T_{n-1}(y, \tau, s)\, g(s) \}\, ds \right]$$

Now, as in Levitan—Sargsyan [2, p-6] we change the order of integration in the above in such a way that the outer integral is taken with respect to s. It then follows that

$$\begin{aligned} U_n(x, t) = \tfrac{1}{2} \int_{x-t}^{x+t} \Big[&\Big\{ \tfrac{1}{2} \int_{\Omega_{\tau y}} Q(y)\, W_{n-1}(y, \tau, s)\, d\tau\, dy \Big\} f(s) \\ &- \Big\{ \tfrac{1}{2} \int_{\Omega_{\tau y}} Q(y)\, T_{n-1}(y, \tau, s)\, d\tau\, dy \Big\}\, g(s) \Big]\, ds \end{aligned} \tag{2.13}$$

where $\Omega_{\tau y}$ is some domain in the (y, τ) plane; (τ, y depend on x, t).

Put

$$W_n(y, \tau, s) = \tfrac{1}{2} \int_{\Omega_{\tau y}} Q(y)\, W_{n-1}(y, \tau, s)\, d\tau\, dy$$

and

$$T_n(y, \tau, s) = \tfrac{1}{2} \int_{\Omega_{\tau y}} Q(y)\, T_{n-1}(y, \tau, s)\, d\tau\, dy.$$

Then from (2.13), it is obvious that (2.12) holds for $k = n$, whenever it holds for $k = n-1$; $W_1(x, t, s)$ and $T_1(x, t, s)$ being given by (2.9) and (2.10) respectively. It follows by mathematical induction that (2.12) holds for all positive integral values of n $(\geqslant 1)$. A limiting consideration then yields (from (2.6) and (2.8)) that

$$\begin{aligned} U(x, t) &= \tfrac{1}{2} [f(x+t) + f(x-t) + g(x+t) - g(x-t)] \\ &\quad + \tfrac{1}{2} \int_{x-t}^{x+t} \{ W(x, t, s)\, f(s) - T(x, t, s)\, g(s) \}\, ds \end{aligned} \tag{2.14}$$

where

$$W(x, t, s) = \sum_{r=1}^{\infty} (-1)^r\, W_r(x, t, s)$$

and

$$T(x, t, s) = \sum_{r=1}^{\infty} (-1)^r\, T_r(x, t, s),$$

the matrices $W_r(x, t, s)$, $T_r(x, t, s)$ having elements $(W_r)_{ij}$ and $(T_r)_{ij}$ respectively.

The matrices $W(x, t, s)$ and $T(x, t, s)$ may be called the Riemann matrix functions associated with the system (1.1) – (1.2).

If possible, let $U_1(x, t)$, $U_2(x, t)$ be two different solutions of the same problem (1.1)—(1.2). Then $\Phi = U_1 - U_2$ is also a solution. But $\Phi \mid_{t=0} = U_1 \mid_{t=0} - U_2 \mid_{t=0} = 0 = f$ and similarly, $g = 0$. Then $\Phi = 0$ and the solution of the Cauchy type problem is unique.

We define the modulus of a matrix M, represented by $\mid M \mid$, as the sum of the moduli of all of its elements. Then the following inequalities hold.

Since $x - t \leqslant s \leqslant x + t$, therefore, $s \leqslant \frac{1}{2}(s + x + t) \leqslant x + t$
and $s \geqslant \frac{1}{2}(s + x - t) \geqslant x - t$,
Thereforee, from (2.9) and (2.10)

$$(2.15) \qquad \mid W_1(x, t, s) \mid \leqslant \frac{1}{2} \int_{x-t}^{x+t} \mid Q(\sigma) \mid d\sigma$$

and

$$(2.16) \qquad \mid T_1(x, t, s) \mid \leqslant \int_{x-t}^{x+t} \mid Q(\sigma) \mid d\sigma$$

provided p, q, r are integrable over any finite interval on the real axis.

$$(2.17) \qquad \text{Let } \mid W_{n-1} \mid \leqslant \frac{t^{n-2}}{2^{n-1}(n-2)!} \left[\int_{x-t}^{x+2} \mid Q(\sigma) \mid d\sigma \right]^{n-1}, \quad n > 1$$

Then, since $x - t \leqslant y - \tau \leqslant y + \tau \leqslant x + t$, so that

$x - (t - \tau) \leqslant y \leqslant x + (t - \tau)$, we have

$$\mid W_n \mid \leqslant \frac{1}{2} \int_{\Omega_{\tau y}} \mid Q(y) \, W_{n-1}(y, \tau, s) \mid d\tau \, dy$$

$$\leqslant \frac{1}{2} \int_{\Omega_{\tau y}} \mid Q(y) \mid dy \, d\tau \frac{\tau^{n-2}}{2^{n-1}(n-2)!} \left[\int_{y-\tau}^{y+\tau} \mid Q(\sigma) \mid d\sigma \right]^{n-1}$$

(by (2.17))

$$\leqslant \frac{1}{2^n (n-2)!} \left[\int_{x-t}^{x+t} \mid Q(\sigma) \mid d\sigma \right]^{n-1} \int_{\Omega_{\tau y}} Q(y) \, \tau^{n-2} dy \, d\tau$$

$$\leqslant \frac{1}{2^n (n-2)!} \left[\int_{x-t}^{x+t} \mid Q(\sigma) \mid d\sigma \right]^{n-1} \int_0^t \tau^{n-2} d\tau \int_{x-t}^{x+t} \mid Q(\sigma) \mid d\sigma$$

Since $\Omega_{y\tau} : \{0 \leqslant \tau \leqslant t \,;\; x - t + \tau \leqslant y \leqslant x + t - \tau\}$

$\subseteq \Omega'_{y\tau} : \{0 \leqslant \tau \leqslant t \,;\; x - t \leqslant y \leqslant x + t\}$

therefore, we have finally

$$\mid W_n \mid \leqslant \frac{t^{n-1}}{2^n (n-1)!} \left[\int_{x-t}^{x+t} \mid Q(\sigma) \mid d\sigma \right]^n .$$

Hence by mathematical induction, the inequality holds for all positive integral values of n.

Therefore, from

$$W(x, t, s) = \sum_{r=1}^{\infty} (-1)^r\, W_r(x, t, s)$$

it follows that

$$(2.18) \qquad |W(x, t, s)| \leqslant \tfrac{1}{2} \int_{x-t}^{x+t} |Q(\sigma)|\, d\sigma \exp\left[\frac{t}{2} \int_{x-t}^{x+t} |Q(\sigma)|\, d\sigma\right]$$

Similarly,

$$(2.19) \qquad |T(x, t, s)| \leqslant \int_{x-t}^{x+t} |Q(\sigma)|\, d\sigma \exp\left[\frac{t}{2} \int_{x-t}^{x+t} |Q(\sigma)|\, d\sigma\right]$$

3. The Goursat Problem.

We now investigate the Goursat problem, i.e., we find the conditions to be satisfied by $W(x, t, s)$ and $T(x, t, s)$ so that (2.14) may represent the solution of the Cauchy type problem (1.1) – (1.2). From (2.14), when $t = 0$, $U(x, t)\,|_{t=0} = f(x)$ on account of the continuity of f and g.

Again differentiating (2.14) with respect to t and then putting $t = 0$, it follows that

$$\left.\frac{\partial U(x, t)}{\partial t}\right|_{t=0} = g'(x) + W(x, 0, x) f(x) + T(x, 0, x)\, g(x),$$

provided W and T are continuous at $t = 0$.

Since $\left.\dfrac{\partial}{\partial t} U(x, t)\right|_{t=0} = g'(x)$ and f and g can take values independent of each other, we have for all x,

$$(3.1) \qquad W(x, 0, x) = T(x, 0, x) = 0.$$

Let $U(x, t) = U(x, t\,; f, g)$ given by (2.14) satisfy the differential system (1.1).

Put $D_t = \dfrac{d}{dt}$, $D_x = \dfrac{d}{dx}$; $L_t = \begin{pmatrix} D_t^2 & 0 \\ 0 & D_t^2 \end{pmatrix}$

$$M_x = \begin{pmatrix} D_x^2 & 0 \\ 0 & D_x^2 \end{pmatrix} - \begin{pmatrix} p & r \\ r & q \end{pmatrix} \text{ and } F = \begin{pmatrix} f_1 & g_1 \\ f_2 & g_2 \end{pmatrix}, \text{ where}$$

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

Then,

$$M_x F = \begin{pmatrix} D_x^2 f_1 - pf_1 - rf_2 & D_x^2 g_1 - pg_1 - rg_2 \\ D_x^2 f_2 - qf_2 - rf_1 & D_x^2 g_2 - qg_2 - rg_1 \end{pmatrix}$$

Let $(A)_{*j}$ represent the jth column vector of a matrix A.

Then from (2.14)

(3.2) $$U(x, t\,;\,(M_x F)_{*1}, (M_x F)_{*2}) \mid_{t=0} = (M_x F)_{*1}$$

Also,

(3.3) $$\frac{\partial}{\partial t} U(x, t\,;\, M_x F)_{*1}, (M_x F)_{*2} j \mid_{t=0} = \frac{d}{dx} (M_x F)_{*2} = \mu(x), \text{ say}$$

Again,

(3.4) $$(L_t - M_x)\, U(x, t\,;(M_x F)_{*1}, (M_x F)_{*2}) = \begin{pmatrix} D_t^2 u - D_x^2 u + pu + rv \\ D_t^2 v - D_x^2 v + qv + ru \end{pmatrix} = 0$$

since $U(x, t\,;\,(M_x F)_{*1}, (M_x F)_{*2}) = \begin{pmatrix} u \\ v \end{pmatrix}$ is a solution of the differential system (1.1).

It follows therefore that $U(x, t\,;\,(M_x F)_{*1}, (M_x F)_{*2})$ is the solution of the Cauchy type problem

(3.5) $$\begin{cases} M_x V = L_t V, \\ V \mid_{t=0} = (M_x F)_{*1}\,, \ \dfrac{\partial V}{\partial t}\Big|_{t=0} = \mu(x) \end{cases}$$

Similarly, it can be verified that $M_x\, U(x, t\,;\,F)$ also satisfics the same Cauchy type problem (3.5).

Thus from the uuiqueness of the solution of the problem,

(3.6) $$L_t U = M_x U(x, t, F) = U(x, t\,;\,(M_x F)_{*1}, (M_x F)_{*2}) = U(x, t\,;\,M_x F), \text{ say}$$

Differentiating the relation (2.14) twice with respect to t we obtain

(3.7) $$\begin{aligned} L_t U(x, t) &= \tfrac{1}{2} [f''(x+t) + f''(x-t) + g''(x+t) - g''(x-t)] \\ &+ \tfrac{1}{2} [W(x, t, x+t) f'(x+t) + W_t'(x, t, x+t) f(x+t) - W(x, t, x-t) f'(x-t) \\ &+ W_t'(x, t, x-t) f(x-t) - T(x, t, x+t) g'(x+t) - T_t'(x, t, x+t) g(x+t) \\ &+ T(x, t, x-t) g'(x-t) - T_t'(x, t, x-t) g(x-t) + W_t(x, t, s) \mid_{s=x+t} f(x+t) \\ &- T_t(x, t, s) \mid_{s=x+t} g(x+t) + W_t(x, t, s) \mid_{s=x-t} f(x-t) - T_t(x, t, s) \mid_{s=x-t} g(x-t)] \\ &+ \tfrac{1}{2} \int_{x-t}^{x+t} [W_{tt}(x, t, s) f(s) - T_{tt}(x, t, s)\, g(s)]\, ds \end{aligned}$$

(where $X_t'(\,.\,) = \dfrac{d}{dt} X(\,.\,)$, $X_t(\,.\,) = \dfrac{\partial}{\partial t} X(\,.\,)$ and $X_{tt}(\,.\,) = \dfrac{\partial^2}{\partial t^2} X(\,.\,)$ etc.)

Replace f and g by $(M_x F)_{*1}$ and $(M_x F)_{*2}$ respectively in (2.14) and adopt the same procedure as above and integrate the involved integrals, twice by parts.

Then

(3.8)
$$U(x,t\,;\,M_x F)=\tfrac{1}{2}\,[f''(x+t)+f''(x-t)+g''(x+t)-g''(x-t)$$
$$-Q(x+t)f(x+t)-Q(x+t)\,g(x+t)-Q(x-t)f(x-t)$$
$$+Q(x-t)\,g(x-t)]+\tfrac{1}{2}\,[\,W(x,t,s)\,f'(s)-T(x,t,s)\,g'(s)\,]_{x-t}^{x+t}$$
$$-\tfrac{1}{2}\,[\,W_s(x,t,s)\,f(s)-T_s(x,t,s)\,g(s)\,]_{x-t}^{x+t}$$
$$-\tfrac{1}{2}\int_{x-t}^{x+t}[\,W(x,t,s)\,Q(s)f(s)-T(x,t,s)\,Q(s)\,g(s)\,]\,ds$$
$$+\tfrac{1}{2}\int_{x-t}^{x+t}[\,W_{ss}(x,t,s)\,f(s)-T_{ss}(x,t,s)\,g(s)\,]\,ds$$

Substituting in (3.6) by the relations (3.7) and (3.8) and taking into consideration the fact that f and g are arbitrary, it follows by equating the co-efficients of $f(x\pm t)$, $g(x\pm t)$ and the terms under the integral signs, that

(3.9)
$$\begin{cases} W'_t(x,t,x\pm t)+W_t(x,t,s)\,|_{s=x\pm t}=-\,Q(x\pm t)\mp W_s(x,t,s)\,|_{s=x\pm t} \\ -T'_t(x,t,x\pm t)-T_t(x,t,s)\,|_{x=x\pm t}=-\,Q(x,\pm t)\pm T_s(x,t,s)\,|_{s=x\pm t} \end{cases}$$

and

(3.10)
$$\begin{cases} W_{tt}(x,t,s)=W_{ss}(x,t,s)-W(x,t,s)\,Q(s) \\ T_{tt}(x,t,s)=T_{ss}(x,t,s)-T(x,t,s)\,Q(s) \end{cases}$$

Now $\quad \dfrac{d}{dt}\,W(x,t\;x\pm t)=W_t(x,t,s)\,|_{s=x\pm t}\pm W_s(x,t,s)\,|_{s=x\pm t}$

with a similar result for $\dfrac{d}{dt}\,T(x,t,x\pm t)$.

Then from (3.9) we obtain
$$2\,\frac{dW(x,t,x\pm t)}{dt}=-\,Q(x,\pm t)$$
$$\text{and}\quad 2\,\frac{dT(x,t,x\pm t)}{dt}=Q(x\pm t)$$

Thus

(3.11)
$$\begin{cases} W(x,t,x\pm t)=-\tfrac{1}{2}\displaystyle\int_0^t Q(x\pm s)\,ds \\ T(x,t,x\pm t)=\tfrac{1}{2}\displaystyle\int_0^t Q(x\pm s)\,ds \end{cases}$$

Consequently

(3.12) $$W(x, t, x \pm t) + T(x, t, x \pm t) = 0$$

The following theorem therefore holds.

Theorem 3.1. The Riemann matrix functions $W(x, t, s)$, $T(x, t, s)$ appearing in the olutions (2.14) of the Cauchy type problem (1.1) and (1.2) satisfy the partial differential equations (3.10) and are expressed in terms of the matrix Q by the relations (3.11). Morever t $s = x \pm t$, the relation (3.12) between W and T holds.

4. **Continuation of $f(x)$, $h(x)$ defined in $(0, \infty)$ to $(-\infty, -\infty)$.**

In the problem (1.1)—(1.2) we impose the additional conditions

(4.1) $$AU_x(x, t)\,|_{x=0} + BU(x, t)\,|_{x=0} = \phi(t)$$

where A, B are suitable non-singular constant (2×2) matrices. Since the solution $U(x, t)$ of the system (1.1)—(1.2) satisfies (4.1), we have from the first relation of (1.2) and the relation (4.1),

(4.2) $$Af'(+0) + Bf(+0) = \phi(+0)$$

Again differentiating (4.1) and then utilizing (1.2) we have

(4.3) $$Ah'(+0) + Bh(+0) = \phi'(+0)$$

where $f(x)$, $\phi(x)$, $h(x)$ are real valued functions of x defined for $x \geqslant 0$ such shat

$$f(x),\ \phi(x) \in C^2[0, \infty) \quad \text{and} \quad h(x) \in C^1[0, \infty),$$

When $h(x) = 0$, we choose the condition corresponding to (4.1) as

(4.4) $$CU_x(x, t)\,|_{x=0} + DU(x, t)\,|_{x=0} = \psi(t)$$

where as before C, D are suitable (2×2) non-singular constant matrices and $U(x, t)$ is now the solution of (1.1)—(1.2) with $h(x) = 0$. The relations corresponding to (4.2) and (4.3) are now

(4.5) $$\begin{cases} \psi(+0) = Cf'(+0) + Df(+0) \\ \psi'(+0) = 0 \end{cases},$$

and

where we assume $\psi(x) \in C^2[0, \infty)$.

A necessary condition for the system (1.1) with (1.2) and (4.1) to have solutions continuous at zero is therefore given by (4.2) and (4.3). The corresponding condition for the system (1.1)—(1.2) (with $h(x) = 0$) and (4.1) is (4.5).

The solution (2.14) of (1.1)—(1.2) satisfies (4.1). By differentiating (2.14) with respect to x so as to obtain $U_x(x, t)$, substituting in (4.1) for $U(x, t)$ given by (2.14) and $U_x(x, t)$ as derived above we have, after some reductions,

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(4.6)
$$2\phi(t) = A\,[f'(t) + f'(-t) + h(t) - h(-t) + W(0,t,t)f(t)$$
$$- W(0,t,-t)f(-t) - T(0,t,t)\,g(t) + T(0,t,-t)\,g(-t)$$
$$+\int_{-t}^{t}\{ W_x(x,t,s)\,|_{x=0}f(s) - T_x(x,t,s)\,|_{x=0}\,g(s)\}\,ds] + B\,[f(t)$$
$$+ f(-t) + g(t)\,g(-t) + \int_{-t}^{t}\{ W(0,t,s)f(s) - T(0,t,s)\,g(s)\}\,ds]$$

where $g(x) = \int_0^x h(y)\,dy$ and is continuous for all values of x including $x = 0$.

Writing $\int_{-t}^{t}(.)\,ds = \int_0^t (.)\,ds + \int_{-t}^{0}(.)\,ds$ and then changing s to $-s$ in the second part we have from (4,6)

(4.7)
$$G_1(t) = Af'(-t) - [AW(0,t,-t)f(-t) - Bf(-t)]$$
$$+ A\,[T(0,t,-t)\,g(-t) - h(-t)] + \int_0^t [K_1(t,s)f(-s) + K_2(t,s)\,g(-s)]\,ds$$

where

$$G_1(t) = 2\phi(t) - A\,[f'(t) + h(t) + W(0,t,t)f(t) - T(0,t,t)\,g(t)$$
$$- \int_0^t \{ W_x(x,t,s)\,|_{x=0}f(s) - T_x(x,t,s)\,|_{x=0}\,g(s)\}\,ds] - B\,[f(t) + g(t)$$
$$- g(-t) - \int_0^t \{ W(0,t,s)f(s) - T(0,t,s)\,g(s)\}\,ds]\,;$$
$$K_1(t,s) = A\,W_x(x,t,-s)\,|_{x=0} + B\,W(0,t,-s)\,;$$
$$K_2(t,s) = -AT_x(x,t,-s)\,|_{x=0} - BT(0,t,-s).$$

Integrating between the limits $(0, t)$ and making some suitable substitutions, (4.7) reduces to

(4.8)
$$G_{11}(t) = A\,[f(-t) - f(-0) - g(-t) + g(-0)] + \int_0^t K_3(t,s)f(-s)$$
$$+ K_4(t,s)\,g(-s)]\,ds.$$

Since $G_{11}(t)$ is known, (4.8) is an integral equation involving the unknown functions $f(-t)$ and $g(-t)$ with known kernels K_3 and K_4. Similarly, when $h(x) = 0$ and the solution of (1.1) – (1.2) (with $h(x) \doteq 0$) satisfies (4.4), we have

$$G_2(t) = 2\,\psi(t) - C[\,[f'(t) + W^*(0, t, t)\,f(t) - \int_0^t W_x^*(x, t, s)\,|_{x=0}\, f(s)\,ds\,]$$

$$- D\,[f(t) - \int_0^t W^*(0, t, s)\,f(s)\,ds\,]$$

(4.9) $$= C\,[f'(-t) - W^*(0, t, -t)\;f(-t)] + Df(-t) + \int_0^t K(t, s)\,f(-s)\,ds$$

where, $K(t, s) = CW_x^*(x, t, -s)\,|_{x=0} + DW^*(0, t, -s)$,

and $W^*(.)$ behaves in the same way as $W(.)$ in the solution (2.14) of the original problem. The integral equation analogous to (4.8) is now

(4.10) $$G_{12}(t) = C\,[f(t) - f(-0)\,] + \int_0^t K_5(t, s)\,f(-s)\,ds$$

in the unknown $f(-t)$ and being a Volterra type equation, can be solved for $f(-t)$ (see Whittaker and Watson — [6]). Therefore $g(-t)$ is determined from (4.8).

Evidently $f(-t)$, $g(-t)$ are continuous for $t > 0$ and we have to test for continuity at $t = 0$ only.

From (4.10) and therefore from (4.8), $f(t)$, $g(t)$ are continuous at $t = 0$ and $f(+0) = f(-0)$, $g(+0) = g(-0)$.

Putting $t = 0$ in (4.9), we have, using the relation (3.1) (holding also for $W^*(.)$ for every x)

$$2\psi(+0) - Cf'(+0) - Df(+0) = Cf'(-0) + Df(-0),$$

which from (4.5) leads to

$f'(+0) = f'(-0)$, ensuring the continuity of $f'(x)$ at $x = 0$.

A similar consideration with (4.7) leads to the continuity of $h(x)$ at $x = 0$.

Differentiate (4.6) with respect to t, make use of the relations (3.1) (when $x = 0$) and the relations

$$W_t(0, t, t)\,|_{t=0} - W_t(0, t, -t)\,|_{t=0} = 0$$

and $$T_t(0, t, t)\,|_{t=0} - T_t(0, t, t)\,|_{t=0} = 0$$

easily deducible from (3.11) and (3.12).

Then after some reductions, we have, when $t = 0$, the relation (f and h being continuous at $x = 0$)

(4.11) $$2\,\phi'(0) = A\,\{f''(+0) - f''(-0) + h'(+0) + [\,W_x(x, t, s)\,|_{x=0}]_{s=\pm t=0}\; f(0)$$

$$- [\,T_x(x, t, s)\,|_{x=0}]_{s=\pm t=0}\; g(0)\,\} + 2B\,h(0)$$

where we use the notation

$$\Phi \mid_{s=\pm t=0} = \{\Phi \mid_{s=t} + \Phi \mid_{s=-t}\}_{t=0}$$

By using (4.5), the corresponding result when $h(x) = 0$ is given by

(4.12) $$f''(+0) - f''(-0) = -[W_x^*(x, t, s) \mid_{\substack{x=0 \\ s=\pm t=0}}] f(0)$$

Differentiate (2.14) first with respect to x and then differentiate again the result so obtained with respect to t. Then putting $x = 0$, $t = 0$, we have, using (3.1),

(4.13) $$2\frac{\partial^2 U}{\partial x \partial t}\Big|_{\substack{x=0\\t=0}} = f''(+0) - f''(-0) + g''(+0) + g''(-0) + \{W_x(x, t, s) \mid_{s=x+t}$$
$$+ W_x(x, t, s)_{s=x-t}\}_{\substack{x=0\\t=0}} f(0) - \{T_x(x, t, s) \mid_{s=x+t} + T_x(x, t, s)_{s=x-t}\}_{\substack{x=0\\t=0}} g(0).$$

Now, using the relation (3.11) and the condition

$$\frac{\partial}{\partial t} U(x, t) \mid_{t=0} = h(x), \text{ as given in (1.2),}$$

we have,

$$\frac{\partial^2 U}{\partial x \partial t}\Big|_{\substack{x=0\\t=0}} = h'(0).$$

Hence (4.13) reduces to

(4.14) $$h'(+0) = f''(+0) - f''(-0) + h'(-0)$$

The corresponding formula with $h(x) = 0$ is

$$f''(+0) = f''(-0) \text{ ; so that } f''(x) \text{ is continuous at } x = 0.$$

Hence from (4.14), $h'(+0) = h'(-0)$, and h' is continuous at $x = 0$.

Therefore from (4.6) and (4.11) the relations (4.2) and (4.3) follow. We therefore obtain the theorem.

Theorem 4.1. If $f(t) \in C^2[0, \infty)$, $h(t) \in C^1[0, \infty)$, $\phi(t) \in C^2[0, \infty)$; then a necessary and sufficient condition that $f(t)$ and $h(t)$ may be continued to the negative half line by means of the equation (4.7) such that $f(t) \in C^2(-\infty, \infty)$ and $h(t) \in C^1(-\infty, \infty)$, is

$$\phi(+0) = Af'(+0) + Bf(+0)$$
$$\text{and} \quad \phi'(+0) = Ah'(+0) + Bh(+0)$$

where A, B (not both zero) are suitable (2×2) non-singular constant matrices.

5. Continuation of $f(x)$, $h(x)$ defined in $(0, \pi)$ to $(-\infty, \infty)$.

We now consider the problem of the finite interval $[0, \pi]$ formed of (1.1), (1.2), (4.1) and the further condition at $x = \pi$, viz,

(5.1) $$\alpha\, U_x(x, t)\,|_{x=\pi} + \beta\, U(x, t)\,|_{x=\pi} = \psi(t)$$

where α, β are suitable (2×2) non-singular non-zero constant matrices.

Let $f(x) \in C^2[0, \pi]$, $h(x) \in C^1[0, \pi]$, $\phi(x)$, $\psi(x) \in C^2[0, \pi]$ and $Q(x)$ continuous on any finite interval over $(-\infty, \infty)$.

Then a necessary and sufficient condition that $f(x)$ and $h(x)$ defined for $[0, \pi]$ may be extended to $[-\pi, 0]$ is that the condition (4.2) and (4.3) are satisfied.

Now let the solution of (1.1), (1.2), (4.1) satisfy (5.1). Then we have, as before,

(5.2) $$\alpha f'(\pi - 0) + \beta f(\pi - 0) = \psi(+0)$$

(5.3) $$\text{and} \quad \alpha h'(\pi - 0) + \beta h(\pi - 0) = \psi'(+0).$$

We find $U_x(x, t)$ from the solution (2.14) of (1.1), (1.2) and substitute for U_x, U in (5.1). Then a process similar to that in § 4 leads to

(5.4) $$\begin{aligned} G_1(t) = {} & \alpha\,[f'(\pi + t) + g'(\pi + t) + W(\pi, t, \pi + t)\, f(\pi + t) \\ & - T(\pi, t, \pi + t)\, g(\pi + t) + \int_\pi^{\pi+t} \{W_x(x, t, s)\,|_{x=\pi}\, f(s) \\ & - T_x(x, t, s)\,|_{x=\pi}\, g(s)\}\, ds] + \beta\,[f(\pi + t) + g(\pi + t) \\ & + \int_\pi^{\pi+t} \{W(\pi, t, s)\, f(s) - T(\pi, t, s)\, f(s)\}\, ds] \end{aligned}$$

where

$$\begin{aligned} G_1(t) = {} & 2\psi(t) - \alpha\,[f'(\pi - t) - g'(\pi - t) - W(\pi, t, \pi - t)\, f(\pi - t) \\ & + T(\pi, t, (\pi - t)\, g(\pi - t) + \int_{\pi-t}^{\pi} \{W_x(x, t, s)\,|_{x=\pi}\, f(s) \\ & - T_x(x, t, s)\,|_{x=\pi}\, g(s)\}\, ds] - \beta\,[f(\pi - t) - g(\pi - t) \\ & + \int_{\pi-t}^{\pi} \{W(\pi, t, s)\, f(s) - T(\pi, t, s)\, g(s)\}\, ds]. \end{aligned}$$

Replace s by $\pi + \sigma$ and then integrate between the limits $(0, t)$. Then with suitably defined kernels $K_1(t, s)$, $K_2(t, s)$ in terms of W and T and their derivatives, we obtain as before, the integral equation

$$\begin{aligned} G_{11}(t) = {} & \alpha\,[f(\pi + t) - f(\pi + 0) + g(\pi + t) - g(\pi + 0)] \\ & + \int_0^t [K_1(t, s)\, f(\pi + s) + K_2(t, s)\, g(\pi + s)]\, ds, \end{aligned}$$

If $h(x) = 0$, so that $g(x) = 0$ and the boundary condition (5.1) is given by

(5.6) $$\alpha_1 U_x(x, t) \mid_{x=\pi} + \beta_1 U(x, t) \mid_{x=\pi} = \chi(t),$$

then we have similarly the integral equation

$$G_{12}(t) = \alpha_1 [f(\pi + t) - f(\pi + 0)] + \int_0^t K_3(t, s) f(\pi + s)\, ds.$$

An argument similar to that in § 4 then shows that

$f(\pi + 0) = f(\pi - 0)$, $g(\pi + 0) = g(\pi - 0)$, showing that f, g are continuous at $x = \pi$.

Putting $t = 0$ and observing that $W(\pi, 0, \pi) = T(\pi, 0, \pi) = 0$ it follows from (5.4) and (5.2), that

$$f'(\pi + 0) - f'(\pi - 0) + h(\pi + 0) - h(\pi - 0) = 0$$

and from a similar consideration with $h(x) = 0$ we have

$$f'(\pi + 0) = f'(\pi - 0).$$

We, therefore, obtain that $f'(x)$ and $h(x)$ are continuous at $x = \pi$.

Differentiate $U(x, t)$ as given by (2.14) first with respect to x and then with respect to t and then put $x = 0$, $t = 0$. We obtain the relations

(A) $$W_t(\pi, t, \pi \pm t) \mid_{t=0} = -\tfrac{1}{2} Q(\pi), \quad T_t(\pi, t, \pi \pm t) \mid_{t=0} = \tfrac{1}{2} Q(\pi)$$

from (3.11)

(B) $$\frac{\partial^2 U(x, t)}{\partial x\, \partial t} \Bigg|_{\substack{t=0\\ x=\pi}} = h'(\pi - 0), \quad \text{from (1.2) (second condition)}$$

and (C) $$\left(W_x(x, t, s) \mid_{\substack{s=x\pm t\\ x=\pi;\ t=0}}\right) f(\pi \pm 0) = 0$$

$$\left(T_x(x, t, s) \mid_{\substack{s=x\pm t\\ x=\pi,\ t=0}}\right) g(\pi \pm 0) = 0$$

from (3.1), where we use the notation

$$[\Phi_x(x, t, s) \mid_{\substack{s=x\pm t\\ x=\pi,\ t=0}}]\ h(\xi \pm \eta) = [\Phi_x(x, t, s) \mid_{s=x+t} h(\xi + \eta)$$
$$+ \Phi_x(x, t, s) \mid_{s=x-t} h(\xi - \eta)]_{\substack{x=\pi\\ t=0}}$$

Then by virtue of (3.1) holding for $x = \pi$, we obtain,

$$2h'(\pi - 0) = f''(\pi + 0) - f''(\pi - 0) + h'(\pi + 0) + h^1(\pi - 0)$$

When $h(x) = 0$, a similar analysis gives

$$f''(\pi + 0) = f''(\pi - 0)$$

Thus $f''(x)$ and $h'(x)$ are continuous at $x = \pi$. Differentiate (5.4) with respect to t after replacing s by $\pi + \sigma$ and then put $t = 0$. Then by virtue of (3.1), (3.11) and the continuity condition on $f''(x)$ at $x = \pi$, the relation (5.3) follows. We have thus the theorem :

Theorem 5.1. If $f(t) \in C^2[0, \pi]$, $\phi(t) \in C^2[0, \pi]$, $h(t) \in C^1[0, \pi]$ and $\psi(t) \in C^2[0, \pi]$, then $f(t)$, $h(t)$ can be continued to the interval $[-\pi, \pi]$ by means of the equation (5.4) such that $f''(t) \in C^2[-\pi. \pi]$, $h'(t) \in C^1[-\pi, \pi]$, if and only if (4.2), (4.3), (5.2) and (5.3) hold.

The process can be continued and we can argue in the Levitan—Sargsyan way [2, p . 12] to obtain the extension of $f(x)$, $h(x)$ to classes $C^2(-\infty, \infty)$ and $C^1(-\infty, \infty)$ respectively. Substituting for $f(x)$, $g(x)$ obtained in this way to (2.14), the continuation of the solution of (1.1), (1.2), (4.1) and (5.1) may be derived.

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REFERENCES

[1] Chakraborty, N. K.—'Some problems on eigenfunction expansions (I)'—*Quart. J. Math. Oxford* (2) **16** (1965) 135-50.

[2] Levitan, B. M. and Sargsyan, I, S.—Some problems in the theory of a Sturm—Liouville equation—*Russian Math, Surveys*— (15) 1960.

[3] Povzner A —'On differential equations of Strum—Liouville type on a half axis'—*Matematicheskii Sbornik* (N. S.) **23** (65), 3-52 (1948) (*Amer. Math. Soc. Transl No.* 5 (1950)

[4] Smirnov, V. I.—Integral Equations and Partial Differential equations—Vol. IV—(Pergamon) 1964.

[5] Sobolev, S. L.—'Partial Differential Equations of Mathematical Physics'—(Pergamon) 1964,

[6] Whittaker. E. T. and Watson, G. N.—'A Course of Modern Analysis'—*Cambridge University Press*—(Fourth Edition), 1952.

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Dept. of Pure Math.
Calcutta University

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ON A FORMAL IDENTITY OF READ

TAHA IBRAHIM SULTAN
(*an Egyptian Student*)

In [1] R. C. Read proposed the following formal identity

$$(1) \qquad \prod_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{1}{n!} \left[H_n\left(\frac{i}{2\sqrt{m}}\right)\right]^2 (-\tfrac{1}{2})^n$$

$$= (1-t)^{-1/2}(1-t^2)^{-1}(1-t^3)^{-1/2}(1-t^4)^{-1} \ldots,$$

where $H_n(x)$ is the Hermite polynomial of degree n.

In solving the above indentity *L.* Carlitz [1] deduced the following more general identity

$$(2) \qquad \prod_{m=1}^{\infty} \left[\sum_{n=0}^{\infty} (-1)^n H_n\left(\frac{x}{\sqrt{m}}\right) H_n\left(\frac{y}{\sqrt{m}}\right) \frac{t^{mn}}{2^n n!} \right]$$

$$= \prod_{m=1}^{\infty} (1-t^{2m})^{x^2+y^2-1/2}(1-t^{2m-1})^{2xy}$$

by means of Mehler's formula.

The object of this note is to extend the result (2) of Carlitz in the following form

$$(3) \qquad \prod_{m=1}^{\infty} \left[\sum_{n=0}^{\infty} (-1)^n H_n\left(\frac{x}{\sqrt{m}}\right) H_{n+k}\left(\frac{y}{\sqrt{m}}\right) \frac{t^{mn}}{2^n n!} \right]$$

$$= \prod_{m=1}^{\infty} (1-t^{2m})^{x^2+y^2-(k+1)/2}(1-t^{2m-1})^{2xy} \cdot H_k\left(\frac{y+xt^m}{[m(1-t^{2m})]^{1/2}}\right),$$

by following the same method of Carlitz. It may be noted that (3) reduce to (2) on using $k=0$.

Now to prove (3) we first notice that [3]

$$\text{(4)} \qquad \sum_{n=0}^{\infty} H_n(x)\, H_{n+k}(y)\, t^n/n!$$

$$= (1-4t^2)^{-(k+1)2} \exp\left[y^2 - \frac{(y-2xt)^2}{1-4t^2} \right] . H_k\left(\frac{y-2xt}{(1-4t^2)^{1/2}} \right),$$

in a slightly corrected form.

On changing t into $(-t/2)$ we get

$$\text{(5)} \qquad \sum_{n=0}^{\infty} (-1)^n H_n(x)\, H_{n+k}(y)\, t^n/(2^n n!)$$

$$= (1-t^2)^{-(k+1)/2} \exp\left[\frac{-2xyt-(x^2+y^2)\,t^2}{1-t^2} \right] . H_k\left(\frac{y+xt}{(1-t^2)^{1/2}} \right)$$

Now we have

$$\prod_{m=1}^{\infty}\left[\sum_{n=0}^{\infty} (-1)^n H_n\left(\frac{x}{\sqrt{m}}\right) H_{n+k}\left(\frac{y}{\sqrt{m}}\right) t^{mn}/(2^n n!) \right]$$

$$= \prod_{m=1}^{\infty} (1-t^{2m})^{-(k+1)/2} H_k\left(\frac{y+xt^m}{[m(1-t^{2m})]^{1/2}} \right)$$

$$. \prod_{m=1}^{\infty} \exp\left[\frac{-2xyt^m-(x^2+y^2)\,t^{2m}}{m(1-t^{2m})} \right]$$

$$= \prod_{m=1}^{\infty} (1-t^{2m})^{-(k+1)/2} H_k\left(\frac{y+xt^m}{[m(1-t^{2m})]^{1/2}} \right)$$

$$. \exp \sum_{m=1}^{\infty} \left[\frac{-2xyt^m}{m(1-t^{2m})} - \frac{(x^2+y^2)\,t^m}{m(1-t^{2m})} \right]$$

$$= \prod_{m=1}^{\infty} (1-t^{2m})^{-(k+1)/2} H_k\left(\frac{y+xt^m}{[m(1-t^{2m})]^{1/2}} \right)$$

$$\cdot \exp \sum_{m=1}^{\infty} \left[-\frac{1}{m} 2xy \sum_{r=1}^{\infty} t^{m(2r-1)} - \frac{1}{m} (x^2 + y^2) \sum_{r=1}^{\infty} t^{2mr} \right]$$

$$= \prod_{m=1}^{\infty} (1 - t^{2m})^{-(k+1)/2} H_k \left(\frac{y + xt^m}{[m (1 - t^{2m})]^{1/2}} \right)$$

$$\cdot \exp \sum_{r=1}^{\infty} [2xy \log (1 - t^{2r-1}) + (x^2 + y^2) \log (1 - t^{2r})]$$

$$= \prod_{m=1}^{\infty} (1 - t^{2m})^{-(k+1)/2} H_k \left(\frac{y + xt^m}{[m (1 - t^{2m})]^{1/2}} \right)$$

$$\cdot \prod_{m=1}^{\infty} \exp \log [(1 - t^{2m-1})^{2xy} (1 - t^{2m})^{x^2 + y^2}]$$

$$= \prod_{m=1}^{\infty} (1 - t^{2m})^{x^2+y^2-(k+1)/2} (1 - t^{2m-1})^{2xy} \cdot H_k \left(\frac{y + xt^m}{[m (1 - t^{2m})]^{1/2}} \right),$$

which is our desired result (3).

It may be of interest to note that (2) can be extended further in the form

$$\prod_{p=1}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k t^{pk}}{2^k k!} \sum_{r=0}^{\infty} 2^r r! \binom{m}{r} \binom{n}{r} \left(\frac{-t}{1-t^2} \right)^r \tag{6}$$

$$\cdot H_{k+m-r} \left(\frac{x}{\sqrt{p}} \right) H_{k+n-r} \left(\frac{y}{\sqrt{p}} \right)$$

$$= \prod_{p=1}^{\infty} (1 - t^{2p})^{(x^2+y^2)-1/2(m+n-1)} \cdot (1 - t^{2p-1})^{2xy}$$

$$\cdot H_m \left(\frac{x + yt^p}{[p (1 - t^{2p})]^{1/2}} \right) H_n \left(\frac{y + xt^p}{[p (1 - t^{2p})]^{1/2}} \right),$$

in exactly the same way on utilizing the following formula of Carlitz [2]:

$$\sum_{k=0}^{\infty} \frac{t^k}{2^k k!} \sum_{r=0}^{\min(m,n)} 2^r r! \binom{m}{r} \binom{n}{r} \left(\frac{-t}{1-t^2} \right)^r \cdot \tag{7}$$

$$\cdot H_{k+m-r} (x) H_{k+n-r} (y)$$

$$= (1 - t^2)^{-1/2\,(m+n+1)} \exp\left[\frac{2xyt - (x^2 + y^2)\,t^2}{1 - t^2}\right]$$

$$\cdot\, H_m\left(\frac{x - yt}{(1 - t^2)^{1/2}}\right) H_n\left(\frac{y - xt}{(1 - t^2)^{1/2}}\right).$$

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REFERENCES

[1] American Mathematical Monthly, 70 (1963), 901.

[2] Carlitz' L.—An extension of Mehler's formula, Boll, Un. Mat. Ital. N. 1 (1970), 43–46.

[3] Mc Bride, E. B.—Obtaining generating functions. Springer–Verlag (1971),

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Dept. of Pure Math.
Calcutta University

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NON-SEPARABILITY OF SECOND-ORDER DIFFERENTIAL EXPRESSIONS IN THE LIMIT-CIRCLE CASE

JYOTI DAS AND JAYASRI SETT

1. **Introduction.** In this paper we consider the formally self-adjoint second-order differential expression

$$M[y] \equiv -(py')' + qy, \quad \left(' = \frac{d}{dx}, \quad 0 \leqslant x < \infty \right) \tag{1.1}$$

where p and q are real-valued functions satisfying

$$\left\{ \begin{array}{ll} \text{(i)} & q \in L(0, X) \text{ and } q \in L^2(0, X), \text{ for all } X > 0, \\ \text{(ii)} & p \text{ is absolutely continuous on } [0, X], \\ & \text{for all } X > 0. \\ \text{(iii)} & p(x) > 0 \text{ for all } x \in [0, \infty). \end{array} \right. \tag{1.2}$$

The manifold $\Delta(p, q)$ is defined at that subspace of the Hilbert function space $L^2(0, \infty)$ determined by $f \in \Delta(p, q)$ if

(1.3) (i) $f \in L^2(0, \infty)$, (ii) f' is absolutely continuous on $[0, X]$, for all $X > 0$, (iii) $M[f] = -(pf')' + qf \in L^2(0, \infty)$.

$M[\,.\,]$ is said to be in the limit-circle case when all solutions of the differential equation $M[y(x)] = \lambda\, y(x)$, $(\operatorname{im} \lambda \neq 0)$ are $L^2(0, \infty)$, otherwise $M[\,.\,]$ is said to be in the limit-point case. We say that $M[\,.\,]$ is separated if $f \in \Delta(p, q)$ implies both $(pf')'$ and qf are separately Lebesgue square integrable.

In this note we consider $M[\,.\,]$ to be in the limit-circle case and show that $M[\,.\,]$ is non-separable under two sets of mutually exclusive conditions on p

$$\text{(i)} \int_0^\infty p^{-\frac{1}{2}}(t)\, dt = \infty \quad \text{and} \quad \text{(ii)} \int_0^\infty p^{-\frac{1}{2}}(t)\, dt < \infty, \tag{1.4}$$

although we assume $q < 0$ in (1.4) (ii).

In § 2 we discuss the notions of "strong limit-point at ∞" and a class "$p(\gamma)$". In § 3 the theorem required to prove is stated and proved.

2. The Green's formula for $M[\,.\,]$ is given by [1, (1.9) of Ch 9]

$$\int_{t_1}^{t_1} (g M[f] - f M[g])\, dt = [fg](t_2) - [fg](t_1), \tag{2.1}$$

where the associated billinear form

(2.2) $\quad [fg](t) = p(t)\{f(t)\, g^{(1)}(t) - f^{(1)}(t)\, g(t)\}$

is defined for differentiable functions $f, g \in \Delta(p, q)$ and for all $x \in (0, \infty)$.

It is known [2, § 6] that $M[.]$ is limit-point at infinity if and only if

(2.3) $$\lim_{x \to \infty} [fg](x) = \lim_{x \to \infty} p(x)\{f(x)\, g^{(1)}(x) - f^{(1)}(x)\, g(x)\} = 0$$

for all $f, g \in \Delta(p, q)$.

We say $M[.]$ is strong limit-point at ∞ if for all $f, g \in \Delta(p, q)$

(2.4) $$\lim_{x \to \infty} p(x)\, f(x)\, g^{(1)}(x) = 0$$

Since (2.4) implies (2.3), strong limit-point implies limit-point. Obviously (2.3) does not imply (2.4). We say that $M[.]$ is weak limit-point at ∞ if (2.3) holds but (2.4) is false for some $f, g \in \Delta(p, q)$.

Following the definition given in [4, § 1] we say that the real valued coefficient q satisfying (1.2) belongs to the class $P(\gamma)$, where γ is a non-negative real number, if $f \in \Delta(p; q) \Rightarrow |q|^{\gamma} f \in L^2(0, \infty)$.

We then write $q \in p(\gamma)$. Thus from § 1, $M[.]$ is separated if $q \in P(1)$. We note the following property of $P(\gamma)$. If $q \in P(\gamma)$, then $q \in P(\beta)$ for all $\beta \in [0, \gamma]$

The proof of the above property is the same as in [4, § 3 lemma 1] and it is easily seen that the proof remains unaltered even if $p \neq 1$ in (1.1).

(2.5) Thus $q \in P(1)$ implies $q \in p(\frac{1}{2})$, i.e., $|q|^{\frac{1}{2}} f \in L^2(0, \infty)$ for all $f \in \Delta(p, q)$.

3. The result of the present paper can be expressed as follows :

THEOREM. Let $M(f) = (pf')' + qf$ be in the limit-circle case at ∞ and let p, q satisfy (1.2). If further p and q satisfy either of the following :

(i) $\int_0^\infty p^{-\frac{1}{2}}(t)\, dt = \infty$ and q satisfying the conditions (1.2) or (ii) $\int_0^\infty p^{-\frac{1}{2}}(t)\, dt < \infty$ and $q < 0$ and unbounded below, then $M[.]$ is not separated.

Proof of the theorem

Suppose $M[.]$ is separated, i.e., $q \in P(1)$. Hence $q \in P(\frac{1}{2})$ from (2.5) i. e.,

(3.1) $$qf^2 \in L(0, \infty).$$

It may be verified on integration by parts, that

(3.2) $$\int_0^X \{pf^{(1)} g^{(1)} + qfg\}\, dx = [pfg^{(1)}]_0^X + \int_0^X M[g]\, f\, dx$$

which is valid for all $f, g \in \Delta(p, q)$ and for all $X > 0$,

(3.3) If in (3.2) we take $f = g \in \Delta(p, q)$ then

$$\int_0^X \{ p f^{(1)^2} + q f^2 \} \, dx = p(X) f(X) f^{(1)}(X) - p(0) f(0) f^{(1)}(0)$$
$$+ \int_0^X f M[f] \, dx$$

From (3.3), for any $f \in \Delta(p.q)$,

$$\int_0^X p f^{(1)^2} \, dt = p(X) f(X) f^{(1)}(X) - p(0) f(0) f^{(1)}(0) - \int_0^X q f^2 \, dt + \int_0^X f M[f] \, dt$$

Since the integrand on the left is non-negative ($p(x) > 0$ for all $x \in [0, \infty]$) and the integrals on the right exist, the above equality indicates that $\int_0^X p f^{(1)^2} \to \infty$ as $X \to \infty$ if and only if $p(X) f(X) f^{(1)}(X) \to \infty$ as $X \to \infty$. However $p(X) f(X) f^{(1)}(X)$ cannot tend to ∞ with X, for then $f(X)$ and $f^{(1)}(X)$ would have the the same sign for all sufficiently large X and this would prevent f belonging to the space $L^2(0, \infty)$.

It follows then that

$$\lim_{X \to \infty} p(X) f(X) f^{(1)}(X) \text{ exists and is finite.}$$

Also

(3.4) $$p^{\frac{1}{2}} f^{(1)} \in L^2(0, \infty).$$

Arguing similarly we can establish from (3.2) that

(3.5) $$\lim_{X \to \infty} p(X) f(X) g^{(1)}(X) \text{ exists and is finite.}$$

We now show that this limit is zero.

We first consider the *case* (*i*) :

(3.6) $$\int_0^\infty p^{-\frac{1}{2}}(t) \, dt = \infty.$$

From (3.5) if $\lim_{x \to \infty} p(X) f(X) g^{(1)}(X)$ is not zero for a pair of $f, g \in \Delta(p, q)$, then for some positive δ and some $x_0 \geqslant 0$ we would have

$$| p(x) f(x) g^{(1)}(x) | \geqslant \delta \quad (X \geqslant X_0) \quad \text{and so}$$

(3.7) $$\int_{X_0}^X | p^{\frac{1}{2}} f g^{(1)} | \, dx \geqslant \int_{X_0}^X p^{-\frac{1}{2}}(x) \, dx \quad (X \geqslant X_0).$$

Now from (3.4), $p^{\frac{1}{2}} g^{(1)} \in L^2(0, \infty)$; and $f \in L^2(0, \infty)$ as $f \in \Delta(p, q)$.

Thus $\int_{X_0}^X p^{\frac{1}{2}} f g^{(1)}$ remains finite as $X \to \infty$.

Again by the assumption (3.6) the right hand side of (3.7) is ∞. This is a contradiction. Hence

(3.8) $$\lim_{X\to\infty} p(X) f(X) g^{(1)}(X) = 0.$$

Following the definition from § 2, (3.8) implies that $M[f]$ is in the strong limit-point case at ∞. But we have assumed $M[\,.\,]$ to be in the limit-circle case at ∞. Hence a contradiction.

Therefore $M[\,.\,]$ cannot be separated.

Case (ii) $\int_0^\infty p^{-\frac{1}{2}}(t)\,dt < \infty$.

As we have $q(x) < 0$ and $q(x)$ is bounded above by any positive number $K > 0$ i. e.,

$$q(x) < 0 < K \quad \text{for every} \quad K > 0.$$

Then $|q(x)| > -K$ for every $K > 0$.

(3.9) Let $Q(x) = -q(x) > 0$. Then $Q(x) = |q(x)| > -K$

(3.10) Again let $Q_1(x) = Q(x) + K$.

(3.11) Then $Q_1(x) > K > 0$, for all $K > 0$.

Suppose $M[\,.\,]$ is separated. Then $|q|^{\frac{1}{2}} f \in L^2(0, \infty)$ for every $f \in \Delta(p, q)$. Then from (3.9),

(3.12) $$Q^{\frac{1}{2}} f \in L^2(0, \infty), \text{ i. e. } Q f^2 \in L(0, \infty).$$

(3.13) Again from (3.10), $Q_1(x) f^2 = ((Q(x) + k) f^2 \in L(0, \infty)$ by (3.12)

(3.14) Then $Q_1^{\frac{1}{2}} f \in L^2(0, \infty)$.

Now we consider the differential expression $M_1[f]$, where

$$M_1[f] = -(pf')' - Q_1 f = M[f] - Kf \quad \text{for all} \quad K > 0.$$

Assuming (3.14) i. e., $M[\,.\,]$ is separated, our aim is to show that $M[\,.\,]$ is in the strong limit-point case at ∞, which would imply $M_1[f]$ to be in the limit-point case at ∞. Now $M_1[f]$ being in the limit-point case at ∞ would mean $M[f]$ is in the limit-point case at ∞ as $M_1[f] = M[f] - Kf$, for $K > 0$. But we have assumed $M[f]$ to be in the limit-circle case at ∞ and this then would bring a contradiction.

So to show the desired contradiction, we can prove the theorem by taking $M_1[f]$ instead of $M[f]$ for every $f \in \Delta(p, q)$.

Arguing similarly as in (3.5) with $M_1[f]$ and assuming (3.14) we have here

$\lim_{X\to\infty} p(X) f(X) g^{(1)}(x)$ exists and is finite.

To show that this limit is zero, we proceed exactly in the same way as in [3;6] using $\int_0^\infty p^{-\frac{1}{2}}(t)\,dt < \infty$ and (3.11).

Then $\lim_{X \to \infty} p(X) f(X) g^{(1)}(X) = 0$, would bring a contradiction, as argued above.

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REFERENCES

[1] Coddington E. A. and Levinson N. Theory of ordinary differential equations (New York and London. 1955).

[2] Everitt W. N. "A note on the self-adjoint domains of second-order differential equations". Quart. J. of Math. (Oxford) (2) 14 (1963). 41-45.

[3] Everitt W, N., "On the limit-point classification of second order differential operators". J. Lond. Math. Soc. 41 (1966) 531-34.

[4] Everitt W. N., Giertz M. and Weidmann J., "Some remarks on a separation and limit-point criterion of second-order ordinary differential expression" Math. Ann. 200 (1973). 335-346.

[5] Everitt W. N., A note on the Dirichlet condition for second-order differential expressions. U.D.D M. Report.

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Calcutta University

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FINITE SUMS INVOLVING HERMITE POLYNOMIALS

MANIK CHANDRA MUKHERJEE

In a recent paper [1], using the notion of forward difference E. Hansen has deduced three variants of finite sums involving functions satisfying three-term recursion relation. As an illustration, the well known three-term recursion relation for the Hermite polynomials is used and the following results are proved :

$$(1)\quad \frac{z^m}{m\,!} H_m - \frac{z^k}{k\,!} H_k = \sum_{n=k}^{m-1} \frac{z^n}{(n+1)\,!} H_{n+1}$$

$$(2)\quad (2z)^{1-m} H_m - (2z)^{1-k} H_k = - \sum_{n=k}^{m-1} 2n\,(2z)^{-n} H_{n-1}$$

$$(3)\quad \sum_{n=k}^{m-1} \frac{(-4)^{-n} H_{2n+s+1}}{\Gamma(n+\frac{s}{2}+\frac{3}{2})} = \frac{2}{z} \left[\frac{(-4)^k H_{2k+s}}{\Gamma(K+\frac{s}{2}+\frac{1}{2})} - \frac{(-4)^{-m} H_{2m+s}}{\Gamma(m+\frac{s}{2}+\frac{1}{2})} \right]$$

The object of this present work is to point out that the above results involving the Hermite polynominals can be deduced in a quite easy and natural way without using the notion of forward difference.

First to prove (1), we notice the following obvious equality

$$(4)\quad \sum_{n=k+1}^{m-1} \frac{z^n H_n(z)}{n\,!} = \sum_{n=k}^{m-2} \frac{z^{n+1}}{(n+1)\,!} H_{n+1}(z)$$

Now we have

$$\frac{z^m}{m\,!} H_m(z) - \frac{z^k}{k\,!} H_k(z)$$

$$= \left(\sum_{n=k}^{m-2} \frac{z^{n+1}}{(n+1)\,!} H_{n+1}(z) + \frac{z^m}{m\,!} H_m(z) \right)$$

$$- \left(\sum_{n=k+1}^{m-1} \frac{z^n H_n(z)}{n!} + \frac{z^k}{k!} H_k(z) \right)$$

$$= \sum_{n=k}^{m-1} \frac{z^{n+1}}{(n+1)!} H_{n+1}(Z) - \sum_{n=k}^{m-1} \frac{z^n H_n(z)}{n!}$$

$$= \sum_{n=k}^{m-1} \frac{z^n}{2(n+1)!} \left\{ 2z H_{n+1}(z) - 2(n+1) H_n(z) \right\}$$

$$= \sum_{n=k}^{m-1} \frac{z^n}{2(n+1)!} H_{n+2}(z),$$

(by the recursion relation of the Hermite polynomials), which is (1).

To prove (2), we notice the following obvious equality

$$(5) \quad \sum_{n=k}^{m-2} (2z)^{-n} H_{n+1}(z) = \sum_{n=k+1}^{m-1} (2z)^{1-n} H_n(z)$$

Now we have

$$(2z)^{1-m} H_m(z) - (2z)^{1-k} H_k(z)$$

$$= \left((2z)^{1-m} H_m(z) + \sum_{n=k}^{m-2} (2z)^{-n} H_{n+1}(z) \right)$$

$$- \left((2z)^{1-k} H_k(z) + \sum_{n=k+1}^{m-1} (2z)^{1-n} H_n(z) \right)$$

$$= \sum_{n=k}^{m-1} (2z)^{-n} H_{n+1}(z) - \sum_{n=k}^{m-1} (2z)^{1-n} H_n(z)$$

$$= - \sum_{n=k}^{m-1} (2z)^{-n} \{2z H_n(z) - H_{n+1}(z)\}$$

$$= - \sum_{n=k}^{m-1} 2n (2z)^{-n} H_{n-1}(z),$$

(by the recursion relation of the Hermite polynomials), which is (2).

In order to prove (3), we notice the following equality :

$$\sum_{n=k+1}^{m-1} \frac{(-4)^{-n} H_{2n+s}(z)}{\Gamma\left(n+\frac{s}{2}+\frac{1}{2}\right)} = \sum_{n=k}^{m-2} \frac{(-4)^{-(n+1)} H_{2n+s+2}(z)}{\Gamma\left(n+\frac{s}{2}+\frac{3}{2}\right)} \tag{6}$$

Now we have

$$\frac{2}{z}\left[\frac{(-4)^{-k} H_{2k+s}(z)}{\Gamma\left(k+\frac{s}{2}+\frac{1}{2}\right)} - \frac{(-4)^{-m} H_{2m+s}(z)}{\Gamma\left(m+\frac{s}{2}+\frac{1}{2}\right)}\right]$$

$$= \frac{2}{z}\left[\frac{(-4)^{-k} H_{2k+s}(z)}{\Gamma\left(k+\frac{s}{2}+\frac{1}{2}\right)} + \sum_{n=k+1}^{m-1} \frac{(-4)^{-n} H_{2n+s}(z)}{\Gamma\left(n+\frac{s}{2}+\frac{1}{2}\right)}\right.$$

$$\left. + \tfrac{1}{4}\, \frac{(-4)^{-(m-1)} H_{2+ms}(z)}{\Gamma\left(m+\frac{s}{2}+\frac{1}{2}\right)} + \sum_{n=k}^{m-1} \frac{(-4)^{-n} H_{2n+s+2}(z)}{4.\, \Gamma\left(n+\frac{s}{2}+\frac{3}{2}\right)}\right]$$

$$= \frac{2}{z}\left[\sum_{n=k}^{m-1} \frac{(-4)^{-n} H_{2n+s}(z)}{\Gamma\left(n+\frac{s}{2}+\frac{1}{2}\right)} + \sum_{n=k}^{m-1} \frac{(-4)^{-n} H_{2n+s+2}(z)}{4\,\Gamma\left(n+\frac{s}{2}+\frac{3}{2}\right)}\right]$$

$$= \frac{2}{z}\left[\sum_{n=k}^{m-1} \left(n+\frac{s}{2}+\frac{1}{2}\right) \frac{(-4)^{-n} H_{2n+s}(z)}{\Gamma\left(n+\frac{s}{2}+\frac{3}{2}\right)}\right.$$

$$\left. + \sum_{n=k}^{m-1} \frac{1}{4}.\, \frac{(-4)^{-n} H_{2n+s+2}(z)}{\Gamma\left(n+\frac{s}{2}+\frac{3}{2}\right)}\right]$$

$$= \sum_{n=k}^{m-1} \frac{(-4)^{-n}}{\Gamma\left(n+\frac{s}{2}+\frac{3}{2}\right)} \cdot \frac{2}{z}\left\{\left(n+\frac{s}{2}+\frac{1}{2}\right) H_{2n+s}(z) + \tfrac{1}{4} H_{2n+s+2}(z)\right\}$$

$$= \sum_{n=k}^{m-1} \frac{(-4)^{-n}}{\Gamma\left(n+\frac{s}{2}+\frac{3}{2}\right)} H_{2n+s+1}(z),$$

(By the recursion relation of the Hermite polynomials), which is (3).

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REFERENCE

[1] Hansen, E—Sums of functions satisfying recursion relation. The Amer. Math. Monthly, 88 (1981) 678-679.

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Netajinagar Bidyamandir
Calcutta – 700 092

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K TH DIFFERENCES OF FUNCTIONS SATISFYING THREE-TERM RECURSION RELATIONS (II)

N. BARIK

1. Introduction : In the earlier work [1], we have extended the first and second variations of E. Hansen [3] for some function $f_n(z)$ satisfying a three term recursion relation by using the notion of the usual k th forward difference [2] of a function in order to generate finite sums.

The object of the present paper is to extend the third variation of Hansen, where the usual k th forward difference will not work at all. For this reason we have introduced the following kth difference

$$\delta^k\, w_{2n+s} \equiv \sum_{r=0}^{k} (-1)^r \binom{k}{r} w_{2n+s+2(k-r)}$$

and discusseed the function $f_n(z)$ satisfying a three-term recurison relation, viz.

(1.1) $\quad a_n f_n(z) + b_n f_{n+1}(z) + c_n f_{n+2}(z) = 0$

with special concentration on the first and third terms of the recuision relation, in order to derive extended finite sums involving $f_n(z)$. It may be of interest to note that the following result of third variation due to Hansen

$$C_{2m+s}\, f_{2m+} - C_{2l+s}\, f_{2l+s} = \sum_{n=l}^{m-1} b_{2n+s} f_{2n+s+1}\, C_{2n+s}\, \frac{1}{a_{2n+s}}$$

follows at once as a particular case when we consider $k = 1$ in our result (2.5b).

2. Extension of third variation of Hansen :

Let us consider the following difference of the function W_{2n+s} :

$$\delta\, W_{2n+s} \equiv W_{2n+2+s} - W_{2n+s}.$$

Writing F for $1 + \delta$, we have

$$F\, W_{2n+s} = (1+\delta)\, W_{2n+s} = W_{2n+2+s}.$$

Again

$$\begin{aligned} F^2 W_{2n+s} &= F.FW_{2n+s} \\ &= FW_{2n+2+s} \\ &= (1+\delta)\, W_{2n+2+s} \\ &= W_{2n+2+s} + \delta W_{2n+2+s} \\ &= W_{2n+s+4} \end{aligned}$$

Proceeding in this way we get

$$E^k\, W_{2n+s} = W_{2n+s+2k}$$

$$\begin{aligned} \text{Now } \delta^k\, W_{2n+s} &= (F-1)^k\, W_{2n+s} \\ &= (-1)^k\, (1-F)^k\, W_{2n+s} \end{aligned}$$

In other words,

(2.1) $$\delta^k\, W_{2n+s} = W_{2n+s+2k} - \binom{k}{1} W_{2n+s+2(k-1)} + \binom{k}{2} W_{2n+s+2(k-2)} - \dots + (-1)^k \binom{k}{k} W_{2n+s}$$

Now we can express the function W_n in the following way

(2.2) $W_n = C_n f_n,$

where C_n is so chosen that

$$\frac{C_{n+2}}{C_n} = -\frac{c_n}{a_n}.$$

Now choosing $C_s = 1$, we find

(2.3) $$C_{2r+s} = \prod_{n=0}^{r-1}\left(-\frac{c_{2n+s}}{a_{2n+s}}\right).$$

Next substituting (2.2) and (2.3) into (2.1) and summing n from l to m-1, we have

$$\sum_{n=l}^{m-1} \sum_{r=0}^{k} (-1)^k \binom{k}{r} W_{2n+s+2(k-r)}$$

(2.4) $$= \sum_{n=l}^{m-1} \sum_{r=0}^{k} (-1)^k \binom{k}{r} \left(\prod_{p=0}^{n+k-r-1} - \frac{c_{p+s}}{a_{p+s}} \right) f_{2n+s+2(k-r)},$$

which can be expressed as either of the following two forms by means of (1.1)

$$\sum_{n=l}^{m-1}\sum_{r=0}^{k}(-1)^r\binom{k}{r}W_{2n+s+2(k-r)}$$

$$\begin{aligned}&=\sum_{n=l}^{m-1}\left(\prod_{p=0}^{n-1}-\frac{c_{2p+s}}{a_{2p+s}}\right)\Bigg[\sum_{t=0}^{k-2}(-1)^t\binom{k-1}{t}\frac{1}{a_{2n+2k+s-2(1+t)}}\cdot\\&\cdot\left(\prod_{p=n}^{n+k-(2+t)}-\frac{c_{2p+s}}{a_{2p+s}}\right)b_{2n+2k+s-2(1+t)}\,f_{2n+2k+s-2(1+t)+1}\\&+(-1)^{k-1}\binom{k-1}{k-1}\frac{1}{a_{2n+s}}\cdot b_{2n+s}\,f_{2n+s+1}\Bigg].\end{aligned}\tag{2.5a}$$

$$\sum_{n=l}^{m-1}\sum_{r=0}^{k}(-1)^r\binom{k}{r}W_{2n+s+2(k-r)}$$

$$\begin{aligned}&=\sum_{n=l}^{m-1}\sum_{t=0}^{k-1}(-1)^t\binom{k-1}{t}\frac{1}{a_{2n+2k+s-2(1+t)}}\left(\prod_{p=0}^{n+k-(2+t)}-\frac{c_{2p+s}}{a_{2p+s}}\right)\cdot\\&\cdot b_{2n+2k+s-2(1+t)}\,f_{2n+2k+s-2(1+t)+1}\end{aligned}\tag{2.5b}$$

We now observe the following particular cases.

If k = 1, then we immediately get from (2.5b)

$$W_{2m+s}-W_{2l+s}=\sum_{n=l}^{m-1}\frac{b_{2n+s}\,f_{2n+s+1}\,c_{2n+s}}{a_{2n+s}}\tag{2.6}$$

which is the result (6.3) of [3], by Hansen.

If $k=2$, then from (2.5a), we get

$$\begin{aligned}&(W_{2m+s+2}-W_{2m+s})-(W_{2l+s+2}-W_{2l+s})\\&=-\sum_{n=l}^{m-1}\frac{C_{2n+s}}{a_{2n+s}\,a_{2n+s+2}}\left(c_{2n+s}\,b_{2n+s+2}\,f_{2n+s+3}+a_{2n+s+2}\,b_{2n+s}\,f_{2n+s+1}\right)\end{aligned}\tag{2.7}$$

Similarly for $k=3$, we obtain from (2.5a)

$$(W_{2m+s+4}-2\,W_{2m+s+2}+W_{2m+s})-(W_{2l+s+4}-2\,W_{2l+2}+W_{2l+s})$$

$$(2.8)\qquad = \sum_{n=l}^{m-1} \frac{C_{2n+s}}{a_{2n+s}\, a_{2n+s+2}\, a_{2n+s+4}} \quad c_{2n+s}\, c_{2n+s+2}\, b_{2n+s+4}\, f_{2+s+5}$$

$$+ 2\, c_{2n+s}\, b_{2n+s+2}\, a_{2n+s+4}\, f_{2n+s+3} + a_{2n+s+2}\, a_{2n+s+4}\, b_{2n+s}\, f_{2n+s+1}).$$

Adding (2.6) and (2.7) we get

$$(2.9)\qquad W_{2m+s+2} - W_{2l+s+2} = - \sum_{n=l}^{m-1} \frac{C_{2n+s}}{a_{2n+s}\, a_{2n+s+2}} \quad (c_{2n+s}\, b_{2n+s+2}\, f_{2n+s+3}).$$

Again adding (2.7) and (2.8), we get

$$(W_{2m+s+4} - W_{2m+s+2}) - (W_{2l+s+4} - W_{2l+s+2})$$

$$(2.10)\qquad = \sum_{n=l}^{m-1} \frac{C_{2n+s}}{a_{2n+s}\, a_{2n+s+2}\, a_{2n+s+4}} \quad (c_{2n+s}\, c_{2n+s+5}\, b_{2n+s+4}\, f_{2n+s+5}$$

$$+ c_{2n+5}\, b_{2n+s+2}\, a_{2n+s+4}\, f_{2n+s+3}).$$

Now comparing (2.6) and (2.9) we obtain

$$\frac{C_{2m+s}}{a_{2m+s}} (b_{2m+s}\, f_{2m+s+1}) - \frac{C_{2l+s}}{a_{2l+s}} \quad (b_{2l+s}\, f_{2l+s+1})$$

$$(2.11)\qquad = - \sum_{n=l}^{m-1} \frac{C_{2n+s}}{a_{2n+s}\, a_{2n+s+2}} \quad (c_{2n+s}\, b_{2n+s+2}\, f_{2n+s+3} + a_{2n+s+2}\, b_{2n+s}\, f_{2n+s+1}).$$

Again comparing (2.7) and (2.10) we obtain

$$\frac{C_{2l+s}}{a_{2l+s}\, a_{2l+s+2}} \quad (c_{2l+s}\, b_{2l+s+2}\, f_{2l+s+3} + a_{2l+s+2}\, b_{2l+s}\, f_{2l+s+1})$$

$$- \frac{C_{2m+s}}{a_{2m+s}\, a_{2m+s+2}} \quad (c_{2m+s}\, b_{2m+s+2}\, f_{2m+s+3} + a_{2m+s+2}\, b_{2m+s}\, f_{2m+s+1})$$

$$(2.12)\qquad = \sum_{n=l}^{m-1} \frac{C_{2n+s}}{a_{2n+s}\, a_{2n+s+2}\, a_{2n+s+4}} \quad (c_{2n+s}\, c_{2n+s+2}\, b_{2n+s+4}\, f_{2n+s+5}$$

$$+ 2\, c_{2n+s}\, b_{2n+s+2}\, a_{2n+s+4}\, f_{2n+s+3} + a_{2n+s+2}\, a_{2n+s+4}\, b_{2n+s}\, f_{2n+s+1})$$

Adding (2.9) and (2.10), we get

$$(2.13)\qquad W_{2m+s+4} - W_{2l+s+4} = \sum_{n=l}^{m-1} \frac{C_{2n+s}}{a_{2n+s}\ a_{2n+s+2}\ a_{2n+s+4}} \cdot$$

$$\cdot\ (c_{2n+s}\quad c_{2n+s+2}\ b_{2n+s+4}\, f_{2n+s+5}).$$

Now we substitute $m+1$ for m and $l+1$ for l into (2.9) and then comparing with (2.13) we obtain

$$\frac{C_{2l+s}}{a_{2l+s}\ a_{2l+s+2}}\ (c_{2l+s}\ b_{2l+s+s}\ f_{2l+s+3}) - \frac{C_{2m+s}}{a_{2m+s}\ a_{2m+s+2}} \cdot$$

$$\cdot\ (c_{2m+s}\quad b_{2m+s+2}\, f_{2m+s+3})$$

$$(2.14)\qquad = \sum_{n=1}^{m-1} \frac{C_{2n+s}}{a_{2n+s}\ a_{2n+s+2}\ a_{2n+s+4}}\ (c_{2n+s}\ c_{2n+s+2}\ b_{2n+s+4}\ f_{2n+s+5}$$

$$+\ a_{2n+s+4}\ c_{2n+s}\ b_{2n+s+2}\ f_{2n+s+3}).$$

Lastly substituting $m+2$ for m and $l+2$ for l into (2·6) and comparing with (2.13) we get

$$(b_{2m+s}\, f_{2m+s+1})\ \frac{C_{2m+s}}{a_{2m+s}} + (b_{2m+s+2}\, f_{2m+s+3})\ \frac{C_{2m+s+2}}{a_{2m+s+2}}$$

$$-\ (b_{2l+s}\ f_{2l+s+1})\ \frac{C_{2l+s}}{a_{2l+s}} - (b_{2l+s+2}\, f_{2l+s+3})\ \frac{C_{2l+s+2}}{a_{2l+s+2}}$$

$$(2.15)\qquad = \sum_{n=l}^{m-1} \frac{C_{2n+s}}{a_{2n+s}\ a_{2s+s+2}\ a_{2n+s+4}}\ (c_{2n+s}\ c_{2n+s+2}\ b_{2n+s+4}\, f_{n+s+5}$$

$$-\ a_{2n+s+2}\ a_{2n+s+4}\ b_{2n+s}\, f_{2n+s+1}).$$

3. Application of extended third variation :

We use the recursion relation for the Hermite polynomials,

$$2(n+1)\ H_n - 2z\ H_{n+1} + H_{n+2} = 0.$$

From (2.3), we get

$$(3.1)\qquad C_{2r+s} = \frac{(-4)^{-r}\ \Gamma\left(\frac{s}{2}+\frac{1}{2}\right)}{\Gamma\left(r+\frac{s}{2}+\frac{1}{2}\right)},$$

so that (2.5b) becomes

$$\sum_{n=l}^{m-1} \sum_{r=0}^{k} (-1)^r \binom{k}{r} \frac{(-4)^{-(n+k-r)}\, \Gamma\left(\frac{s}{2}+\frac{1}{2}\right)}{\Gamma\left(n+k-r\frac{s}{2}+\frac{1}{2}\right)} H_{2n+s+2(k-r)}.$$

$$=\sum_{n=1}^{m-1} \sum_{t=0}^{k-1} (-1)^t \binom{k-1}{t} \frac{(-4)^{-(n+k-t-1)}\, \Gamma\left(\frac{s}{2}+\frac{1}{2}\right)}{\Gamma\left(n+k-t-\frac{1}{2}+\frac{s}{2}\right)} \frac{l}{a_{2n+2k+s+2-(1+t)}}$$

$$\cdot b_{2n+2k+s-2(1+t)} \quad H_{2n+2k+s-2l-1},$$

which does not seem to appear before.

If $k=1$, then it follows immediately

$$\sum_{n=l}^{m-1} \frac{1}{\Gamma\left(n+\frac{s}{2}+\frac{3}{2}\right)} (-4)^{-n}\, H_{2n+s+1}$$

$$=\frac{2}{z}\left(\frac{(-4)^{-l}}{\Gamma\left(l+\frac{s}{2}+\frac{1}{2}\right)} H_{2l+s} - \frac{(-4)^{-m}}{\Gamma\left(m+\frac{s}{2}+\frac{1}{2}\right)} H_{2m+s}\right)$$

which is the third example of [3] by Hansen.

Next for $k=2$, substituting (3.1) into (2.7) we get

$$\frac{(-4)^{-(m+1)}}{\Gamma\left(m+\frac{s}{2}+\frac{3}{2}\right)} H_{2m+s+2} - \frac{(-4)^{-m}}{\Gamma\left(m+\frac{s}{2}+\frac{1}{2}\right)} H_{2m+s}$$

$$-\frac{(-4)^{-(l+1)}}{\Gamma\left(l+\frac{s}{2}+\frac{3}{2}\right)} H_{2l+s+2} + \frac{(-4)^{-l}}{\Gamma\left(l+\frac{s}{2}+\frac{1}{2}\right)} H_{2l+s}$$

$$=2z \sum_{n=l}^{m-1} \frac{(-4)^{-n-2}}{\Gamma\left(n+\frac{s}{2}+\frac{5}{2}\right)} \left(H_{2n+s+3}+(4n+2s+6)\, H_{2n+s+1}\right).$$

For $l=s=0$, this becomes

$$\sum_{n=0}^{m-1} \frac{(-4)^{-n-2}}{\Gamma\left(n+\frac{5}{2}\right)} \left(H_{2n+3}+(4n+6)\, H_{2n+1}\right)$$

$$= \frac{z}{\sqrt{\pi}} - \frac{1}{2z} \frac{(-4)^{-m}}{\Gamma(m+\frac{1}{2})} \left(\frac{H_{2m+2}}{4(m+\frac{1}{2})} + H_{2m} \right)$$

For $s = 1$ and $l = 0$ we obtain

$$\sum_{n=0}^{m-1} \frac{(-4)^{-n-2}}{(n+2)!} \left(H_{2n+4} + (4n+8) H_{2n+s} \right)$$

$$= z^2 - \frac{1}{2} + \frac{1}{2z} \frac{(-4)^{-m}}{(m)!} \left(\frac{(-4)^{-1}}{(m+1)} H_{2m+2} - H_{2m+1} \right).$$

For $k = 3$, substituting (3.1) into (2.8), we obtain

$$\frac{(-4)^{-(m+2)}}{\Gamma\left(m + \frac{s}{2} + \frac{5}{2}\right)} H_{2m+s+4} - 2 \frac{(-4)^{-(m+1)}}{\Gamma\left(m + \frac{s}{2} + \frac{3}{2}\right)} H_{2m+s+2}$$

$$+ \frac{(-4)^{-m}}{\Gamma\left(m + \frac{s}{2} + \frac{1}{2}\right)} H_{2m+s} - \frac{(-4)^{-(l+2)}}{\Gamma\left(l + \frac{s}{2} + \frac{5}{2}\right)} H_{2l+s+4}$$

$$+ 2 \frac{(-4)^{-(l+1)}}{\Gamma\left(l + \frac{s}{2} + \frac{3}{2}\right)} H_{2l+s+2} - \frac{(-4)^{l}}{\Gamma\left(l + \frac{s}{2} + \frac{1}{2}\right)} H_{2l+s}$$

$$= 2z \sum_{n=l}^{m-1} \frac{(-4)^{-n-3}}{\Gamma\left(n + \frac{s}{2} + \frac{7}{2}\right)} \left(H_{2n+s+5} + 4(2n + s + 5) H_{2n+s+3} \right.$$

$$\left. + (4n\, 2s + 6)(4n + 2s + 10) H_{2n+s+1} \right).$$

Now for $s = l = 0$,

$$\frac{(-4)^{-m}}{\Gamma(m+\frac{1}{2})} \left(\frac{(-4)^{-2}}{(m+\frac{3}{2})(m+\frac{1}{2})} H_{2m+4} + \frac{1}{2(m+\frac{1}{2})} H_{2m+2} + H_{2m} \right) - \frac{4}{3\sqrt{\pi}} z^4$$

$$= 2z \sum_{n=0}^{m-1} \frac{(-4)^{-n-3}}{\Gamma(n+\frac{7}{2})} \left(H_{2n+5} + 2(4n+10) H_{2n+3} + (4n+6)(4n+10) H_{2n+1} \right)$$

Next for $s = 1$ and $l = 0$, we get

$$2z \sum_{n=0}^{m-1} \frac{(-4)^{-n-3}}{(n+3)!} \left(H_{2n+6} + 2(4n+12) H_{2n+4} + (4n+8)(4n+12) H_{2n+2} \right)$$

$$= \frac{(-4)^{-m}}{(m)!}\left(\frac{(-4)^{-2}}{(m+1)(m+2)} H_{2n+5} + \frac{1}{2(m+1)} H_{2m+3} + H_{3m+1}\right)$$

$$- \left(\frac{H_5}{32} + \frac{H_3}{2} + H_1\right)$$

Again substituting (3.1) into (2.9) and simplifying we get

$$\frac{(-4)^{-(m+1)}}{\Gamma\left(m+\frac{s}{2}+\frac{3}{2}\right)} H_{2m+s+2} - \frac{(-4)^{-(l+1)}}{\Gamma\left(l+\frac{s}{2}+\frac{3}{2}\right)} H_{2l+s+2}$$

$$= 2z \sum_{n=l}^{m-1} \frac{(-4)^{-n-2} H_{2n+s+3}}{\Gamma\left(n+\frac{s}{2}+\frac{5}{2}\right)}$$

then for $s=1=0$, we wet

$$\frac{(-4)^{-(m+1)}}{\Gamma\left(m+\frac{3}{2}\right)} H_{2m+2} + \frac{(2Z-1)}{\sqrt{\pi}} = 2z \sum_{n=0}^{m-1} \frac{(-4)^{-n-2} H_{2n+3}}{\Gamma\left(n+\frac{5}{2}\right)}$$

Next for $s=1$ and $l=0$ we have

$$\frac{(-4)^{-(m+1)}}{(m+1)!} H_{2m+3} + z(2z^2-3)$$

$$= 2z \sum_{n=0}^{m-1} \frac{(-4)^{-n-2} H_{2n+4}}{(n+2)!}$$

Next substituting (3.1) into (2.10) and simplifying we get

$$\frac{(-4)^{-(m+2)}}{\Gamma\left(m+\frac{s}{2}+\frac{5}{2}\right)} H_{2m+s+4} - \frac{(-4)^{-(m+1)}}{\Gamma\left(m+\frac{s}{2}+\frac{3}{2}\right)} H_{2m+s+2}$$

$$- \frac{(-4)^{-(l+2)}}{\Gamma\left(l+\frac{s}{2}+\frac{5}{2}\right)} H_{2l+s+4} + \frac{(-4)^{-(l+1)}}{\Gamma\left(l+\frac{s}{2}+\frac{3}{2}\right)} H_{2l+s+2}$$

$$= 2z \sum_{n=l}^{m-1} \frac{(-4)^{-n-3}}{\Gamma\left(n+\frac{s}{2}+\frac{7}{2}\right)} \left(H_{2n+s+5} + 2(2n+s+5) H_{3n+s+3}\right),$$

assuming $l=s=0$, we have

$$\frac{(-4)^{-(m+2)}}{\Gamma(m+\frac{5}{2})} H_{2m+4} - \frac{(-4)^{-(m+1)}}{\Gamma(m+\frac{3}{2})} H_{2m+2} - \frac{2z^2(\frac{2}{3}z^2-1)}{\sqrt{\pi}}$$

$$= 2z \sum_{n=0}^{m-1} \frac{(-4)^{-n-3}}{\Gamma(n+\frac{7}{2})} \left(H_{2n+5} + (4n+10) H_{2n+3} \right)$$

Next assuming $l=0$ and $s=1$, we get

$$\frac{(-4)^{-(m+2)}}{(m+2)\,!} H_{2m+5} - \frac{(-4)^{-(m+1)}}{(m+1)\,!} H_{2m+3} - \frac{z\,H_4}{16}$$

$$= 2z \sum_{n=0}^{m-1} \frac{(-4)^{-n-3}}{(n+3)\,!} \left(H_{2n+6} + (4n+12) H_{2n+4} \right).$$

Again substituting (3.1) into (2.11), we get

$$\frac{(-4)^{-l}}{(4l+2s+2)\,\Gamma\left(l+\frac{s}{2}+\frac{1}{2}\right)} H_{2l+s+1} - \frac{(-4)^{-m}}{(4m+2s+2)\,\Gamma\left(m+\frac{s}{2}+\frac{1}{2}\right)} H_{2m+s+1}$$

$$= \sum_{n=l}^{m-1} \frac{(-4)^{-n-2}}{\Gamma\left(n+\frac{s}{2}+\frac{5}{2}\right)} \left(H_{2n+s+3} + 2(2n+s+3)\, H_{2n+s+1}\right).$$

Let $l=s=0$, then we have

$$\sum_{n=0}^{m+1} \frac{(-4)^{-n-2}}{\Gamma(n+\frac{5}{2})} \left(H_{2n+3} + 2(2n+3)\, H_{2n+1}\right)$$

$$= \frac{z}{\sqrt{\pi}} - \frac{(-4)^{-m}}{2(2m+1)\,\Gamma(m+\frac{1}{2})} H_{2m+1}$$

Next assuming $l=0$ and $s=1$, we get

$$z^2 - \tfrac{1}{2} - \frac{(-4)^{-m}}{4(m+1)\,!} H_{2m+2} = \sum_{n=0}^{m-1} \frac{(-4)^{-n-2}}{(n+2)\,!} \left(H_{2n+4} + 4(n+2)\, H_{2n+2}\right).$$

Similarly substituting (3.1) into (2.12), and simplifying we get

$$\sum_{n=l}^{m-1} \frac{(-4)^{-n-3}}{\Gamma\left(n+\frac{s}{2}+\frac{7}{2}\right)} \left(H_{2n+s+5} + 4(2n+s+5)\, H_{2n+s+2} \right.$$
$$\left. + 4(2n+s+3)(2n+s+5)\; H_{2n+s+1}\right)$$

$$= \frac{(-4)^{-m-2}}{\Gamma\left(m+\frac{s}{2}+\frac{5}{2}\right)} \left(H_{2m+s+3} + 2\,(2m+s+3)\; H_{2m+s+1}\right)$$
$$- \frac{(-4)^{-l-2}}{\Gamma\left(l+\frac{s}{2}+\frac{5}{2}\right)} \left(H_{2l+s+3} + 2(2l+s+3)\; H_{2l+s+1}\right)$$

Assuming $l = s = 0$, we have

$$\sum_{n=0}^{m-1} \frac{(-4)^{-n-3}}{\Gamma\left(m+\frac{7}{2}\right)} \left(H_{2n+5} + 4(2n+5)\, H_{2n+3} + 4(2n+3)(2n+5)\, H_{2n+1}\right)$$

$$= \frac{(-4)^{-m-2}}{\Gamma\left(m+\frac{5}{2}\right)} \left(H_{2m+3} + 2(2m+3)\, H_{2m+1}\right) - \frac{2}{\sqrt{\pi}}\, z^3.$$

Next assuming $l = 0$ and $s = 1$, we have

$$\sum_{n=0}^{m-1} \frac{(-4)^{-n-3}}{(n+3)\,!} \left(H_{2n+6} + 8(n+3)\, H_{2n+4} + 16(n+2)(n+3)\, H_{2n+2}\right)$$

$$= \frac{(-4)^{-m-2}}{(m+2)\,!} \left(H_{2m+4} + 4(m+2)\, H_{2m+2} - \tfrac{1}{8}\left(4z^4 - 4z^2 - 1\right)\right)$$

Again applying (3.1) into (2.13) we get

$$2z \sum_{n=l}^{m-1} \frac{(-4)^{-n-3}}{\Gamma\left(n+\frac{s}{2}+\frac{7}{2}\right)} H_{2n+s+5}$$

$$= \frac{(-4)^{-(m+2)}}{\Gamma\left(m+\frac{s}{2}+\frac{5}{2}\right)} H_{2m+s+4} - \frac{(-4)^{-(l+2)}}{\Gamma\left(l+\frac{s}{2}+\frac{5}{2}\right)} H_{2l+s+4}$$

Let $s = l = 0$, then we have

$$\frac{(-4)^{-(m+2)}}{\Gamma(m+\frac{5}{2})} H_{2m+4} - \frac{H_4}{12\sqrt{\pi}} = 2z \sum_{n=0}^{m-1} \frac{(-4)^{-n-3}}{\Gamma(n+\frac{7}{2})} H_{2n+}$$

Next let $l = 0$ and $s = 1$, then we obtain

$$\frac{(-4)^{-(m+)}}{(m+2)!} H_{2m+5} - \frac{H_5}{32} = 2z \sum_{n=0}^{m-1} \frac{(-4)^{-n-3}}{(n+3)!} H_{2n+6}$$

Again substituting (3.1) into (2.14) and simplifying we get

$$\sum_{n=l}^{m-1} \frac{(-4)^{-n-3}}{\Gamma\left(n+\frac{s}{2}+\frac{7}{2}\right)} \left(H_{2n+s+2} + 2\,(2n+s+5)\, H_{2n+s+3} \right)$$

$$= \frac{(-4)^{-m-2}}{\Gamma\left(m+\frac{s}{2}+\frac{5}{2}\right)} H_{2m+s+3} - \frac{(-4)^{-l-2}}{\Gamma\left(l+\frac{s}{2}+\frac{5}{2}\right)} H_{2l+s+3}.$$

Let $l = s = 0$, then we have

$$\sum_{n=0}^{m-1} \frac{(-4)^{-n-3}}{\Gamma(n+\frac{7}{2})} \left(H_{2n+5} + 2\,(2n+5)\, H_{2n+3} \right)$$

$$= \frac{(-4)^{-m-2}}{\Gamma(m+\frac{5}{2})} H_{2m+3} - \frac{H_3}{12\sqrt{\pi}}$$

Next assuming $l = 0$, $s = 1$, we get

$$\sum_{n=0}^{m-1} \frac{(-4)^{-n-3}}{(n+3)!} \left(H_{2n+6} + 4\,(n+3)\, H_{2n+4} \right)$$

$$= \frac{(-4)^{-m-2}}{(m+2)!} H_{2m+4} - \frac{H_4}{32},$$

At last substituting (3.1) into (2.15), we obtain

$$\frac{H_{2m+s+1}}{\Gamma\left(m+\frac{s}{2}+\frac{3}{2}\right)} (-4)^{-m} + \frac{H_{2m+s+3}}{\Gamma\left(m+\frac{s}{2}+\frac{5}{2}\right)} (-4)^{-(m+1)}$$

$$-\frac{H_{2l+s+1}}{\Gamma\left(l+\frac{s}{2}+\frac{3}{2}\right)}(-4)^{-l}-\frac{H_{2l+s+3}}{\Gamma\left(l+\frac{s}{2}+\frac{5}{2}\right)}(-4)^{-(l+1)}$$

$$=\sum_{n=l}^{m-1}\frac{(-4)^{-n}}{16\,\Gamma\left(n+\frac{s}{2}+\frac{7}{2}\right)}\left(H_{2n+s+5}-4(2n+s+3)(2n+s+5)H_{2n+s+1}\right).$$

Assuming $l=s=0$, we get

$$\frac{H_{2m+1}}{H(m+\frac{3}{2})}(-4)^{-m}+\frac{H_{2m+3}}{H(m+\frac{5}{2})}(-4)^{-(m+1)}-\frac{4z}{\sqrt{\pi}}+\frac{4z\,(2z^2-3)}{3\sqrt{\pi}}$$

$$=\sum_{n=0}^{m-1}\frac{(-4)^{-n}}{16\,H(n+\frac{7}{2})}\left(H_{2n+5}-4(2n+3)(2n+5)H_{2n+1}\right)$$

Next assuming $l=0$ and $s=1$, we

$$\frac{H_{2m+2}}{(m+1)\,!}(-4)^{-m}+\frac{H_{2m+4}}{(m+2)\,!}(-4)^{-(m+1)}$$

$$=\sum_{n=-2}^{m-1}\frac{(-4)^{-n}}{16\,(n+3)\,!}\left(H_{2n+6}-16(n+2)(n+3)H_{2n+2}\right),$$

then substituting $n-2$ for n, we at once obtain

$$\sum_{n=m+1}^{m+2}(-4)^{-(n-1)}\frac{H_{2n}}{n\,!}=\sum_{n=0}^{m+1}\frac{(-4)^{-n}}{(n+1)\,!}\,H_{2n+2}-16n\,(n+1)\,H_{2n-2}$$

which are our results.

Conclusion. Applications of our result on extended third variation to other special functions satisfying three term recursion relations are merely routine work.

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REFERENCES

[1] Barik, N. — k th differences of fuuctious satisfying three-term recursion relation (I),—Pure Math. Manuscript Vol. 3 (1984), 11-21.

[2] Boole, G.—A treatise on the calculus of finite differences, G. E. Stechert and Co., New York (1931). 676-679.

[3] Hansen, E.—Sums of functions satisfying recursion relation. The Amer. Math, Monthly, 88 (1981). 676 -679

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Gurudas College of Commerce
Calcutta-54

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ORDINARY DIFFERENTIAL EQUATIONS OF SECOND ORDER INVARIANT UNDER ONE-PARAMETER GROUPS

S. K. CHATTERJEA

1. Introduction : Although Lie suggested a most ingenious method for finding differential equation of second order invariant under a given one-parameter group many years ago, yet in recent year O. Blazo [1] has studied in details the general form of ordinary differential equations of second order invariant under the one-parameter group whose generator is

$$(ax^2 + bx + c) \frac{\partial}{\partial x} + (ly^2 + my + n) \frac{\partial}{\partial y},$$

where a, b, c, l, m and n are constants. It may be of interest to note that the final form of $g(x, y, y')$ would remain the same provided one could set

$$\int \frac{2l\phi(x, c_1) + m}{A} = \Phi_1(x, c_1)$$

in the work of Blazo.

The object of the present paper is to point out that the study of Blazo can be extended to the more general one-parameter group generated by

$$\xi(x) \frac{\partial}{\partial x} + \eta(y) \frac{\partial}{\partial y}.$$

Moreover, one can study the general form of ordinary differential equations of second order invariant under the one-parameter group generated by

$$\eta(y) \frac{\partial}{\partial x} + \xi(x) \frac{\partial}{\partial y}.$$

2. Ordinary differential equations of second order invariant under the group generated by $\xi(x) \frac{\partial}{\partial x} + \eta(y) \frac{\partial}{\partial y}$:

We consider the following second order differential equation

(2.1) $y'' = g(x, y, y')$.

Now the problem is to find a general form of the function $g(x, y, y')$ so that the equation (2.1) admits the group generated by $\xi(x)\frac{\partial}{\partial x} + \eta(y)\frac{\partial}{\partial y}$.

The once-extended group corresponding to the above group is given by

(2.2) $$U \equiv \xi(x)\frac{\partial}{\partial x} + \eta(y)\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial y'}$$

where $\zeta = (\eta' - \xi')y'$, $\xi' = \frac{d\xi}{dx}$ and $\eta' = \frac{d\eta}{dy}$.

The equation (2.1) is equivalent to the following system of differential equations

(2.3) $$dy = y'\,dx, \quad dy' = g(x, y, y')\,dx.$$

and the above system corresponds to a single partial differential equation

(2.4) $$Y(f) \equiv f_x + y' f_y + g f_{y'} = 0.$$

Now the necessary and sufficient condition in order that (2.4) admits the group given by (2.2) is

(2.5) $$[Y, U]f \equiv k(x, y, y')\,Y(f),$$

where $[Y, U] = YU - UY$,

which reduces to

(2.6) $$Y(\xi)\,f_x + \{Y(\eta) - U(y')\}f_y + \{Y(\zeta) - U(g)\}f_{y'}$$
$$= k(x, y, y')\,(f_x + y' f_y + g f_{y'}).$$

From the above identity we have on eliminating k

(2.7) $$\begin{cases} Y(\eta) - y'\,Y(\xi) = U(y') \\ Y(\zeta) - gY(\xi) = U(g). \end{cases}$$

Now the first condition of (2.7) is an identity, whereas the second condition implies that

(2.8) $$\xi g_x + \eta g_y + (\eta' - \xi')y' g_{y'} = (\eta'' y' - \xi'')y' + (\eta' - 2\xi')g,$$

which is equivalent to the system

(2.9) $$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dy'}{(\eta' - \xi')y'} = \frac{dg}{(\eta'' y' - \xi'')y' + (\eta' - 2\xi')g}.$$

From the first two ratios we obtain

(2.10) $X - Y = c_1$, where $X(x) = \int \frac{dx}{\xi}$ and $Y(y) = \int \frac{dy}{\eta}$,

wherefrom we write the following explicit form

(2.11) $y = \phi(x, c_1)$.

Then from the first and third ratios we get

(2.12) $$\frac{dy'}{y'} + \frac{\xi'}{\xi}\, dx = \frac{\eta'(\psi(x, c_1))}{\xi}\, dx.$$

Putting $\phi_1(x, c_1) = \int \frac{\eta'(\phi(x, c_1))}{\xi}\, dx$, we derive

(2.13) $y' = c_2\, \phi_2(x, c_1)$,

where

(2.14) $\phi_2(x, c_1) = \xi^{-1} e^{\phi_1(x, c_1)}$.

Lastly from the first and fourth ratios we obtain

(2.15) $$\frac{dg}{dx} + \frac{2\xi' - \eta'(\phi(x, c_1))}{\xi}\, g = \frac{(\eta''(\phi(x, c_1))\, y' - \xi'')\, y'}{\xi},$$

which being a linear equation, gives us

(2.16) $$g = \frac{\phi_2}{\xi}\left[c_3 + c_2^2 H(x, c_1) + c_2 \xi'\right],\ H(x, c_1) = \int \frac{\eta'' e^{\phi_1(x, c_1)}}{\xi}\, dx,$$

so that

(2.17) $$c_3 = \xi^2 g e^{-\phi_1(x, X - Y)} - \xi^2 (y')^2 e^{-2\phi_1(x, X - Y)} H(x, X - Y) + \xi y' e^{-\phi_1(x, X - Y)} \xi'$$

From the general solution of (2.9) we have

(2.18) $c_3 = \Phi(c_1, c_2)$, where Φ is arbitrary,

so that we derive

(2.19) $$g(x, y, y') = e^{\phi_1(x, X - Y)} \xi^{-2} \Phi(X - Y, \xi y' e^{-\phi_1(x, X - Y)}) + (y')^2 e^{-\phi_1(x, X - Y)} H(x, X - Y) - \xi' y' \xi^{-1},$$

Hence the desired general form of differential equations is

(2.20) $$y'' = e^{\phi_1 (x, X - Y)} \xi^{-2} \Phi (X - Y, \xi y' e^{- \phi_1 (x, X - \quad)}) + (y')^2 e^{- \phi_1 (x, X - Y)} H (x, X - Y) - \xi' y' \xi^{-1}$$

3. Ordinary differential equations of second order invariant under the group generated by $\eta (y) \frac{\partial}{\partial x} + \xi (x) \frac{\partial}{\partial y}$:

Suppose the following differential equation

(3.1) $$y'' = g(x, y, y')$$

admits the one-parameter group generated by $\eta(y) \frac{\partial}{\partial x} + \xi (x) \frac{\partial}{\partial y}$. Now the once-extended group corresponding to the above group is given by

(3.2) $$U = \eta (y) \frac{\partial}{\partial x} + \xi (x) \frac{\partial}{\partial y} + \zeta \frac{\partial}{\partial y'} :$$

where $\zeta = \xi' - \eta' (y')^2$. where $\xi' = \frac{d\xi}{dx}$ and $\eta' = \frac{d\eta}{dy}$.

Proceeding exactly in the same manner as in § 2, we observe that the necessary and sufficient condition that the partial differential equation

(3.3) $$Y (f) \equiv f_x + y' f_y + g f_{y'} = 0$$

admits the group given by (3.2) is

(3.4) $$Y (\eta) f_x + [Y (\xi) - U (y')] f_y + [Y (\zeta) - U (g)] f_{y'} = k(x, y, y') (f_x + y' f_y + g f_{y'}).$$

By virtue of the identity (3.4) we have

(3.5) $$\begin{cases} Y (\eta) = k (x, y, y') \\ Y (\xi) - U (y') = k (x, y, y') y' \\ Y (\zeta) - U (g) = k (x, y, y') g \end{cases}$$

Eliminating $k (x, y, y')$ between the equations of (3.5) we get

(3.6) $$\begin{cases} Y (\xi) - y' \, Y (\eta) = U (y') \\ Y (\zeta) - g \, Y (\eta) = U (g) \end{cases}$$

Now the first condition of (3.6) implies that

$$\frac{d\xi}{dx} - (y')^2 \frac{d\eta}{dy} = \zeta,$$

which is already stated in (3.2).

The second condition of (3.6) implies that

(3.7) $\eta g_x + \xi g_y + (\xi' - \eta' (y')^2) g_{y'} = \xi'' - \eta'' (y')^3 - 3 g y' \eta'$,

which is equivalent to

(3.8) $$\frac{dx}{\eta} = \frac{dy}{\xi} = \frac{dy'}{\zeta' - \eta'(y')^2} = \frac{dg}{\xi'' - \eta'' (y')^3 - 3gy'\eta'} .$$

From the first two rations, we obtain

(3.9) $X - Y = c_1$, where $X(x) = \int \xi \, dx$ and $Y(y) = \int \eta \, dy$,

wherefrom we write the explicit form

(3.10) $y = \phi(x, c_1)$.

Then from the first and third rations we get

(3.11) $$\frac{dy'}{dx} + \frac{\eta'}{\eta} (y')^2 = \frac{\xi'}{\eta},$$

which is Riccati's equation. From the theory of one-parameter group, we know that $y' = \dfrac{\xi(x)}{\eta(\phi(x, c_1))}$ is a particular solution of the Riccati's equation.

(3.12) $$\frac{dy'}{dx} + \frac{\eta'(\phi(x, c_1))}{\eta(\phi(x, c_1))} (y')^2 = \frac{\xi'}{\eta(\phi(x, c_1)} .$$

Now the transformation

$$y' = \frac{1}{z} + \frac{\xi(x)}{\eta(\phi(x, c_1))}$$

changes the equation (3.12) into

(3.13) $$\frac{dz}{dx} - 2 \frac{\eta'(\phi(x, c_1))\xi}{\eta^2(\phi(x, c_1))} z = \frac{\eta'(\phi(x, c_1))}{\eta(\phi(x, c_1))},$$

which being a linear equation, gives us

(3.14) $z = e^{-\phi_1(x, c_1)} [c_2 + \phi_2(x, c_1)]$,

where $$\phi_1(x, c_1) = \int \frac{-2\eta'(\phi(x, c_1))\xi}{\eta'(\phi(x, c_1))} dx$$

and $$\phi_2(x, c_1) = \int \frac{\eta'(\phi(x, c_1))}{\eta(\phi(x, c_1))} e^{\phi_1(x, c_1)} dx.$$

Thus we obtain

(3.15) $$y' = \frac{e^{\phi_1(x, c_1)}}{c_2 + \phi_2(x, c_1)} + \frac{\xi(x)}{\eta(\phi(x, c_1))},$$

so that

$$(3{\cdot}16)\qquad c_2=\frac{e^{\phi_1(x,\,c_1)}}{y'-\xi(x)/\eta(\phi(x,\,c_1))}-\phi_2(x,\,c_1)$$

Lastly from the first and fourth ratios we get

$$(3.17)\qquad \frac{dg}{dx}+\frac{3y'\eta'(\phi(x,\,c_1))}{\eta(\phi(x,\,c_1))}\,g=\frac{\xi''-\eta''(\phi(x,\,c_1))(y')^3}{\eta(\phi(x,\,c_1))},$$

which being a linear equation, gives us

$$(3.18)\qquad g=e^{3/2\,\phi_1(x\,c_1,)-\phi_3)\,(x,\,c_1,\,c_2)}\,[c_3+\phi_4(x,\,c_1,\,c_2)\,],$$

where $\phi_3(x,c_1,c_2)=\int\frac{3\,\eta'(\phi(x,\,c_1))}{\eta(\phi(x,\,c_1))}\cdot\frac{e^{\phi_1(x,\,c_1)}}{c_2+\phi_2(x,\,c_1)}\,dx$

and $\phi_4(x,\,c_1,\,c_2)=\int\frac{\xi''-\eta''(\phi(x,\,c_1))(y')^3}{\eta(\phi(x,\,c_1))}\,e^{\phi_3(x,\,c_1,\,c_2)-3/2\,\phi_1(x,\,c_1)}\,dx.$

It follows therefore from (3.18) that

$$(3.19)\qquad c_3=g\,e^{\phi_3(x,\,c_1,\,c_2)\,-\,3/2\,\phi_1(x,\,c_1)}-\phi_4(x,c_1,\,c_2).$$

Now from the general solution of (3.8) we have

$$(3.20)\qquad c_3=\Phi(c_1,\,c_2),\ \text{where } \Phi \text{ is orbitrary.}$$

Hence the general form of the desired differential equation is

$$(3.21)\qquad \begin{aligned} y''=\exp\Big[\,&3/2\,\phi_1(x,X-Y)-\phi_3\Big(x,X-Y,\,e^{\phi_1(x,X-Y)}.\\ &\cdot\Big\{y'-\frac{\xi}{\eta(\phi(x,X-Y))}\Big\}^{-1}-\phi_2(x,X-Y)\Big)\Big].\\ &\cdot\Big[\Phi\Big(X-Y,\,e^{\phi_1(x,\,c_1)}\Big\{y'-\frac{\xi}{\eta(\phi(x,X-Y))}\Big\}^{-1}-\phi_2(x,\,c_1)\Big)\\ &+\phi_4\Big(x,X-Y,\,e^{\phi_1(x,\,c_1)}\Big\{y'-\frac{\xi}{\eta(\phi(x,X-Y))}\Big\}^{-1}-\phi_2(x,c_1)\Big)\Big]\end{aligned}$$

Remark. Since the present work which was to appear in Math. Balkanica vol. 10 (1980) with some applicatiions, still remains unpublished, the author wishes to publish it in the departmental journal only stating the main results.

REFERENCE

[1] **Blazo, O.** — La forme ge'ne'rale de L' E quation difee'rentielle dn second ordre admettant la transformation infinite'simale conforme prolonge's, Math, Balkanica 4 (1974), 469—472,

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Dept. of Pure Math.
Calcutta University.

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AN APPROACH TO FIXED POINT THEOREMS ON 2-METRIC SPACES

KANAN MAJUMDAR

The notion of a 2-metric space was introduced by S. Gahler [1]. Recently, fixed point theorems in 2-metric space have received much attention.

We ([3], [4]) have already studied some fixed point theorems in 2-metric space.

The object of this paper is to use the notion of orbitally continuous self-mapping due to K. Iseki [2] in 2-metric space in proving two fixed point theorems which are stated below :

Theorem 1. Let $\{f_n\}$ $(n = 1, 2 \ldots)$ be a sequence of mappings of a complete 2-metric space X into itself. If the following conditions hold :

(i) f_0 is orbitally continuous

(ii) for some λ with $0 < \lambda < 1$ and every x, y, a of X

$$\min [d(f_0 x, f_n y, a)\ d(x, f_0 x, a)\ d(y, f_n y, a)]$$
$$-\min [d(x, f_n y, a)\ d(y, f_0 x, a)] \leqslant \lambda\, d(x, y, a)],$$

then there exists a common fixed point of f_n $(n = 0, 1, 2, \ldots)$ in X.

Theorem 2. Let $f : X \to X$ (X being a complete 2-metric space) be orbitally conti uous and for some positive $q < 1$ and every $x, y, a \in X$

(i) $d(fx, fy, a) \leqslant q \max \{ d(x, y, a),\ d(x, fx, a),\ d(y, fy, a),\ d(y, fx, a) \}$.

Then f has a fixed point.

Proof of theorem 1. Let x_0 be an arbitrary point of X. We define a sequence as $x_0, x_1 = f_0 x_0, x_2 = f_1 x_1, x_3 = f_0 x_2, x_4 = f_2 x_3 \ldots$

Now by the condition (ii) we have

$$\min [d(f_0 x_0 f_1 x_1, a),\ d(x_0, f_0 x_0, a),\ d(x_1, f_1 x_1, a)]$$
$$-\min [d(x_0, f_1 x_1, a),\ d(x_1, f_0 x_0, a)] \leqslant \lambda\, d(x_0, x_1, a)$$

or $\min [d(x_1, x_2, a), d(x_0, x_1, a), d(x_1, x_2, a)]$

$- \min [d(x_0, x_2\, a), d(x_1, x_1, a)] \leqslant \lambda\, d(x_0, x_1, a)$.

Since X is a 2-metric space,

$$d(x_0, x_1, a) \neq 0 \text{ for some } a \in X.$$

Hence $\lambda\, d(x_0, x_1, a) < d(x_0, x_1, a)$ for $\lambda < 1$

implies $d(x_1, x_2, a) \leqslant \lambda\, d(x_0, x_1, a)$.

Similarly $d(x_2, x_3, a) \leqslant \lambda\, d(x_1, x^2, a)$

$$\leqslant \lambda^2\, d(x_0, x_1, a)$$

....

$$d(x_n, x_{n+1}, a) < \lambda^n\, d(x_0, x_1, a).$$

Thus $\{x_n\}$ is a Cauchy sequence. As X is complete, $\{x_n\}$ converges to a limit u (say).

$$\therefore \lim_{n \to \infty} d(x_n, u, a) = 0 \text{ for all } a \in X.$$

As f_0 is orbitally continuous,

$$\lim_{n \to \infty} d(f_0 x_n, f_0 u, a) = 0 \text{ for all } a \in X.$$

$$\begin{aligned} \therefore\ d(u, f_0 u, a) &\leqslant d(u, f_0 u, f_o x_n) + d(u, f_o x_n, a) \\ &\quad + d(f_0 x_n, f_0 u, a) \\ &\to 0 \text{ as } n \to \infty \text{ for all } a \in X. \end{aligned}$$

Hence $d(u, f_0 u, a) = 0$ for all $a \in X$.

If $u \neq f_o u$, then $d(u, f_0 u, a) \neq 0$ for some $a \in X$ by the definition of 2-metric space.

$$\therefore \quad u \ \cdot f_0 u.$$

Now we put $x = y = u$ in (ii) and we get

$\min \{ d(f_0 u, f_n u, a),\ d(u, f_0 u, a),\ d(u, f_n u, a) \}$

$- \min \{ d(u, f_n u, a),\ d(u, f_0 u, a) \} \leqslant \lambda\ d(u, u, a)$

implies $d(u, f_n u, a) = 0$.

If $u \neq f_n u$, then $d(u, f_n u, a) \neq 0$ for some $a \in X$.

$$\therefore \quad u = f_n u.$$

Therefore u is a common fixed point of f_n ($n = 0, 1, 2, \ldots$).

Proof of theorem 2. Let x_0 be an arbitrary point of X. We define $\{x_n\}$ as follows

$$x_0,\ x_1 = x_0,\ \ldots\ x_n = fx_{n-1}.$$

Case 1. If for some m, $x_{m-1} = x_m$, then $\{x_n\}$ is a Cauchy sequence and the limit of $\{x_n\}$ is a fixed point of f.

Case 2. Let $x_m \neq x_{m-1}$, for every $m = 0, 1, 2 \ldots$

By (i) we have

$$d(fx_{n-1}, fx_n, a) \leqslant q \max\{d(x_{n-1}, x_n, a),\ d(x_{n-1}, fx_n, a),\ d(x_n, fx_n, a),\ d(x_n, fx_{n-1}, a)\}$$

or, $$d(x_n, x_{n+1}, a) \leqslant q \max\{d(x_{n-1}, x_n, a),\ d(x_{n-1}, x_n, a),\ d(x_n, x_{n+1}, a)\ d(x_n, x_n, a)\}.$$

Since X is a 2-metric space,

$q\, d(x_n, x_{n+1}, a) < d(x_n, x_{n+1}, a)$ for $q < 1$.

$\therefore$ In both the cases,

$$d(x_n, x_{n+1}, a) \leqslant q\ d(x_{n-1}, x_n, a)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\leqslant q^n\, d(x_0, x_1, a).$$

Thus $\{x_n\}$ is a Cauchy sequence converging to a point $u \in X$, since X is complete.

$$\therefore \lim_{n \to \infty} d(f^n x_0, u, a) = 0 \text{ for all } a \in X.$$

Orbital continuity of f implies

$$\lim_{n \to \infty} d(f^{n+1} x_0, fu, a) = 0 \text{ for all } a \in X.$$

$$\therefore\ d(u, fu, a) \leqslant d(u, fu, f^{n+1} x_0) + d(u, f^{n+1} x_0, a) + d(f^{n+1} x_0, fu, a)$$
$$\to 0 \text{ as } n \to \infty$$

Thus $d(u, fu, a) = 0$ for all $a \in X$.

Since $u \neq f(u)$ implies $d(u, fu, a) \neq 0$ for some $a \in X$, by definition of 2-metric space, we have

$$u = f(u)$$

i. e. u is a fixed point of f.

This completes the proof.

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REFERENCES

[1] Gahlere S—2-metrische Raume and ihre topologische Struktur, Math. Nachr, 26 (1963/64) 115—148.

[2] Isekie K—Fixed point theorem in 2-metric spaces. Mathematics Seminar Nothes 3 (1975) 131—132.

[3] Majumdar K—A fixed point theorem in 2-metric spaces Pure Math. Manuscript 2 (1983) 35—38.

[4] ———— Common fixed point theorem in 2-metric spaces—Journal of Pure Mathematics 2 (1982) 67–69.

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Dept. of Pure Mathematics
University of Calcutta
35, Ballygunge Circular Road
Calcutta—700 019
INDIA.

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OPERATIONAL DERIVATION OF SOME GENERATING FUNCTIONS FOR THE EVEN AND ODD GENERALISED HERMITE POLYNOMIALS

BANDANA GHOSH

1. **Introduction.** The Hermite polynomials $H_n(x)$ can be reduced to the Laguerre polynomials $L_n^{(\alpha)}(x)$ with the parameters $\alpha = \pm \frac{1}{2}$ by means of formulas [4] :

(1.1) $$H_{2n}(x) = (-1)^n\, 2^{2n}\, n!\, L_n^{(-1/2)}(x^2).$$

(1.2) $$H_{2n+1}(x) = (-1)^n\, 2^{2n+1}\, n!\, x\, L_n^{(1/2)}(x^2).$$

Dickinson and Warsi [2] have generalised the relations (1.1) and (1.2) between the Hermite and the Laguerre polynomials in the form :

(1.3) $$H_{2n}^{(\alpha)}(x) = (-1)^n\, 2^{2n}\, n!\, L_n^{(\alpha)}(x^2)$$

and

(1.4) $$H_{2n+1}^{(\alpha)}(x) = (-1)^n\, 2^{2n+1}\, n!\, x\, L_n^{(\alpha+1)}(x^2).$$

The object of the present paper is to derive some generating functions for generalised Hermite polynomials by means of a different operational technique.

2. **Generalised Hermite polynomials.** Following O. V. Viskov's [5] operational representation for the Laguerre polynomials :

(2.1) $$L_n^{(\alpha)}(x) = \frac{(-1)^n}{n!}\, e^x\, (x D^2 + \alpha D + D)^n\, e^{-x}\ ;\ D \equiv d/dx$$

we derive operational representations for the generalised Hermite polynomials in the form

(2.2) $$H_{2n}^{(\alpha)}(\sqrt{x}) = 2^{2n}\, e^x\, (x D^2 + \alpha D + D)^n\, e^{-x}$$

(2.3) $$H_{2n+1}^{(\alpha)}(\sqrt{x}) = 2^{n+1}\, \sqrt{x}\, e^x\, (x D^2 + \alpha D + 2 D)^n\, e^{-x}.$$

To derive the generating functions of the said polynomials we shall require the following operational formula [3] :

(2.4) $$e^{-t\,(xD^2+\alpha D+D)}\,e^{-kx}=(1-kt)^{-(\alpha+1)}\,e^{\frac{kx}{kt-1}}.$$

Now using (2.2) and (2.3) we shall derive some generating functions for the said polynomials.

At first we note

$$\sum_{n=0}^{\infty} H_{2n}^{(\alpha)}(\sqrt{x})\,\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty} 2^{2n}\,e^{x}\,(xD^2+\alpha D+D)^n\,e^{-x}\,\frac{t^n}{n!}$$

$$=e^{x}\,e^{4t\,(xD^2+\alpha D+D)}\,e^{-x}$$

$$=(1+4t)^{-(\alpha+1)}\,e^{x}\,e^{\frac{-x}{1+4t}}$$

$$=(1+4t)^{-(\alpha+1)}\,e^{\frac{4tx}{1+4t}}$$

Thus we obtain

(2.5) $$\sum_{n=0}^{\infty} H_{2n}^{(\alpha)}(x)\,\frac{t^n}{n!}=(1+4t)^{-(\alpha+1)}\,e^{\frac{4tx^2}{1+4t}}$$

Next we have

$$\sum_{n=0}^{\infty} H_{2n+1}^{(\alpha)}(\sqrt{x})\,\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty} 2^{2n+1}\,\sqrt{x}\,e^{x}\,(xD^2+\alpha D+2D)^n\,e^{-x}\,\frac{t^n}{n!}$$

$$=2e^{x}\sqrt{x}\sum_{n=0}^{\infty}\frac{(4t)^n}{n!}\,(xD^2+\alpha D+2D)^n\,e^{-x}$$

$$=2e^{x}\sqrt{x}\,e^{4t\,(xD^2+\alpha D+2D)}\,e^{-x}$$

$$=2e^{x}\sqrt{x}\,(1+4t)^{-(\alpha+2)}\,e^{\frac{-x}{1+4t}}$$

$$=2\sqrt{x}\,(1+4t)^{-(\alpha+2)}\,e^{\frac{4tx}{1+4t}}$$

Thus we derive

(2.6) $$\sum_{n=0}^{\infty} H^{(\alpha)}_{2n+k}(x)\frac{t^n}{n!} = 2x(1+4t)^{-(\alpha+2)} e^{\frac{4tx^2}{1+4t}}$$

Lastly we derive the sum

$$\sum_{n=0}^{\infty} H^{(\alpha)}_{2n+k}(x).\frac{t^n}{n!}$$

At first we put $k = 2m$. Then we have

$$\sum_{n=0}^{\infty} H^{(\alpha)}_{2n+2m}(\sqrt{x})\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} 2^{2n+2m} e^x (xD^2 + \alpha D + D)^{n+m} e^{-x}\frac{t^n}{n!}$$

$$= e^x 2^{2m}(xD^2 + \alpha D + D)^m \sum_{n=0}^{\infty}\frac{(4t)^n}{n!}(xD^2 + \alpha D + D)^n e^{-x}$$

$$= e^x 2^{2m}(xD^2 + \alpha D + D)^m e^{4t}(xD^2 + D + \alpha D)e^{-x}$$

$$= (1-4t)^{-\alpha-1} e^x 2^{2m}(xD^2 + \alpha D + D)^m e^{\frac{-x}{1+4t}}$$

Now putting $\dfrac{x}{1+4t} = z$, the right hand member becomes,

$$(1+4t)^{-(\alpha+m+1)} e^{4tz} 2^{2m} e^z (zD_z^2 + \alpha D_z D_z)^m e^{-z}$$

$$= (1+4t)^{-(\alpha+m+1)} e^{4tz} H_{2m}^{(\alpha)}(\sqrt{z}).$$

$$= (1+4t)^{-(\alpha+m+1)} e^{\frac{4tx}{1+4t}} H^{(\alpha)}_{2m}\left(\sqrt{\frac{x}{1+4t}}\right).$$

Thus we have shown that

(2.7) $$\sum_{n=0}^{\infty} H^{(\alpha)}_{2n+2m}(x)\frac{t^n}{n!}$$

$$= (1+4t)^{-(\alpha+m+1)} e^{\frac{4tx^2}{1+4t}} H^{(\alpha)}_{2m}\left(\frac{x}{\sqrt{1+4t}}\right)$$

Next we substitute $k = 2m + 1$. Then we get

$$\sum_{n=0}^{\infty} H^{(\alpha)}_{2n+2m+1} (\sqrt{x}) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} 2^{2n+2m+1} \sqrt{x}\, e^x (xD^2 + \alpha D + 2D)^{n+m} e^{-x} \frac{t^n}{n!}$$

$$= \sqrt{x}\, e^x\, 2^{2m+1} \sum_{n=0}^{\infty} (xD^2 + \alpha D + 2D)^m \frac{(4t)^n}{n!} (xD^2 + \alpha D + 2D) e^{-x}$$

$$= \sqrt{x}\, e^x\, 2^{2m+1} (xD^2 + \alpha D + 2D)^m e^{4t(xD^2 + \alpha D + 2D)} e^{-x}$$

$$= (1+4t)^{-(\alpha+2)} \sqrt{x}\, e^x\, 2^{2m+1} (xD^2 + \alpha D + 2D)^m e^{\frac{-x}{1+4t}}$$

Now putting $\frac{x}{1+4t} = z$, the right hand member becomes,

$$(1+4t)^{-(\alpha+m+3/2)} \sqrt{z}\, e^{4tz}\, 2^{2m+1} e^z (zD_z^{\,2} + \alpha D_z + 2D_z)^m e^{-z}$$

$$= (1+4t)^{-(\alpha+m+3/2)} e^{4tz} H^{(\alpha)}_{2m+1} (\sqrt{z})$$

$$= (1+4t)^{-(\alpha+m+3/2)} e^{\frac{4tx}{1+4t}} H^{(\alpha)}_{2m+1} \left(\sqrt{\frac{x}{1+4t}}\right).$$

Thus we have shown

$$(2.8) \qquad \sum_{n=0}^{\infty} H^{(\alpha)}_{2n+2m+1} (x) \frac{t^n}{n!}$$

$$= (1+4t)^{-(\alpha+m+3/2)} e^{\frac{4tx^2}{1+4t}} H^{(\alpha)}_{2m+1} \left(\frac{x}{\sqrt{1+4t}}\right).$$

Combining (2.7) and (2.8) we obtain

$$(2.9) \qquad \sum_{n=0}^{\infty} H^{(\alpha)}_{2n+k} (x) \frac{t^n}{n!}$$

$$= (1+4t)^{-(\alpha+1+k/2)} e^{\frac{4tx^2}{1+4t}} H^{(\alpha)}_{k} \left(\frac{x}{\sqrt{1+4t}}\right).$$

3. **Applications.** It is interesting to note that if we put $\alpha = -\frac{1}{2}$ in the above generating relations of generalised Hermite polynomials we may easily obtain differeent generating relations for the odd and even Hermite polynomials.

At first putting $\alpha = -\frac{1}{2}$ in (2.5) we have

$$\sum_{n=0}^{\infty} H_{2n}(x)\frac{t^n}{n!} = (1+4t)^{-\frac{1}{2}} e^{\frac{4tx^2}{1+4t}} \tag{3.1}$$

Next substituting $\alpha = -\frac{1}{2}$ we obtain from (2.6)

$$\sum_{n=0}^{\infty} H_{2n+1}(x)\frac{t^n}{n!} = 2x(1+4t)^{-3/2} e^{\frac{4tx^2}{1+4t}}. \tag{3.2}$$

Again putting $\alpha = -\frac{1}{2}$ we may easily derive from (2.7) and (2.8) the even and odd generating functions for Hermite polynomials respectively.

$$\sum_{n=0}^{\infty} H_{2n+2m}(x)\frac{t^n}{n!} \tag{3.3}$$

$$= (1+4t)^{-(m+1/2)} e^{\frac{4tx^2}{1+4t}} H_{2m}\left(\frac{x}{\sqrt{1+4t}}\right)$$

$$\sum_{n=0}^{\infty} H_{2n+2m+1}(x)\frac{t^n}{n!} \tag{3.4}$$

$$= (1+4t)^{-(m+1)} e^{\frac{4tx^2}{1+4t}} H_{2m+1}\left(\frac{x}{\sqrt{1+4t}}\right).$$

It may be remarked that (3.3) and (3.4) can be easily combined into the following single result :

$$\sum_{n=0}^{\infty} H_{2n+k}(x)\frac{t^n}{n!} \tag{3.5}$$

$$= (1+4t)^{-(k+1)/2} e^{\frac{4tx^2}{1+4t}} H_k\left(\frac{x}{\sqrt{1+4t}}\right)$$

which can be compared with the result (1.7) of W. A. Al-Salam [1].

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REFERENCES

[1] Al-Salam, W. A.—Operational derivation of some formulas for the Hermite and Laguerre polynomials, Boll Un. Mat. Ital., 18 (1963), 358-363.

[2] Dickinson, D and Warsi, S, A.—On a generalised Hermite polynomials and a problem of Carlitz, Boll, Un. Mat. Ital, 18 (1963). 256-259.

[3] Ghosh, B.—Operational derivation of some generating functions for the Laguerre polynomials. Jour. Pure Math. 3 (1983), 43-47.

[4] Szego, G.—Orthogonal polynomials, New York, 1959, p. 105.

[5] Viskov, O. V.—On an identity of L. B. Redei for Laguerre polynomials ; (in Russian) Acta. Sci. Math. (Szeged) 39 (1977), 27-28,

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Dept. of Pure Math.
Calcutta University
Calcutta — 700 019

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WEISNER METHODIC SURVEY FOR GEGENBAUER POLYNOMIALS

TAHA IBRAHIM SULTAN
(an Egyptian student)

AND

S. K. CHATTERJEA

1. Introduction : In a recent work [3] E. B. McBride has studied Gegenbauer polynomials $C_n^\nu(x)$ to obtain generating functions by following Weisner's group-theoretic method by means of a suitable interpretation to the index of the polynomial. The raising and lowering operators R and L for Gegenbauer polynomials are as follows :

$$R = (x^2-1)\,y\,\frac{\partial}{\partial x} + xy^2\,\frac{\partial}{\partial y} + 2\nu\,xy$$

$$L = (x^2-1)\,y^{-1}\frac{\partial}{\partial x} - x\,\frac{\partial}{\partial y},$$

such that

$$R\,[C_n^\nu(x)\,y^n] = (n+1)\,C_{n+1}^\nu(x)\,y^{n+1}$$

and $L\,[C_n^\nu(x)\,y^n] = -(2\nu-1+n)\;C_{n-1}^\nu(x)\,y^{n-1}.$

The operators $I \equiv y\,\dfrac{\partial}{\partial y}$, R, L satisfy the following commutator relations :

$$[I, L] = -L,\quad [I, R] = R,\quad [L, R] = -2I - 2\nu,$$

so that the operators 1, I, R, L generate a Lie group. In [4] B. Viswanathan has studied Gegenbauer function group-theoretically.

Later S. Das [2] has studied Gegenbauer polynomials $C_n^\nu(x)$ to obtain generating functions by following Weisner's group-theoretic method by means of a suitable interpretation to the parameter ν of the polynomial.

The raising and lowering operators in this connection are as follows :

$$R = xy\frac{\partial}{\partial x} + 2y^2\frac{\partial}{\partial y} + ny$$

$$L = x(x^2-1)\,y^{-1}\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y} - (n(x^2-1)+1)\,y^{-1},$$

such that

$$R\left[C_n^{\nu}(x)\,y^{\nu}\right] = 2\nu\, C_n^{\nu+1}(x)\,y^{\nu+1}$$

and $L\left[C_n^{\nu}(x)\,y^{\nu}\right] = \dfrac{(2\nu+n-1)\ (2\nu+n-2)}{4(\nu-1)}\, C_n^{\nu-1}(x)\,y^{\nu-1}$

The operators $I \equiv y\dfrac{\partial}{\partial y}$, R, L satisfy the following commutator relations

$$[I, L] = -L,\quad [I, R] = R,$$

$$[L, R] = -2(n-4)\,I + 4(n-1),$$

so that the operators 1, I, R, L generate a Lie group.

Also A. K. Chongdar [1] has studied Gegenbauer polynomials $C_n^{\nu}(x)$ to obtain generating functions by following Weisner's group-theoretic method by means of suitable interpretations to both the index and the parameter of the polynomial.

First we like to make some remarks on the works of Mc Bride, Das and Chongdar. Finally, we like to study modified Gegenbauer polynomials $C_n^{\nu-n}(x)$ for obtaining generating functions by following Weisner's group-theoretic method by means of a suitable interpretation to the index (n) of the polynomial and furthermore we like to point out that double interpretation like Chongdar is not necessary when a single interpretation to the modified Gegenbauer polynomials gives rise to the same generating functions.

2. Remarks on the works of Mc Bride, Das and Chongdar :

In [1] we notice that

$$(2.1)\qquad \rho^{-2\nu-k}\, C_k^{\nu}\left(\frac{1-x\nu}{\rho}\right) = \sum_{n=0}^{\infty} \frac{(2\nu+n)_k}{k!}\, C_n^{\nu}(x)\, y^n.$$

where

$$\rho = \sqrt{(1 - 2xy + y^2)}$$

It may be of interest to remark that the left member of (2.1) gives rise to a different relation. In fact from [3, p. 53] we have

$$e^{C}\, e^{-B} [\, C_k^{\nu} (x)\, y^k \,] = (y^2 - 2xy + 1)^{-\nu}\, C_k^{\nu} (\xi)\, \eta^k \tag{2.2}$$

where

$$\xi = (1 - xy) / \sqrt{(1 - 2\,xy + y^2)}$$

and

$$\eta = 1 / \sqrt{(1 - 2\,xy + y^2)}.$$

On the other hand we have

$$e^{C}\, e^{-B} [\, C^{\nu}_k (x)\, y^k \,]$$

$$= \sum_{n=0}^{\infty} \frac{C^n}{n\,!} \sum_{m=0}^{\infty} \frac{(-1)^m}{m\,!} B^m [\, C_k^{\nu} (x)\, y^k \,]$$

$$= \sum_{n=0}^{\infty} \frac{C^n}{n\,!} \sum_{m=0}^{\infty} \frac{(2\nu + k - 1)_m}{m\,!} C^{\nu}_{k-m} (x)\, y^{k-m}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (-2\nu - k + 1)_m}{n\,!\; m\,!} C^n [\, C^{\nu}_{k-m} (x)\, y^{k-m} \,]$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (-2\gamma - k + 1)_m (k - m + 1)_n}{n\,!\; m\,!} C^{\nu}_{k-m+n}\, y^{k+n-m}$$

$$= \sum_{n=0}^{\infty} \frac{y^n}{n\,!} \sum_{m=0}^{k} \frac{(-1)^m (-2\nu - k + 1)_m (k - m + 1)_n}{m\,!} C^{\nu}_{k+n-m} (x)\, y^{k-m} \tag{2.3}$$

$$(\because (k - m + 1)_n = 0,\ m > k,\ n > 0)$$

Equating the results (2.3) and (2.2), we get

(2.4) $$\rho^{-2\nu-k}\, C_k^{\nu}\left(\frac{1-xy}{\rho}\right)$$

$$= \sum_{n=0}^{\infty} \frac{y^n}{n\,!} \sum_{m=0}^{k} \frac{(-1)^m\,(-2\nu-k+1)_m\,(k-m+1)_n}{m\,!}\, C_{k+n-m}^{\nu}\,(x)\; y^{k-m},$$

which may be compared with (2.1).

Since $[\,B, C\,] \neq 0$, we can well apply the operator $e^{-B}\, e^{C}$ on $C_k^{\nu}\,(x)\, y^k$ in order to derive a generating relation analogous to (2.4). In fact, we have

$$e^{-B}\, e^{C}\,[\, C_k^{\nu}\,(x)\; y^k\,]$$

$$= e^{-B}\left[\,(y^2 - 2xy + 1)^{-\nu}\left(\frac{y}{\sqrt{y^2-2xy+1}}\right)^k C_k^{\nu}\left(\frac{x-y}{\sqrt{y^2-2xy+1}}\right)\right]$$

$$= e^{-B}\left[\, y^k\,(y^2 - 2xy + 1)^{-\nu-k/2}\, C_k^{\nu}\left(\frac{x-y}{\sqrt{y^2-2xy+1}}\right)\right].$$

$$\because\quad e^{-B} f(x,\, y) = f\left(\frac{xy+1}{\sqrt{y^2+2xy+1}},\ \sqrt{y^2+2xy+1}\right),$$

we obtain

(2.5) $$e^{-B}\, e^{C}\,[\, C_k^{\nu}\,(x)\; y^k\,]$$

$$= (y^2 + 2xy + 1)^{k/2}\; y^{-2\nu-k}\; C_k^{\nu}\left(\frac{-(x+y)}{\sqrt{y^2+2xy+1}}\right).$$

On the other hand from the left member, we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n\,!}\, B^n \sum_{m=0}^{\infty} \frac{C^m}{m\,!}\;[\,C_k^{\nu}\,(x)\; y^k\,]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n\,!}\, B^n \sum_{m=0}^{\infty} \frac{(k+1)_m}{m\,!}\, C_{k+m}^{\nu}\,(x)\; y^{k+m}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^n\,(k+1)_m}{n\,!\,m\,!}\, B^n\,[\, C_{k+m}^{\nu}\,(x)\; y^{k+m}\,]$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^n (k+1)_m (-2\nu - k - m + 1)_n}{n!\ m!} C^{\nu}_{k+m-n}(x)\ y^{k+m-n}$$

$$= \sum_{m=0}^{\infty} \frac{(k+1)_m\, y^m}{m!} \sum_{n=0}^{k+m} \frac{(-1)^n (-2\nu - k - m + 1)_n}{n!} C^{\nu}_{k+m-n}(x)\ y^{k-n}.$$

Equating the above results, we get

$$(y^2 + 2xy + 1)^{k/2}\ y^{-2\nu-k}\ C_k^{\nu}\left(\frac{-(x+y)}{\sqrt{y^2+2xy+1}}\right)$$

$$= \sum_{m=0}^{\infty} \frac{(k+1)_m\, y^m}{m!} \sum_{n=0}^{k+m} \frac{(-1)^n (-2\nu - k - m + 1)_n}{n!} C^{\nu}_{k+m-n}(x)\ y^{k-n},$$

which may also be compared with (2.1).

In [2] S. Das has proved the following general theorem by means of the raising operator :

Theorem : If there exists a linear generating relation of the form

$$G(x, w) = \sum_{m=0}^{\infty} \frac{a_m\, w^m}{m!} C_n^m(x),$$

then there exists the following bilateral generating relation

$$(1-w)^{-n/2}\ G\left(\frac{x}{1-w}, \frac{wz/2}{1-w}\right)$$

$$= \sum_{m=0}^{\infty} \frac{w^m}{m!} C_n^m(x)\ g_m(z),$$

where

$$g_m(z) = \sum_{k=0}^{m} a_k \binom{m}{k} (k)_{m-k} (z/2)^k.$$

It may be of interest to remark that a similar theorem can well be established by means of the lowering operator.

In fact, let

$$(2.6)\qquad F(x, w) = \sum_{m=0}^{\infty} \frac{a_m w^m}{m!} C_n^m (x).$$

Replacing w by wyz in (2.6) and operating both members of the derived equation with $\exp(wB)$, where

$$B = x(x^2-1)y^{-1}\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y} - (n(x^2-1)+1)y^{-1},$$

$$B[C_n^{\nu}(x)y^{\nu}] = \frac{(2\nu+n-1)(2\nu+n-2)}{2(\nu-1)} C_n^{\nu-1}(x)\, y^{\nu-1},$$

and

$$e^{bB} f(x, y)$$

$$= (1+2b/y)^{-1/2}\left(1+2b\,\frac{1-x^2}{y}\right)^{n/2} f\left(\frac{x}{\sqrt{1+2b\,\dfrac{1-x^2}{y}}},\ y+2b\right),$$

we obtain

$$(2.7)\qquad \begin{aligned} &(1+2w/y)^{-1/2}\left(1+2w\,\frac{1-x^2}{y}\right)^{n/2} F\left(\frac{x}{\sqrt{1+2w\,\dfrac{1-x^2}{y}}},\ wz(y+2w)\right) \\ &= \sum_{m=0}^{\infty} \frac{a_m (wz)^m}{m!} \sum_{k=0}^{\infty} \frac{\left(-\frac{w}{2}\right)^k (-2m-n+1)_{2k}}{k!\,(1-m)_k} C_n^{m-k}(x)\, y^{m-k}. \end{aligned}$$

If we put $w = 1$, we get

$$(1+2/y)^{-1/2}\left(1+2\,\frac{1-x^2}{y}\right)^{n/2} F\left(\frac{x}{\sqrt{1+2\,\dfrac{1-x^2}{y}}},\ z(y+2)\right)$$

$$= \sum_{m=0}^{\infty} \frac{a_m z^m}{m!} \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})^k (-2m-n+1)_{2k}}{k!\,(1-m)_k} C_n^{m-k}(x)\, y^{m-k}.$$

In [1] Chongdar has considered the following operators ;

$$A_1 = y\frac{\partial}{\partial y}, \quad A_2 = z\frac{\partial}{\partial z},$$

$$A_3 = (x^2-1)\, y^{-1} z\frac{\partial}{\partial x} + 2xz\frac{\partial}{\partial y} - x\, y^{-1} z,$$

$$A_4 = y\, z^{-1}\frac{\partial}{\partial x},$$

such that

$$A_3\left[\, C_n^{\nu}(x)\, y^{\nu} z^{n} \,\right] = \frac{(n+1)(2\nu+n-1)}{2(\nu-1)}\, C_{n-1}^{\nu-1}(x)\, y^{\nu-1} z^{n+1}$$

and

$$A_4\left[\, C_n^{\nu}(x)\, y^{\nu} z^{n} \,\right] = 2\nu\, C_{n-1}^{\nu+1}(x)\, y^{\nu+1} z^{n-1}.$$

But, by interpreting the index n and the parameter ν of the Gegenbauer polynomials, one can consider the following operators

$$(2.8)\quad \begin{cases} A_1 = y\dfrac{\partial}{\partial y}, \quad A_2 = z\dfrac{\partial}{\partial z}, \\ B = x(x^2-1)\, y^{-1}\dfrac{\partial}{\partial x} + 2\dfrac{\partial}{\partial y} - y^{-1}(x^2-1)\, z\dfrac{\partial}{\partial z} - y^{-1}, \\ G = xy\dfrac{\partial}{\partial x} + 2y^2\dfrac{\partial}{\partial y} + yz\dfrac{\partial}{\partial z}, \\ M = xy\, z^{-2}\dfrac{\partial}{\partial x} - yz^{-1}\dfrac{\partial}{\partial z}, \\ N = x(x^2-1)\, y^{-1} z^2\dfrac{\partial}{\partial x} + 2x^2 z^2\dfrac{\partial}{\partial y} + (x^2-1)\, y^{-1} z^3\dfrac{\partial}{\partial z}, \end{cases}$$

such that

$$(2.9)\quad \begin{cases} B[C_n^{\nu}(x)\, y^{\nu} z^{n}] = \dfrac{(2\nu+n-1)(2\nu+n-2)}{2(\nu-1)}\, C_n^{\nu-1}(x)\, y^{\nu-1} z^{n}, \\ G[C_n^{\nu}(x)\, y^{\nu} z^{n}] = 2\nu\, C_n^{\nu+1}(x)\, y^{\nu+1} z^{n}, \\ M[C_n^{\nu}(x)\, y^{\nu} z^{n}] = 2\nu\, C_{n-2}^{\nu+1}(x)\, y^{\nu+1} z^{n-2}, \\ N[C_n^{\nu}(x)\, y^{\nu} z^{\nu}] = \dfrac{(n+1)(n+2)}{2(\nu-1)}\, C_{n+2}^{\nu-1}(x)\, y^{\nu-1} z^{n+2}. \end{cases}$$

The operators A_1, A_2, B, G, M, N satisfy the following commutator relations :

$$(2.10)\quad \begin{cases} [A_1, A_2] = 0, \quad [A_1, M] = M, \quad [A_1, N] = -N, \quad [A_1, B] = -B, \\ [A_1, G] = G, \quad [A_2, B] = 0, \quad [A_2, G] = 0, \quad [A_2, M] = -2M, \\ [A_2, N] = 2N, \quad [M, N] = 4A_2 + 2, \quad [B, N] = 0, \\ [B, M] = 0, \quad [G, N] = 0, \quad [G, M] = 0, \quad [B, G] = 8A_1 + 4A_2 - 2. \end{cases}$$

Thus the operators 1, A_1, A_2, B, G, M, N generate a Lie group LG and 1, A_2, M, N and 1, A_1, A_2, B, G generate subgroups of LG.

One can notice that

$$x^2 L = -MN + 3A_2 + A_2^2 + 2$$

$$x^2 L = -BG + 4A_1^2 + 2A_1 + 4A_1A_2 + A_2^2 + A_2,$$

where

$$L = (1 - x^2)\frac{\partial^2}{\partial x^2} - 2xy\frac{\partial^2}{\partial x \partial y} - x\frac{\partial}{\partial x} + 2yz\frac{\partial^2}{\partial y \partial z} + z\frac{\partial}{\partial z} + z^2\frac{\partial^2}{\partial z^2}$$

One can calculate

$$(2.11)\quad \begin{cases} e^{bB} f(x, y, z) = (1 + 2by^{-1})^{-1/2} f(\theta^{-1/2}, y + 2b, \theta^{1/2} xz), \\ \text{where} \\ \qquad \theta = 1 - (1 + 2by^{-1})(1 - x^{-2}); \\ e^{wN} f(x, y, z) = (1 + 2wy^{-1}z^2)^{-1/2} \\ \qquad \cdot f\left(\dfrac{x}{z\sqrt{\delta}}, \dfrac{y}{x^2 - 1}\left(\dfrac{x^2}{z^2\delta} - 1\right), \delta^{-1/2}\right), \\ \text{where} \\ \qquad \delta = z^{-2}(1 - 2w(x^2 - 1)y^{-1}z^2), \\ e^{gG} f(x, y, z) = f\left(x/(\sqrt{1 - 2gy}), y/(1 - 2gy), z/(\sqrt{1 - 2gy})\right), \\ e^{mM} f(x, y, z) = f\left(\dfrac{xz}{\sqrt{z^2 - 2my}}, y, \sqrt{z^2 - 2my}\right), \\ e^{a_1 A_1} f(x, y, z) = f(x, e^{a_1} y, z), \\ e^{a_2 A_2} f(x, y, z) = f(x, y, e^{a_2} z). \end{cases}$$

Using the operators $e^{wN}\; e^{bB}\; e^{gG}\; e^{mM}$ on $C_n^{\nu}(x)\, y^{\nu} z^n$, one can obtain :

$$(2.12)\qquad \sum_{k=0}^{\infty} \frac{w^k}{k\,!} \sum_{r=0}^{\infty} \frac{b^r}{r\,!} \sum_{s=0}^{\infty} \frac{g^s}{s\,!} \sum_{p=0}^{\infty} \frac{m^p}{p\,!}.$$

$$\cdot\ (-1)^{r+k}\, (\nu)_{p+s}\, \frac{(-2\nu-2s-n+1)_{2r}\, (n-2p+1)_{2k}}{(-\nu-p-s+1)_{r+k}}.$$

$$\cdot\ (2y)^{p+s-r-k}\quad Z^{2k-2p}\quad C_{n-2p+2k}^{\nu+p+s-r-k}\ (x).$$

$$= (1+2wy^{-1}z^2)^{1/2}\quad (1+2b\eta^{-1})^{\nu-1/2}\quad (y^{-1}\eta)^{\nu}.$$

$$\cdot\ (1-2g\eta-4bg)^{-\nu-n/2}\ (z^{-2}\gamma)^{n/2}\qquad C_n^{\nu}\left(\frac{x/z\delta}{\sqrt{(1-2g\eta-4\,bg)\gamma}}\right).$$

where

$$\begin{cases} \eta = yz^{-2}\ \delta^{-1}\ (1+2\,wy^{-1}z^2) \\ \delta = z^{-2}\,\{\,1-2\,w\,(x^2-1)\ y^{-1}\ z^2\,\} \\ \gamma = z^{-2}\ \delta^{-2}\ (1-2by^{-1}\,(x^2-1)\,)-2m\,\eta-4\,bm. \end{cases}$$

Some particular cases of (2.12) are worthy of notice :

Case I :

If $w=g=m=0$, we obtain from (2.12)

$$\sum_{r=0}^{\infty} \frac{\left(\frac{-b}{2y}\right)}{r\,!}\ \frac{(-2\nu-n+1)_{2r}}{(1-\nu)_r}\, C_n^{\nu-r}\ (x)$$

$$= (1+2by^{-1})^{\nu-1/2}\,(1-2by^{-1}\,(x^2-1)\,)^{n/2}\ C_n^{\nu}\left(\frac{x}{\sqrt{1-2by^{-1}\,(x^2-1)}}\right),$$

which can be compared with the result of Das [2, formula (3.1)].

Case II :

If $w = b = m = 0$, we obtain from (2.12)

$$\sum_{s=0}^{\infty} \frac{(2gy)^s}{s\,!} (\nu)_s\, C_n^{\nu+s}(x)$$

$$= (1-2gy)^{-\nu-n/2}\, C_n^{\nu}\left(\frac{x}{\sqrt{1-2gy}}\right).$$

which can well be compared with the result of Das [2, formula (3.2)].

Case III

If $w = b = g = 0$, we obtain from (2.12)

$$\sum_{p=0}^{\infty} \frac{(2my/z^2)^p}{p\,!} (\nu)_p \quad C_{n-4p}^{\nu+p}(x)$$

$$= ((z^2 - 2my)z^2)^{n/2} \quad C_n^{\nu}\left(\frac{zx}{\sqrt{(z^2-2my)}}\right).$$

Case IV

If $b = g = m = 0$, we obtain from (2.12).

$$\sum_{k=0}^{\infty} \frac{(-w\,z^2/2\,y)^k}{k\,!} \; \frac{(n+1)_{2k}}{(-\nu+1)_k}\, C_{n+2k}^{\nu-k}(x)$$

$$= (1+2wy^{-1}z^2)^{-1/2}\,(1+2b\eta^{-1})^{\nu-1/2} \;.\; (y^{-1}\eta)^{\nu}\,(z^{-4}\delta^{-2})^{n/2}\, C_n^{\nu}(x,$$

Case V

If $b = g = 0$, we obtain from (2.12)

$$\sum_{k=0}^{\infty} \frac{(-z^2 w)^k}{k\,!} \quad \sum_{p=0}^{\infty} (2my/z^2)^p \;\; \frac{(\nu)_p}{p\,!}.$$

$$\cdot \frac{(n-2p+1)_{2k}}{(-\nu-p+1)_k} \quad C_{n-2p+k}^{\nu+p-k}(x)$$

$$= (1+2wy^{-1}z^2)^{-1/2}\,(z^{-2}(z^{-2}\delta^{-2}-2m\eta))^{n/2}\cdot(y^{-1}\eta)^{\nu}$$

$$C_\nu^n\left(\frac{x/z\delta}{\sqrt{z^{-2}\delta^{-2}-2m\eta}}\right).$$

Case VI :

If $w=g=0$, we obtain from (2.12)

$$\sum_{r=0}^{\infty}\frac{(-b/2y)^r}{r\,!}\sum_{p=0}^{\infty}(2my/z^2)^p\frac{(\nu)_p}{p\,!}\frac{(-2\nu-n+1)_{2r}}{(-\nu-p+1)_r}C_{n-2p}^{\nu+p-r}(x)$$

$$=(1+2by^{-1})^{\nu-1/2}\left[1-2by^{-1}(x^2-1)-\frac{2m}{z^2}(y+2b)\right]^{n/2}.$$

$$\cdot\, C_n^\nu\left(\frac{x}{\sqrt{(1-2by^{-1}(x^2-1)-\frac{2m}{z^2}(y+2b))}}\right)$$

Case VII :

If $w=b=0$, we obtain from (2.12)

$$\sum_{s=0}^{\infty}\frac{(2gy)^s}{s\,!}\sum_{p=0}^{\infty}\frac{(2my/z^2)^p}{p\,!}(\nu)_{p+s}\,C_{n-2p}^{\nu+p+s}(x)$$

$$=(1-2gy)^{-\nu-n/2}(1-2myz^{-2})^{-n/2}.\; C_n^{\nu}\left(\frac{x}{\sqrt{(1-2gy)(1-2myz^{-2})}}\right).$$

Case VIII :

If $g=m=0$, we obtain from (2.12)

$$\cdot\sum_{k=0}^{\infty}\frac{(-wz^2/2y)^k}{k\,!}\sum_{r=0}^{\infty}\frac{(-b/2y)^r}{r\,!}\cdot\frac{(-2\nu-n+1)_{2r}\,(n+1)_{2k}}{(1-\nu)_{r+k}}\cdot\; C_{n+2k}^{\nu-r-k}(x)$$

$$=(1+2wy^{-1}z^2)^{-1/2}\;.\;(1+2b\eta^{-1})^{\nu-1/2}.\;.\;(y^{-1}\eta)^{\nu}\;(z^{-4}\delta^{-2}(1-by^{-1}(x^2-1))^{n/2}$$

$$C_n^\nu\left(\frac{x/z\delta}{\sqrt{z^{-2}\delta^{-2}(1-2by^{-1}(x^2-1))}}\right).$$

Similar other two cases when $w = m = 0$ and $b = m = 0$ can be easily calculated from (2.12).

Furthermore the four particular cases $w = 0$, $b = 0$, $g = 0$ and $m = 0$ can be easily calculated from (2.12). For example if $w = 0$ we get

$$\sum_{r=0}^{\infty} \frac{(-b/2y)^r}{r!} \sum_{s=0}^{\infty} \frac{(2gy)^s}{s!} \sum_{p=0}^{\infty} \frac{(2my/z^2)^p}{p!} \cdot (\nu)_{p+s} \frac{(-2\nu-2s-n+1)_{2r}}{(-\nu-p-s+1)_r} \cdot C_{n-2p}^{\nu+p+s-r}(x)$$

$$= (1+2by^{-1})^{\nu-1/2} (1-2gy-4bg)^{-\nu-n/2} [1-2by^{-1}(x^2-1)-2m(y+2b)z^{-2}]^{n/2} \cdot C_n^{\nu}\left(\frac{x}{\sqrt{(1-2gy-4bg)(1-2by^{-1}(x^2-1)-2m(y+2b)z^{-2})}}\right).$$

3. Group Theoretic study of the modified Gegenbauer Polynomials $C_n^{\nu-n}(x)$:

The modified Gegenbauer Polynomials $C_n^{\nu-n}(x)$ satisfies the following differential equation

(3.1) $\quad [(1-x^2) D^2 - (2\nu - 2n + 1) x D + n(2\nu - n)] y = 0$ when $D = d/dx$.

Now, we require the following two independent recurrence relations satisfied by each element of the set $\{C_n^{\nu-n}(x)\}$:

(3.2) $\quad D\, C_n^{\nu-n}(x) = 2(\nu - n)\, C_{n-1}^{\nu-n+1}(x)$

(3.3) $\quad (1-x^2)\, D\, C_n^{\nu-n}(x) = (2\nu - 2n - 1)\, x\, C_n^{\nu-n}(x)$

$$- \frac{(n+1)(2\nu-n-1)}{2(\nu-n-1)} C_{n+1}^{\nu-n-1}(x).$$

In order to use Weisner's method we construct from (3.1) the following partial differential equation by replacing

$\dfrac{d}{dx}$ by $\dfrac{\partial}{\partial x}$, n by $y\dfrac{\partial}{\partial y}$, and $C_n^{\nu-n}(x)$ by $u(x, y)$: —

$$(3.4)\qquad \left[(1-x^2)\frac{\partial^2}{\partial x^2} + 2xy\frac{\partial^2}{\partial x\,\partial y} - y^2\frac{\partial^2}{\partial y^2} - (2\nu+1)\,x\frac{\partial}{\partial x} + (2\nu-1)\,y\frac{\partial}{\partial y}\right] \cdot u\,(x,y) = 0.$$

We observe that

$$u\,(x, y) = C_n^{\nu-n}\,(x)\,y^n$$

is a solution of (3,4).

Now, we rewrite the equation (3.4) in the form $L\,u = 0$, where

$$L = (1-x^2)\frac{\partial}{\partial x^2} + 2xy\frac{\partial^2}{\partial x\,\partial y} - y^2\frac{\partial^2}{\partial y^2} - (2\nu+1)\,x\frac{\partial}{\partial x} + (2\nu-1)\,y\frac{\partial}{\partial y}.$$

Next we seek linear differential operators which will commute with L.

For the modified Gegenbauer Polynomials, we will now actually find the first order linear differential operators B and C such that

$$(3.5)\qquad B\left[C_n^{\nu-n}\,(x)\,y^n \right] = \delta_n\,C_{n-1}^{\nu-n+1}\,(x)\cdot y^{n-1}$$

and

$$(3.6)\qquad C\left[C_n^{\nu-n}\,(x)\,y^n \right] = r_n\,C_{n+1}^{\nu-n-1}\,(x)\,y^{n+1}$$

Now, let

$$B = B_1\frac{\partial}{\partial x} + B_2\frac{\partial}{\partial y} + B_0,$$

where each $B_i\,(i = 0, 1, 2)$ is a function of x and y which is independent of n.

By using (3.2), we get

$$B\left[C_n^{\nu-n}(x)\,y^n \right] = B_1\,y^n\left(2\,(\nu-n)\,C_{n-1}^{\nu-n+1}\,(x)\right) + B_2\,n\,y^{n-1}\,C_n^{\nu-n}\,(x) + B_0\,C_n^{\nu-n}\,(x)\,y^n.$$

To make the coefficient of $C_{n-1}^{\nu-n+1}\,(x)\,y^{n-1}$ independent of x and y, we choose $B_1 = y^{-1}$

$$\therefore\; B\left[C_n^{\nu-n}(x)\,y^n \right] = 2\,(\nu-n)\,C_{n-1}^{\nu-n+1}\,(x)\,y^{n-1} + C_n^{\nu-n}\,(x)\,y^n\left[n\,B_2\,y^{-1} + B_0 \right]$$

Again to make the coefficient of $C_n^{\nu-n}(x)\, y^n$ is equal to zero,

we have

$$B_2 = 0, \; B_0 = 0.$$

$$\therefore B\left[C_n^{\nu-n}(x)\, y^n \right] = 2(\nu - n)\, C_{n-1}^{\nu-n+1}(x)\, y^{n-1}$$

where

$$B = y^{-1} \frac{\partial}{\partial x}.$$

Secondly, by using (3.3), we get

$$C\left[C_u^{\nu-n}(x)\, y^n \right] = \frac{c_1\, y^n}{(1-x^2)} \left[x(?\nu - 2n - 1)\, C_n^{\nu-n}(x) \right.$$

$$\left. - \frac{(n+1)(2\nu - n - 1)}{2(\nu - n - 1)}\, C_{n+1}^{\nu-n-1}(x) \right] + c_2\, C_n^{\nu-n}(x)\, n y^{n-1} + c_0\, C_n^{\nu-n}(x) y^n.$$

To make the coefficient of $C_{n+1}^{\nu-n-1}(x)\, y^{n+1}$ independent of x and y we choose

$$c_1 = y\,(x^2 - 1).$$

$$\therefore C\left[C_n^{\nu-n}(x)\, y^n \right] = \frac{(n+1)(2\nu - n - 1)}{2(\nu - n - 1)}\, C_{n+1}^{\nu-n-1}(x)\, y^{n+1}$$

$$+ C_n^{\nu-n}(x)\, y^n \left[n\, c_2\, y^{-1} + c_0 - xy\,(2\nu - 2n - 1) \right]$$

Again to make the coefficient of $C_n^{\nu-n}(x)\, y^n$ is equal to zero,

we have

$$c_2 = -2xy^2,$$
$$c_0 = (2\nu - 1)\, xy.$$

$$\therefore C\left[C_n^{\nu-n}(x)\, y^n \right] = \frac{(n+1)(2\nu - n - 1)}{2(\nu - n - 1)}\, C_{n+1}^{\nu-n-1}(x)\, y^{n+1}$$

where,

$$C = y\,(x^2 - 1)\frac{\partial}{\partial x} - 2xy^2 \frac{\partial}{\partial y} + (2\nu - 1)\, xy.$$

Group of operators ;

$$\text{Let } A = y\frac{\partial}{\partial y}.$$

Then the operators A, B, C satisfy the following commutator relations

$$[A.\ B] = -B$$

(3.7) $$[A, C] = C$$

$$[B, C] = -2A + 2\nu - 1$$

where

$$[A, B] = AB - BA.$$

Therefore, the operators A, B, C generate a Lie group. To prove that these operators commute with L or $\psi(x)\,L$, when $\psi(x)$ is a suitable function to be determined, and

$$L = (1-x^2)\frac{\partial^2}{\partial x^2} + 2xy\frac{\delta^2}{\partial x\partial y} - y^2\frac{\partial^2}{\partial y^2} - (2\nu+1)\,x\frac{\partial}{\partial x} + (2\nu-1)\,y\frac{\partial}{\partial y},$$

we try to express L or $\psi(x)\,L$ in terms of these operators.

To this end, we note that

$$Lu = (1-x^2)\frac{\partial^2}{\partial x^2} + 2xy\frac{\partial u}{\partial x\partial y} - y^2\frac{\partial^2 u}{\partial y^2} - (2\nu+1)\,x\frac{\partial u}{\partial x} + (2\nu-1)\,y\frac{\partial u}{\partial y}.$$

But

$$A^2u = y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y},$$

and

$$CBu = (x^2-1)\frac{\partial^2 u}{\partial x^2} - 2xy\frac{\partial^2 u}{\partial y^2} + (2\nu+1)\,x\frac{\partial u}{\partial x},$$

so that

(3.8) $$L = -CB - A^2 + 2\nu A.$$

By means of this identity and the commutator relations we note that L commutes with each of the operators A, B, C.

Now to calculate $e^{bB}f(x, y)$, we choose new variables X and Y, such that

$$BX = 1,\ BY = 0$$

$$\therefore\quad y^{-1}\frac{\partial}{\partial x}X = 1\ ;\ y^{-1}\frac{\partial}{\partial x}Y = 0$$

The subsidiary equations of $y^{-1} \frac{\partial}{\partial x} X = 1$ are

$$\frac{dx}{y^{-1}} = \frac{dy}{0} = \frac{dX}{1},$$

from which we choose the particular solution

$$X = xy.$$

Again the subsidiary equations of $y^{-1} \frac{\partial}{\partial x} \; Y = 0$ are

$$\frac{dx}{y^{-1}} = \frac{dy}{0} = \frac{dY}{0}$$

from which, are can choose the particular solution

$$Y = y.$$

Thus we get

$$x = XY^{-1}, \; y = Y.$$

In other words, we have shown that the substitution $X = xy$, and $Y = y$ will transform B into d/dX.

$$\therefore e^{bB} j(x, y) = e^{b d/dX} j(XY^{-1}, Y)$$
$$= j\left((X + b) Y^{-1}, Y\right)$$
$$= j\left(\frac{xy + b}{y}, y\right).$$

Generating function :

If $b = 1$, we get

$$e^{B} j(x, y) = j\left(\frac{xv + 1}{y}, y\right)$$

$$\therefore e^{B} [C_n^{\nu-n}(x) y^n] = y^n C_n^{\nu-n}\left(\frac{xy+1}{y}\right) \quad \ldots \text{(i)}$$

But

$$B[C_n^{\nu-n}(x) \; y^n] = 2(\nu - n) \; C_{n-1}^{\nu-n+1}(x) \cdot y^{n-1}$$

$$\therefore e^{B}[C_n^{\nu-n}(x) \, y^n] = \sum_{k=0}^{\infty} \frac{B^k}{k!}[C_n^{\nu-n}(x) \; y^n]$$

$$= \sum_{k=0}^{n} \frac{2^k (\nu - n)_k}{k!} C_{n-k}^{\nu-n+k} y^{n-k} \quad \ldots \text{(ii)}$$

By equating (i) & (ii), we obtain

$$(3.9) \qquad \sum_{k=0}^{n} \frac{\left(\frac{2}{y}\right)^k}{k\,!} (\nu - n)_k \; C_{n-k}^{\nu-n+k} (x) = C_n^{\nu-n} \left(\frac{xy+1}{y}\right),$$

which can be easily compared with the result of Chongdar [4, formula (1.3)]. In fact, chahging ν into $\nu + n$, and putting $\frac{1}{y} = t$ in (3.9), we obtain the exact form of Chongdar viz.

$$(3.9') \qquad \sum_{k=0}^{n} \frac{2^k}{k\,!} (\nu)_k \; C_{n-k}^{\nu+k} (x) \; t^n = C_n^{\nu} (x + t).$$

Thus it is clear that a double interpretation on both index and parameter of $C_n^{\nu}(x)$ like Chongdar gives rise to the same result vix. (3.9′) by using a single interpretation on the index of the modified Gegenbauer Polynomials $C_n^{\nu-u}(x)$.

Next to calculate $e^{cC} f(x, y)$, we shall seek the function ϕ such that

$$C\phi = 0.$$

$$\therefore \quad (x^2 - 1)\, y \frac{\partial \phi}{\partial x} - 2xy^2 \frac{\partial \phi}{\partial y} = -(2\nu - 1)\, xy\phi$$

For $y \neq 0$, we get

$$(x^2 - 1) \frac{\partial \phi}{\partial x} - 2xy \frac{\partial \phi}{\partial y} = -(2\nu - 1)\, x\phi.$$

The corresponding subsidiary equations are

$$\frac{dx}{(x^2 - 1)} = -\frac{dy}{2xy} = -\frac{d\phi}{(2\nu - 1)\, x\phi},$$

from which we obtain the general integral

$$\therefore \quad \sigma \left(y\,(x^2 - 1), \frac{y^{\nu-1/2}}{\phi} \right) = 0, \quad \text{where } \sigma \text{ is arbitrary.}$$

Thus we can choose the particular solution

$$\phi = y^{\nu+1/2} (x^2 - 1),$$

so that

$$\therefore \quad E = \phi^{-} C\phi$$

$$\Longrightarrow E = (x^2 - 1)\, y \frac{\partial}{\partial x} - 2xy^2 \frac{\partial}{\partial y}.$$

Now, we choose new variables X and Y such that

$$(x^2-1)\,y\,\frac{\partial Y}{\partial x}-2xy^2\,\frac{\partial Y}{\partial y}=0, \qquad \text{(i)}$$

$$(x^2-1)\,y\,\frac{\partial X}{\partial x}-2xy^2\,\frac{\partial X}{\partial y}=1 \qquad \text{(ii)}$$

The subsidiary equations of (i) are

$$\frac{dx}{(x^2-1)}=-\frac{dy}{2xy}=\frac{dY}{0},$$

from which we can chose the particular solution

$$\therefore \quad Y=y\,(x^2-1).$$

Secondly, the subsidiary equations of (ii) are

$$\frac{dx}{y\,(x^2-1)}=-\frac{dy}{2xy^2}=\frac{dX}{1},$$

from which we can choose the particular solution

$$X=\frac{x}{y\,(x^2-1)}\,.$$

Finally, we have

$$Y=y\,(x^2-1),\quad X=\frac{x}{y\,(x^2-1)}.$$

Inversely we hεve

$$x=XY,\quad y=\frac{Y}{X^2\,Y^2-1}$$

Now E is transformed into $\frac{d}{dX}$.

$$\therefore \quad e^{cC}\,f(x,y)=e^{c\phi E\phi^{-1}}\,f(x,y)$$
$$=\phi\,e^{cE}\,\phi^{-1}\,f(x,y)$$
$$=y^{\nu+1/2}\,(x^2-1)\;e^{c\,d/dX}\,(y^{-\nu-1/2}\,(x^2-1)^{-1})\,f(x,y).$$
$$=y^{\nu+1/2}\,(x^2-1)\;e^{c\,d/dX}\,(Y^{-\nu-1/2}\,(X^2\,Y^2-1)^{\nu+1/2})\,.$$
$$\cdot f\left(XY,\;\frac{Y}{X^2\,Y^2-1}\right).$$
$$y^{\nu+1/2}\,(x^2-1)\;[\;Y^{-\nu-1/2}\,(\,(X+c)^2\,Y^2-1)^{\nu+1/2}\,].$$
$$\cdot\, f\left((X+c)\;Y,\;\frac{Y}{(X+c)^2\,Y^2-1)}\right).$$

Thus we obtain

$$e^{cC} f(x, y) = (1 + 2cxy + c^2 y^2 (x^2 - 1))^{\nu - 1/2} \cdot f\left(x + cy(x^2 - 1), \frac{y}{1 + 2cxy + c^2y^2(x^2 - 1)}\right).$$

Generating function :

If $c = 1$, we get

$$e^{C} f(x, y) = (1 + 2xy + y^2 (x^2 - 1))^{\nu - 1/2} \cdot f\left(x + y(x^2 - 1), \frac{y}{1 + 2xy + y^2(x^2 - 1)}\right).$$

$$\therefore\ e^{C}\left[C_n^{\nu - n}(x)\, y^n\right]$$

$$= y^n (1 + 2xy + y^2 (x^2 - 1))^{\nu - n - 1/2} C_n^{\nu - n}(x + y(x^2 - 1)). \qquad \text{(i)}$$

On the other hand, since

$$C\left[C_n^{\nu - n}(x)\, y^n\right] = \frac{(n + 1)(2\nu - n - 1)}{2(\nu - n - 1)} C_{n+1}^{\nu - n - 1}(x)\, y^{n+1}$$

we get,

$$e^{C}\left[C_n^{\nu - n}(x)\, y^n\right] = \sum_{k=0}^{\infty} \frac{C^k}{k!}\left[C_n^{\nu - n}(x)\, y^n\right]$$

$$= \sum_{k=0}^{\infty} \frac{(n + 1)_k (n + 1 - 2\nu)_k}{k!\, 2^k (n + 1 - \nu)_k} C_{n+k}^{\nu - n - k}(x)\, y^{n+k} \qquad \text{(ii)}$$

By equating (i) and (ii), we obtain the desired generating function

$$\text{(3.10)} \quad \sum_{k=0}^{\infty} \frac{(n + 1)_k (n + 1 - 2\nu)_k}{k!\, 2^k (n + 1 - \nu)_k} C_{n+k}^{\nu - n - k}(x)\, y^k$$

$$= (1 + 2xy + y^2 (x^2 - 1))^{\nu - n - 1/2} C_n^{\nu - n}(x + y(x^2 - 1)),$$

which again can be easily compared with the result of Chongdar [4, formula (1 . 2)]. In fact, changing ν into $\nu + n$ and putting $y/2 = t$ in (3.10) we obtain the exact form of Chongdar, viz.

$$\sum_{k=0}^{\infty} \frac{(n+1)_k\,(-2\nu-n+1)_k}{k\,!\,(1-\nu)_k}\; C_{k+k}^{\nu-k}(x)\, t^k$$

$$= (1+4xt+4t^2(x^2-1))^{\nu-1/2}\quad C_n^{\nu}(x+2t(x^2-1)).$$

To calculate $e^{bB}\,e^{cC}\,j\,(x_1\,y)$, we notice that $e^{cC}\,j\,(x,y)$

$$= (1+2c\,xy+c^2y^2\,(x^2-1))^{\nu-1/2}\,j\left(x+cy\,(x^2-1),\ \left(\frac{y}{1+2cxy+c^2y^2\,(x^2-1)}\right)\right.$$

and $e^{bB}\,j\,(x,y) = j\left(\dfrac{xy+b}{y}\ ,\ y\right)$

Then we obtain

$$e^{bB}\,e^{cC}\,j\,(x,y)$$
$$= (1+2cxy\,(1+bc)+bc\,(2+bc)+c^2y^2\,(x^2-1))^{v-1/2}.$$

$$j\left(\frac{xy\,(1+2b)+b\,(1+bc)+cy^2\,(x^2-1)}{y}\ ,\ \frac{y}{1+2cxy\,(1+bc)+bc\,(2+bc)+c^2y^2\,(x^2-1)}\right)$$

Generating function :

$$If\ b=-1,\ c=1$$

$$\therefore\ e^{-B}\,e^{C}\,j\,(x,y) = (y^2\,(x^2-1))^{\nu-1/2}.f\left(-x+y\,(x^2-1),\ \frac{1}{y\,(x^2-1)}\right).$$

We thus obtain

$$e^{-B}\,e^{C}\left[\ C_n^{\nu-n}\ (x)\,y^n\ \right]$$

$$= y^{2\nu-n-1}\ (x^2-1)^{\nu-n-1/2}\ C_n^{\nu-n}\ (-x+y\,(x^2-1)).$$

But the left member of the above equation is equal to

$$\sum_{m=0}^{\infty}\frac{(-1)^m\,B^m}{m\,!}\ \sum_{k=0}^{\infty}\frac{C^k}{k\,!}\left[\ C_n^{\nu-n}\ (x)\ y^n\ \right]$$

$$= \sum_{m=0}^{\infty}\ \sum_{k=0}^{\infty}\ \frac{(n+1)_k\ (n+1-2\nu)^k}{2^k\,m\,!\ k\,!\ (n+1-\nu)_k}\ (_k-1)^m\,B^m\left[\ C_{n+k}^{\nu-n-k}\ (x)\ y^{n+k}\ \right]$$

$$= \sum_{k=0}^{\infty} \sum_{m=0}^{n+k} \frac{(-1)(2)^{m-k}(\nu-n-k)_m (n+1)_k (-2\nu+n+1)_k}{m!\, k!\, (-\nu+n+1)_k} .$$

$$C_{n+k-m}^{\nu-n-k+m}(x)\; y^{n+k-m}.$$

Thus we obtain the desired generating relation

(3.11) $$y^{2\nu-n-1}(x^2-1)^{\nu-n-1/2}\, C_n^{\nu-n}(-x+y(x^2-1))$$

$$= \sum_{k=0}^{\infty} \sum_{m=0}^{n+k} \frac{(-1)^m (2)^{m-k}(\nu-n-k)_m (n+1)_k (-2\nu+n+1)_k}{m!\, k!\, (-\nu+n+1)_k} \cdot$$

$$\cdot C_{n+k-m}^{\nu-n-k+m}(x)\; y^{n+k-m},$$

which can well be compared with the result of Chongdar [1, formula (3.3)]. In fact. changing ν into $\nu+n$ in (3.11) we get

$$\sum_{k=0}^{\infty} \sum_{m=0}^{n+k} \frac{(-1)^m 2^{m-k}(\nu-k)_m (n+1)_k (-2\nu-n+1)_k}{m!\, k!\, (1-\nu)_k}\, y^{k-m}$$

$$.C_{n+k-m}^{\nu-k+m}(x)$$

$$= [\,y^2(x^2-1)\,]^{\nu-1/2}\, C_n^{\nu}(-x+y'(x^2-1)),$$

which is Chongdar's result if one puts $a_3=1$, $a_4=-1$, $\lambda=\nu$, $z=1$ and changes y into y^{-1} in [1, formula (3.3)].

Since $[B, C] \neq 0$, we may well apply the operator $e^C e^{-B}$ to $C_n^{\nu-n} y^n(x)$ and obtain a different generating relation analogous to (3.11).

We obtain

$$e^C e^{-B}\left[\, C_n^{\nu-n}(x)\; y^n \,\right]$$

$$= y^n (1+2xy+y^2(x^2-1))^{\nu-n-1/2}\; C_n^{\nu-n}\left(\frac{-1-x\nu}{y}\right).$$

But the left member of the above equation is equal to

$$\sum_{k=0}^{\infty} \frac{C^k}{k\,!} \sum_{m=0}^{\infty} \frac{(-1)^m B^m}{m\,!} \left[C_n^{\nu-n}(x)\, y^n \right]$$

$$= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m 2^m \ \nu - n)_m}{k\,!\ m\,!} \ C^k \left[C_{n-k}^{\nu-n+k}(x)\, y^{n-k} \right]$$

$$= \sum_{k=0}^{\infty} \sum_{m=0}^{n} \frac{(-1)^m (2)^m (\nu-n)_m\, (n-m+1)_k\, (2\nu-n+k-1)_k}{k\,!\ m\,!\ 2^k (\nu-n+k-1)_k} \cdot$$

$$\cdot\, C_{n-m+k}^{\nu-n+m-k}\, y^{n-m+k}.$$

Then, by equating, we obtain the desired generating relation in the following form

$$(3.12) \qquad (1+2xy+y^2(x^2-1)^{\nu-n-1/2}\, C_n^{\nu-n}\left(\frac{-1-xy}{y}\right)$$

$$= \sum_{k=3}^{\infty} \sum_{m=0}^{n} (-1)^m \left(\frac{y}{2}\right)^{k-m} \frac{(\nu-n)\,(n-m+1)\,(-2\nu+n-m+1)_k}{m\,!\ k\,!\ (-\nu+n-m+1)_k} \cdot$$

$$\cdot\, C_{n-m+k}^{\nu-n+m-k}(x),$$

which can well be compared with the result of Chongdar [1, formula (3.4)]. In fact, changing ν into $\nu+n$ in (3.12) we get

$$(1+2xy+y^2(x^2-1))^{\nu-1/2}\, C_n^{\nu}\left(\frac{-1-xy}{y}\right)$$

$$= \sum_{k=0}^{\infty} \sum_{m=0}^{n} (-1)^m \left(\frac{y}{2}\right)^{k-m} \frac{(\nu)_m\, (n-m+1)_k\, (-2\nu-n-m+\ )_k}{m\,!\ k\,!\ (-\nu-m+1)_k}$$

$$\cdot\, C_{n-m+k}^{\nu+m-k}(x),$$

which is a correct version of Chongdar's result if one puts $a_4 = -1$, $a_3 = 1$, $\lambda = \nu$, $z = 1$ and changes y into y^{-1} in [1, formula (3.4)]. The incorrectness of Chongdar's result occurs in the co-efficients as well as in the inner sum due to the fact $(n-m+1)_k = 0$ for $m > n$.

REFERENCES

[1] Chongdar, A. K.—Group-theoretic study of certain generating functions of Gegenbauer polynomials, Jour. Orissa Math. Soc. 2 (1983), 71-85.

[2] Das, Sarama—Group-theoretic origins of certain generating functions of Gegenbauer polynomials—to appear in Pure Math. Manuscript Vol. 4 (1985).

[3] Mc Bride, E. B.—Obtaining generating functions, Springer-Verlag (1971).

[4] Viswanathan, B.—Generating functions for ultraspherical functions, Canad. Jour. Math. 20 (1968), 120-134.

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Dept. of Pure Math.
Calcutta University

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ON A TYPE OF SEMI-SYMMETRIC METRIC CONNECTION ON A RIEMANNIAN MANIFOLD

M. C. CHAKI AND SWAPAN KUMAR KAR

1. Introduction. Let (M, g) be an n-dimensional $(n > 3)$ Riemannian manifold with metric tensor g. A linear connection $\bar{\nabla}$ on (M, g) is said to be semi-symmetric [4] if its torsion tensor T satisfies the relation

$$T(X, Y) = \pi(Y) X - \pi(X) Y, \tag{1.1}$$

where π is a 1-form and X, Y are any vector fields on (M, g). If, in addition to (1.1), $\bar{\nabla}$ satisfies the condition $\bar{\nabla} g = 0$, then $\bar{\nabla}$ is said to be a semi-symmetric metric connection. This paper deals with a type of semi-symmetric metric connection $\bar{\nabla}$ whose curvature tensor K satisfies the relation

$$(\nabla_W K)(X, Y) Z = B(W) K(X, Y) Z, \tag{1.2}$$

where B is a non-zero 1-form and ∇ denotes the Levi-Civita connection. Denoting the curvature tensor of ∇ by R and the conformal curvature tensor of (M, g) by C it is shown that if (M, g) admits a semi-symmetric metric connection $\bar{\nabla}$ for which the condition (1.2) is satisfied, then in (M, g)

$$R(X, Y) \,.\, C = 0 \quad \text{but} \quad \operatorname{div} C \neq 0,$$

where $R(X, Y)$ is considered as a derivation of the tensor algebra at each point of (M, g) for tangent vectors and divergence of C is with respect to ∇.

2. A useful result. In this section we shall derive a useful result.

The conformal curvature tensor C of (M, g) is defined by

$$\begin{aligned} C(X, Y) Z = {} & R(X, Y) Z + \lambda(Y, Z) X - \lambda(X, Z) Y \\ & + g(Y, Z) LX - g(X, Z) LY \end{aligned} \tag{2.1}$$

where

$$\lambda(Y, Z) = -\frac{1}{n-2} S(Y, Z) + \frac{r}{2(n-1)(n-2)} g(Y, Z) \tag{2.2}$$

and L is a $(1-1)$ tensor field such that

$$g(LY, Z) = \lambda(Y, Z), \tag{2.3}$$

S and r being the Ricci tensor and the scalar curvature respectively of (M, g).

Since $\nabla g = 0$ we have $(\nabla_X g)(LY, Z) = 0$ from which we get in virtue of (2 3)

$$(\nabla_X \lambda)(Y, Z) = g[(\nabla_X L)(Y), Z] \tag{2.4}$$

Again, from (2.1) we have

$$\begin{aligned}(\nabla_W C)(X, Y)Z = {} & (\nabla_W R)(X, Y)Z + [(\nabla_W \lambda)(Y, Z)]X \\ & - [(\nabla_W \lambda)(X, Z)]Y + g(Y, Z)(\nabla_W L)(X) \\ & - g(X, Z)(\nabla_W L)(Y).\end{aligned} \tag{2.5}$$

In virtue of (2.2) it follows from (2.5) that

$$(\text{div } C)(X, Y)Z = (n-3)[(\nabla_Y \lambda)(X, Z) - (\nabla_X \lambda)(Y, Z)] \tag{2.6}$$

Using (2.5) we have

$$\begin{aligned}(\nabla_W C)&(X, Y)Z + (\nabla_X C)(Y, W)Z + (\nabla_Y C)(W, X)Z \\ = {} & [(\nabla_W \lambda)(Y, Z) - (\nabla_Y \lambda)(W, Z)]X \\ & + [(\nabla_X \lambda)(W, Z) - (\nabla_W \lambda)(X, Z)]Y \\ & + [(\nabla_Y \lambda)(X, Z) - (\nabla_X \lambda)(Y, Z)]W \\ & + g(Y, Z)[(\nabla_W L)(X) - (\nabla_X L)(W)] \\ & + g(X, Z)[(\nabla_W L)(Y) - (\nabla_Y L)(W)] \\ & + g(W, Z)[(\nabla_X L)(Y) - (\nabla_Y L)(X)] \\ = {} & \frac{1}{n-3}[\{(div\, C)(Y, W)Z\}X + \{(div\, C)(W, X)Z\}Y \\ & \qquad + \{(div\, C)(X, Y)Z\}W] \\ & + g(Y, Z)[(\nabla_W L)(X) - (\nabla_X L)(W)] + g(X, Z)[(\nabla_W L)(Y) \\ & \qquad - (\nabla_Y L)(W)] \\ & + g(W, Z)[(\nabla_X L)(Y) - (\nabla_Y L)(X)]. \qquad (\text{by } (2.6))\end{aligned} \tag{2.7}$$

Hence in virtue of (2.4), (2.6) and (2.7) we get

$$\begin{aligned}g[(\nabla_W C)&(X, Y)Z + (\nabla_X C)(Y, W)Z + (\nabla_Y C)(W, X)Z, U] \\ = {} & \frac{1}{n-3}[g(X, U)(div\, C)(Y, W)Z + g(Y, U)(div\, C)(W, X)Z \\ & + g(W, U)(div\, C)(X, Y)Z + g(Y, Z)(div\, C)(X, W)U \\ & + g(X, Z)(div\, C)(Y, W)U + g(W, Z)(div\, C)(Y, X)U]\end{aligned} \tag{2.8}$$

We shall use the result (2.8) in section 3.

3. A special type of semi-symmetric metric connection

In this section we consider a semi-symmetric metric connection on (M, g) whose curvature tensor K satisfies the condition

$$(\nabla_W K)(X, Y)Z = B(W)\, K(X, Y)Z, \tag{3.1}$$

where B is a non-zero 1-form.

It has been shown by one of the authors elsewhere [1] that if a semi-symmetric metric connection ∇ satisfies (3.1), then

$$(\nabla_W C)(X, Y)Z = B(W)\, C(X, Y)Z, \tag{3.2}$$

Let us suppose that $C \neq 0$.

We now define a function f on (M, g) by

$$f^2 = g(C, C) \tag{3.3}$$

where the Riemannian metric g is extended to the inner product between the tensor fields in the standard fashion [2], [3].

Using the fact that $\nabla_Y g = 0$ it follows from (3.3) that

$$2f(Yf) = 2f^2 B(Y)$$

or

$$Yf = f B(Y) \qquad \text{(because } f \neq 0). \tag{3.4}$$

From (3.4) we get

$$X(Yf) = \frac{1}{f}(Xf)(Yf) + (XB(Y))\, f.$$

Hence

$$X(Yf) - Y(Xf) = \{XB(Y) - YB(X)\}\, f.$$

Therefore

$$\begin{aligned}\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X, Y]})\, f &= \{XB(Y) - YB(X) - B([X, Y]\}\, f \\ &= \{dB(X, Y)\}\, f.\end{aligned} \tag{3.5}$$

Since the left hand side of (3.5) is zero, and $f \neq 0$, it follows from (3.5) that

$$dB(X, Y) = 0. \tag{3.6}$$

This means that the 1-form B is closed.

Now, from (3.2) we have

$$(\nabla_X\nabla_Y C)(U, V)W = \{XB(Y) + B(X)B(Y)\}\, C(U, V)W.$$

Hence

$$(R(X, Y).C)(U, V)W = \{(dB)(X, Y)\} C(U, V)W$$
$$= 0 \qquad \text{(by (3.6))}.$$

Therefore

(3.7) $$R(X, Y).C = 0.$$

Next we show that div $C \neq 0$.

If possible, let

(3.8) $$\text{div } C = 0.$$

Then from (2.8) it follows that

$$g[(\nabla_W C)(X, Y)Z, U] + g[(\nabla_X C)(Y, W)Z, U] + g[(\nabla_Y C)(W, X)Z, U] = 0.$$

Or,

(3.9) $$B(W)\,'C(X, Y, Z, U) + B(X)\,'C(Y, W, Z, U) + B(Y)\,'C(W, X, Z, U) = 0$$
(by (3.2))

where

(3.10) $$'C(X, Y, Z, U) = g(C(X, Y)Z, U),$$

Let σ be the vector field such that

(3.11) $$g(X, \sigma) = B(X)$$

for any vector field X.

Putting $W = \sigma$ in (3.9) we get

(3.12) $$B(\sigma)\,'C(X, Y, Z, U) + B(X)\,'C(X, \sigma, Z, U) + B(Y)\,'C(\sigma, X, Z, U) = 0$$

In virtue of (3.8) and (3.2) we get

$$B(C(X, Y)Z) = 0.$$

or

(3.13) $$'C(X, Y, Z, \sigma) = 0.$$

Hence in virtue of (3.13), it follows from (3.12) that

(3.14) $$B(\sigma)\,'C(X, Y, Z, U) = 0$$

Since $C \neq 0$, from (3.14) we get $B(\sigma) = 0$,

or, $I(\sigma, \sigma) = 0$. This implies that $\sigma = 0$, because g is positive definite.

Hence from (3.11) we have $B(X) = 0$. But this is contrary to the assumption that B is a non-zero 1-form. Therefore $div\, C \neq 0$.

We can therefore state the following theorem :

Theorem. If a non-conformally flat n-dimensional Riemannian manifold (M, g) $(n>3)$ admits a semi-symmetric metric connection whose curvature tensor is recurrent with respect to the Levi-Civita connection ∇, then in (M, g)

$$R\,(X, Y,).C = 0 \;\text{ but }\; div\; C \neq 0,$$

where $R(X, Y)$ is considered as a derivation of the tensor algebra at each point of (M, g) for its tangent vectors and divergence of C is with respect to ∇.

REFERENCES

[1] Chaki M. C. and Kumar Gouri — On a type of semi-symmetric connection on a Riemannian manifold — An. Stiint Univ. 'Al. I. Cuza', Iasi, Sect I a. Mat (N. S.). (To appear)

[2] Kobayashi S. and Nomizu K. — Foundations of Differential Geometry Vol 1, Interscience Publishers, New York, 1933, 157.

[3] Nomizu K. and Ozeki H. — A theorem on curvature tensor fields — Proc. Nat. Acad. Sci. U. S. A. 48 (1962), 206-207.

[4] Yano K. — On semi-symmetric metric connection — Rev. Roum. Math. Pures et Appl. 15 (1970), 1579—1581.

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Dept. of Pure Math.
Calcutta University

and

Dept. of Mathematics
Rammohan College
Calcutta

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ON AN EXTENSION OF THE THEOREM OF V. A. AMBARZUMYAN

N. K. CHAKRAVARTY
&
SUDIP KUMAR ACHARYYA

Introduction :

In a paper published in 1929, V. A. Ambarzumyan [1] proved the theorem that if $\{\lambda_n\}$, n = 0, 1, 2, ..., be the eigenvalues of the operator

$y'' + (\lambda - q)\, y = 0,\ 0 \leqq x \leqq \pi,\ y'(0) = y'(\pi) = 0,$

q (x) a real valued function of x continuous in $[0, \pi]$, and if $\lambda_n = n^2$, then $q = 0$. This theorem is considered as a first step towards the solution of the inverse problem associated with the Sturm—Liouville operator.

In the present paper we propose to extend this theorem to the matrix differential system

(1.1) $L\phi = \lambda\phi$

where $L \equiv \begin{pmatrix} -D^2+p & r \\ r & -D^2+q \end{pmatrix}$, $D \equiv d/dx$, p, q, r are real valued functions of x such that p, q, r are integrable over $[0, \pi]$ and $\phi = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \equiv \begin{pmatrix} u \\ v \end{pmatrix}$.

The boundary conditions to be satisfied by the solutions ϕ of (1.1) are

$$(1.2)\quad \left.\begin{array}{l} u'(0) = v'(0) = 0 \\ u'(\pi) = v'(\phi) = 0 \end{array}\right\},$$

the 'Neumann boundary conditions' at $x=0$ and $x=\pi$ respectively. The problem (1.1) along with (1.2) may be called the 'Neumann boundary value problem'.

When $p = q = r = 0$, the system (1.1) reduces to

(1.3) $$D^2 \phi + \lambda \phi = 0,$$

where $\phi = \begin{pmatrix} u\ (x) \\ v\ (x) \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$.

When the solutions of (1.3) satisfy the boundary conditions (1.2) we obtain the 'Fourier problem' corresponding to the 'Neumann boundary value problem' (1.1) and (1.2).

It is easy to verify that the eigenvalues for the Fourier problem are precisely given by the set $\{n^2\}$, $n = 0, 1, 2, 3, \ldots$.

It is noted that equations of the form

$$Y'' + \lambda^2 Y = [v(x) + 6x^{-2}P]\, Y,\ 0 < x < \infty$$

where $V = \| v_{jk}(x) \|_1^2$ is a Hermitian matrix satisfying

$$\int_0^\infty x^{1+\theta} \ | V(x) |\, dx < \infty,\ -\epsilon < \theta < \epsilon,\ 0 < \epsilon < 1.$$

and $P = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ arises from the Schrödinger equation for a deuteron (in ground state) if tensor interaction forces are taken into account [Agranovich and Marchenko—The inverse problem in Scattering theory, Gordon Breach, NY, 1963, P.7].

2. Asymptotic estimates:

Let the solutions $\begin{pmatrix} u \\ v \end{pmatrix}$ of (1.1) satisfy the general boundary conditions at $x=0$ and $x=\pi$, viz.,

(2.1) $a_{j1}\, u\,(0) + a_{j2}\, u'\,(0) + a_{j3}\, v\,(0) + a_{j4}\, v'\,(0) = 0$

(2.2) $b_{j1}\, u\,(\pi) + b_{j2}\, u'\,(\pi) + b_{j3}\, v\,(\pi) + b_{j4}\, v'\,(\pi) = 0.$

$j = 1, 2$, where a_{ij}, b_{ij} are real constants (independent of λ) such that

i) $\text{rank}\,(a_{ij}) = \text{rank}\,(b_{ij}) = 2$;

ii) $a_{j1}\, a_{k2} + a_{j3}\, a_{k4} = 0,\ j, k = 1, 2$;

iii) $b_{11}\, b_{22} - b_{12}\, b_{21} + b_{13}\, b_{24} - b_{14}\, b_{23} = 0$;

iv) $b_{j2}\, a_{k1} + b_{j4}\, a_{k3} = 0,\ j = 1,2$;

v) $b_{j2}\, a_{12} + b_{j4}\, a_{14} \neq 0,\ \ b_{j1}\, a_{k2} + b_{j3}\, a_{k4} = 0$; $j = 1, 2$.

Then the system (1.1) together with the boundary conditions (2.1), (2.2) and the conditions (j)—(iv) determines a self adioint eigenvalue problem in $[0, \pi]$.

Let $\{\lambda_n\}$, where $\lim_{n \to \infty} \lambda_n = \infty$, be the eigenvalues of the system (1.1) with the boundary conditions (2.1) and (2.2). Then to solve the present problem we exploit the analysis of Levitan and Gasymov (3, Appendix I, II) as follows.

Let A be the matrix $A = \begin{pmatrix} a_{12} & a_{14} \\ a_{11} & a_{13} \end{pmatrix}$ and let the ith row and the jth column of any matrix M be represented, respectively, by M_{i*} or $(M)_{i*}$ and M_{*j} or $(M)_{*j}$.

Put $X(x,t) \equiv \begin{pmatrix} X_1(x,t) \\ X_2(x,t) \end{pmatrix}$, $X(0,0)=0$, and $Y(x,t) \equiv \begin{pmatrix} Y_1(x,t) \\ Y_2(x,t) \end{pmatrix}$,

$Y(0,0)=0$, where X, Y have absolutely continuous partial derivatives with respect to x and t.

Then a necessary and sufficient condition that the vector

$$\phi(x,\lambda) \equiv \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \tag{2.3}$$

$$= \begin{pmatrix} a_{12} \cos\sqrt{\lambda}\ x + a_{11} \sin\sqrt{\lambda}\ x \\ a_{14} \cos\sqrt{\lambda}\ x + a_{13} \sin\sqrt{\lambda}\ x \end{pmatrix} + \int_0^x \begin{pmatrix} X(x,t)\, S(t) \\ Y(x,t)\, S(t) \end{pmatrix} dt$$

where

$S(x) = \begin{pmatrix} \cos\sqrt{\lambda}\ x \\ \sin\sqrt{\lambda}\ x \end{pmatrix}$ and $X(x,t)S(t) = X_1(x,t) \cos\sqrt{\lambda}\ t + X_2(x,t) \sin\sqrt{\lambda}\ t$, with a similar meaning for $Y(x,t)\, S(t)$, is a solution of the given differential system (1.1) with boundary codditions (2.1) at $x=0$, is that all the conditions (3.6)—(3.9) of Ray Paladhi (4, Theorem 1, P. 172—175) are satisfied. One of the conditions explicitly required in our discussion is the following :

(2.4) $X'(x,x) = 1/2\ F_{*1}(x)$, $Y'(x,x) = 1/2\ F_{*2}(x)$,

where

$$F = \begin{pmatrix} a_{12}\, p + a_{14}\, r & a_{12}\, r + a_{14}\, q \\ a_{11}\, p + a_{13}\, r & a_{11}\, r + a_{13}\, q \end{pmatrix}.$$

Differentiation of (2.3) yields

$$\begin{pmatrix} u'(x) \\ v'(x) \end{pmatrix} = \sqrt{\lambda} \begin{pmatrix} a_{11} \cos\sqrt{\lambda}\ x - a_{12} \sin\sqrt{\lambda}\ x \\ a_{13} \cos\sqrt{\lambda}\ x - a_{14} \sin\sqrt{\lambda}\ x \end{pmatrix} + \begin{pmatrix} X(x,x)\, S(x) \\ Y(x,x)\, S(x) \end{pmatrix} \tag{2.5}$$

$$+ \int_0^x \begin{pmatrix} \dfrac{\partial}{\partial x} X(x,t)\, S(t) \\ \dfrac{\partial}{\partial x} Y(x,t)\, S(t) \end{pmatrix} dt.$$

When $\lambda = \lambda_n$ is an eigenvalue for the problem (1.1) with (2.1) and (2.2) so that $\phi(x,\lambda_n)$ may now represent the corresponding eiganvector, it follows from (2.2) on substitution for $\begin{pmatrix} u \\ v \end{pmatrix}$, $\begin{pmatrix} u' \\ v' \end{pmatrix}$ as given by (2.3) and (2.5), utilization of the relation (iv) satisfied by a_{ij}, b_{ij} and subsequent reductions, that for $j=1,2$,

$$(2.6)\quad \left[(B_j)\underset{2*}{\top} A\underset{1*}{\top} + (B_j)\underset{1*}{\top} \Omega_{*1}(\pi,\pi)\right]\cos\sqrt{\lambda_n}\,\pi$$

$$+\left[(B_j)\underset{2*}{\top} A\underset{2*}{\top} - \sqrt{\lambda_n}\,(B_j)\underset{1*}{\top} A\underset{1*}{\top} + (B_j)\underset{1*}{\top}\Omega_{*2}(\pi,\pi)\right]\sin\sqrt{\lambda_n}\pi$$

$$+\int_0^{\pi}\left\{(B_j)\underset{2*}{\top}\Omega_{*1}(\pi,t) + \frac{\partial}{\partial x}(B_j)\underset{1*}{\top}\Omega_{*1}(x,t)\Big|_{x=\pi}\right\}\cos\sqrt{\lambda_n}\,t\,dt$$

$$+\int_0^{\pi}\left\{(B_j)\underset{2*}{\mathrm{T}}\Omega_{*2}(\pi,t) + \frac{\partial}{\partial x}(B_j)\underset{1*}{\mathrm{T}}\Omega_{*2}(x,t)\Big|_{x=\pi}\right\}\sin\sqrt{\lambda_n}\,t\,dt=0,$$

where $B_j=\begin{pmatrix} b_{j2} & b_{j4}\\ b_{j1} & b_{j3}\end{pmatrix}$, $\Omega\,(x,t)=\begin{pmatrix} X_1(x,t) & X_2(x,t)\\ Y_1(x,t) & Y_2(x,t)\end{pmatrix}$

and A, A_{i*}, A_{*j}, etc. are as defined before. $\alpha\beta$ as usual stands for

$\alpha_1\,\alpha_2+\beta_1\,\beta_2$ where $\alpha=\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix}, \beta=\begin{pmatrix}\alpha_2\\ \beta_2\end{pmatrix}$.

From (2.6), as $n\to\infty$

$$(2.7)\qquad \sin\sqrt{\lambda_n}\;\pi+o\,(\lambda_n^{-\frac{1}{2}})=0$$

Also by adopting the analysis of Titchmarsh (5, P. 19), it is easy to deduce that $\lambda_n\sim n^2$. as n tends to infinity [see for example Chakravarty—Q. J. M 19(74), 1968, P. 216)]. So that a first approximation of λ_n is given by

$\sqrt{\lambda_n}=n+o(1/n)$, as n tends to infinity.

Put

$\sqrt{\lambda_n}=n+a_j/n\;+\Upsilon_n/n$, $j=1,2$, where a_j are constants independent of n and γ_n tends to zero as n tends to infinity, implying that

$$\sqrt{\lambda_n}\sim n+a_j/n.$$

Then

$$(2.8)\qquad \sin\sqrt{\lambda_n}\,\pi=(-1)^n\pi\,(a_j+\Upsilon_n)/n+o(n^{-3})$$
$$\text{and } \cos\sqrt{\lambda_n}\,\pi=(-1)^n(1+o(n^{-2}).$$

Also by the Riemann—Lebesgue theorem, the integral terms in (2.6) vanish as n and therefore λ_n tends to infinity. Hence from (2 6) by using the relations (2.8), we have

$$a_j=\left((B_j)\underset{2*}{\mathrm{T}} A\underset{1*}{\mathrm{T}} + (B_j)\underset{1*}{\mathrm{T}}\Omega_{*1}(\pi,\pi)\right)/[\pi(B_j)\underset{1*}{\mathrm{T}} A\underset{1*}{\mathrm{T}}]$$

$\left((B_j)\underset{1*}{\mathrm{T}} A\underset{1*}{\mathrm{T}} \neq 0\text{, by the condition (v) on } a_{ij}, b_{ij}.\right)$

Since p, q, r are integrable over $[0,\pi]$ (or in particular p,q,r are continuous over $[0,\pi]$), it follows by using the relation (2.4) that

$$(2.9)\quad a_j=\left[(B_j)^T_{2*}A^T_{1*}+1/2\int_0^\pi (B_j)^T_{1*}F^T_{1*}(t)dt)\right]\Big/\left[(\pi\ (B_j)^T_{1*}A^T_{1*}\right],\ j=1,2.$$

The vector with two components a_1, a_2 so obtained may be called the 'boundary characteristic vector' of the given problem. In particular, a_1 may be equal to a_2.

3. Solution of the problem :

If, $(B_j)^T_{2*}A^T_{1*}=0$, and if the elements of $(B_j)^T_{1*}$ as well as the constants a_{12}, a_{14} assume values independent of each other, it follows from (2.9) that

$$(3.1)\quad \int_0^\pi p\,dx=\int_0^\pi q\,dx=\int_0^\pi r\,dx=0.$$

when $a_j=0$.

Let the eigenvalues for the Neumann boundary value problem be given by $\{n^2\}$, n $=0,1,2,\ldots$. Then the vector (a_1,a_2) is null. Also for the Neumann boundary conditions, the requirements relating to $(B_j)^T$, A^T, etc as stated above are satisfied. Therefore in this case (3.1) holds.

Now let c_n be the Fourier coefficient of a vector $f=\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, such that $f\in C^1(I), I=[0,\pi]$, and f' is absolutely continuous over I, If $f'(0)=f'(\pi)=0$, then from Chakravarty and Sen Gupta (2, formula (3.2) P.23),

$$(3.2)\quad D(f,f)=D(f)\geqq\sum_{n=0}^{\infty}\lambda_n c_n^2\geqq\lambda_0\sum_{n=0}^{\infty}c_n^2=\lambda_0\ \|f\|^2$$

where

$$D(f)=\int_0^\pi\left\{|f'|^2+f^T Pf\right\}dx,\ P=\begin{pmatrix} p & r \\ r & q\end{pmatrix},\ \lambda_n\geqq\lambda_0\geqq 0.$$

The equality in (3.2) holds if and only if f is an eigenvector corresponding to the eigenvalue λ_0 for the Neumann boundary value problem over $[0,\pi]$ and now

$$(3.3)\quad \lambda_0=\min\left(D(f,f)/\|f\|^2\right)$$

the minimum being taken over all $f \neq 0 \in D$, satisfying the Neumann boundary conditions at $x=0$ and $x=\pi$, where D is the set of all complex-valued vector functions continuous in $[0, \pi]$ and having piecewise derivatives in the same interval.

Let $f_0 = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$ be an arbitrarily chosen non-null constant vector.

f_0 satisfies the Neumann boundary conditions at $x=0$ and $x=\pi$. Also by utilizing (3.1), it follows that $D(f_0)=0$. Hence from (3.3), f_0 is the eigenvector corresponding to the minimum eigenvalue '0' of the sequence $\{n^2\}$, n=0, 1, 2, 3, ... Therefore from (1.1)

$$pC_1 + rC_2 = 0, \qquad rC_1 + qC_2 = 0,$$

leading to $p=q=r=0$ almost everywhere in $[0, \pi]$.

We thus obtain the following theorem.

Theorem : A necessary and sufficient condition that the system (1.1) with the boundary conditions (1.2) reduces to the corresponding Fourier problem (i.e., the system (1.3) with boundary conditions (1.2)) is that the eigenvalues of the given system are characterised by $\{n^2\}$, n=0, 1, 2, 3, ...

Remark

An elaborate and revised version of the present paper is due to appear in the Proceedings of the Royal Society of Edinburgh, where some of the shortcomings and ambiguities have been corrected.

REFERENCES

[1] Ambarzumyan V. A.—Uber eine Frage der Eigenvert theorie, Z. Physic 53 (1929) 690-695.

[2] Chakrabarty N. K. and Sengupta P. K.—On the distribution of the eigenvalues of a matrix differential operator, Jour. Ind. Inst. Sc. 61 (B) (1979) 19-42.

[3] Chakravarty N. K.—Some problems in eigenfunction expansions (I), Q. J. Math. (Oxford), 1965, 16, 135—150.

[4] Chakravarty N. K.—Ibid (II), Q. J. Math. (Oxford) 19(74), 1968.

[5] Levitan B. M. and Gasymov M. G.—Determination of a differential equation by two of its spectra, Uspekhi Mat. Nauk. 19 (2) (116) (1964) 3—63.

[6] Ray Paladhi Basudeb—The Inverse problem associated with a pair of second order differential equations, Proc. Lond. Math. Soc. (3) XLIII (1981) 169—192.

[7] Titchmarsh E. C.—Eigenfunction expansions associated with second order differential equations (Part I, 2nd Edition), Oxford Univ. Press (1962).

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Dept. of Pure Mathematics
University of Calcutta.

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ON THE RIESZ SUMMABILITY OF A FOURIER TYPE EXPANSION OF A TWO COMPONENT FUNCTION

N. K Chakravarty

Abstract

A theorem on Riesz Means of order $l \geqslant 0$ for a Fourier type series or the F—series of a two component function $(f_1, f_2)^T$ differentiated p times, is obtained under a set of conditions analogous to that of Fe'jer—Lebesgue in ordinary Fourier Series.

1. Introduction : The theorem.

Cousider the differential system

(1.1)
$$d^2u/dx^2+\lambda u=0$$
$$d^2v/dx^2+\lambda v=0$$

in the interval $[0,\pi]$, where λ is the eigenvalue parameter.

Let the solutions $(u, v)^T$ of (1.1) satisfy the following boundary conditions :

At $x=0$,

(1.2) $a_{j1}\,u(0)+a_{j2}\,u'(0)+a_{j3}\,v(0)+a_{j4}\,v'(0)=0,\ j=1,2,$ where

i) rank $(a_{ij})=2,\ i=1,2,\ j=1,2,3,4$;

ii) $a_{j1}\,a_{k2}+a_{j3}\,a_{k4}=0,\ j,k=1,2$;

iii) $(a_{jn}, a_{jm}) \neq (0,0)$, when $j=1,\ n=1,\ m=3$ and when $j=2,\ n=2,\ m=4$;

and at $x=\pi$,

(1.3) $b_{j1}\,u(\pi)+b_{j2}\,u'(\pi)+b_{j3}\,v(\pi)+b_{j4}\,v'(\pi)=0,\ j=1,2,$

iv) $b_{11}\,b_{22}-b_{12}\,b_{21}+b_{13}\,b_{24}-b_{14}\,b_{23}=0$;

v) rank $(b_{ij})=2,\ i=1,2,\ j=1,2,3,4.$

a_{ij}, b_{ij} are real valued constants independent of the parameter λ.

The eigenvalue problem associated with the system (1.1) along with the boundary conditions (1.2) and (1.3) over the function space $L_2(0, \pi)$ is well-known. (See e. g. Chakravarty [2], p.135–150). The problem is self-adjoint by the conditions (ii), (iv) on a_{ij}, b_{ij}. We call this problem, the E—problem.

Let $f(x)=(f_1,f_2)^T$ possess continuous derivatives upto the order two ; or more generally, let f (x) be the integral of an absolutely continuous function on $(0, \pi)$. If f (x) satisfy the boundary conditions (1.2), (1.3), respectively, at $x=0$, $x=\pi$, then f (x) admits of the eigenfunction expansion

$$(1.4) \qquad f(x) = \sum_{0}^{\infty} C_n \Psi_n(x), \; 0 \leqslant x \leqslant \pi,$$

where $C_n = \int_0^\pi (f, \Psi_n)\, dt$, are the Fourier coefficients of f ; Ψ_n (x) are the normalized eigenvectors corresponding to the eigenvalue λ_n of the E—problem. The series is uniformly and absolutely convergent for $0 \leqslant x \leqslant \pi$.

Let a_{ij}, b_{ij} satisfy additional conditions

(1.5) $b_{km_1} a_{jm_2} + b_{kn_1} a_{jn_2} = \alpha_{kj}$, $k, j = 1, 2$, (b_{ij} not all zero), where $\alpha_{kj}=0$, if (m_1, m_2, n_1, n_2) are the arrangements (1, 2, 3, 4), (2, 1, 4, 3) of (1, 2, 3, 4), $\alpha_{kj} \neq 0$ for other arrangements. In view of the conditions (1.2)—(ii), (iii) on a_{ij}, it follows from (1.5) that $(b_{jn}, b_{jm}) \neq (0,0)$, when $j=1$, $n=1$, $m=3$, and when $j=2$, $n=2$, $m=4$.

Put $D_j = \frac{1}{2} \pi^{\frac{3}{2}} D_o^{-\frac{1}{2}} \mid a_j \mid^2, j=1, 2,$

and $D_o = \frac{3}{8}\pi^3 \mid a_1 \mid^2 \mid a_2 \mid^2 \mid a_1 - a_2 \mid^2 > 0$

where $a_j = (a_{j1}, a_{j2}, a_{j3}, a_{j4})$ with usual norm $\mid a_j \mid^2$ and inner product (a_1, a_2).

If

$$(1.6) \quad C_j\ (x,n) = \begin{pmatrix} a_{j2} \cos nx + a_{j1} \sin nx \\ a_{j4} \cos nx + a_{j3} \sin nx \end{pmatrix}.$$

where n is a positive integer, then it can be easily verified that

$$(1.7) \quad \Psi_n(x) = \left\{ D_1 C_2 (x, n) - D_2 C_1 (x, n) \right\} / \pi^{\frac{1}{2}}$$

is a normalized eigenvector corresponding to the eigenvalue n^2 associated with the E-problem. (Compare Acharyya [1], where eigenvectors in this form occur in a different context.)

Definition. The series on the right hand side of (1.4) where Ψ_n (x) has the explicit representation (1.7) is defined as the "Fourier type series corresponding to the two component function f (x)", or shortly, the F-series of our problem.

Evidently, the F-series is an eigenfunction expansion of a two component function f (x) (with none of its components a classical Fourier series) when f satisfies certain

conditions including boundary conditions as above stated. F-series as an eigenfunction expansion in used only for mean convergence considerations needed in the context.

It may be noted that the choice of $\Psi_n(x)$ need not be restricted to the form (1.7). In fact, if

$K_j(x, n) = \begin{pmatrix} A_j \cos nx + B_j \sin nx \\ C_j \cos nx + D_j \sin nx \end{pmatrix}$ and A_j, B_j, C_j, D_j are suitably restricted, it is possible to choose a linear combination of $K_1(x, n)$, $K_2(x, n)$ and a normalizing constant, not so simple as that in (1.7), as the normalized eigenvector for the E-problem.

The summability problems (including Riesz summability) for the Fourier series of scalar functions have been extensively studied ; but no such problems involving the F-series appear to have been taken up. The standard methods available for the single component functions cannot also be readily extended to hold in the present case.

Levitan and Sargsyan ([5] p. 54—57) obatined for the ordinary Fourier series $F\{f(x)\}$ of $f(x)$ a theorem on the summability of $F'\{f(x)\}$ at $x=x_0$ by Riesz means of order one to $f'(x_0)$, when (i) $f'(x)$ exists and is continuous at x_0 and (ii) $f'(x)$ is integrable in the neighbourhood of x_0. Their method is considerably different from the available ones. They use in their investigation the Tauberian theorem :

Theorem A. Let $\sigma(v)$ be a function of bounded variation in every finite interval, such that

$$\mathop{V}_{\mu}^{\mu+1} \sigma(\nu) = o(|\mu|^r), \; r>0, \quad \int_{-\infty}^{\infty} h(\nu)\, d\sigma(\nu) = 0, \text{ where}$$

$$h(\nu) = \frac{1}{2\pi} \int_{-\epsilon}^{\epsilon} K_\epsilon(t) \exp(-i\nu t)\, dt,$$ K_ϵ is an arbitrary function having bounded $(r+2)$ th derivative and K_ϵ vanishes outside $(-\epsilon, \epsilon)$. Then for all $s \geqslant 0$,

$$\int_{-\infty}^{\infty} (1-\nu^2/\mu^2)^s \, d\sigma = o(|\mu|^{r-s}),$$ as μ tends to infinity, the passage to the limit being uniform. (See Levitan and Sargsyan ([5], p. 85, Appendix))

In the present note we investigate the problem involving Riesz means of order $l \geqslant 0$ for a p-times differentiated F-series under a set of conditions satisfied by $f(x) \equiv (f_1, f_2)^T$ analogous to those of Fe'jer—Lebesgue for the classical Fourier Series. We note

that a series $(\Sigma a_n, \Sigma b_n)^T$ is summable (R, λ, k) if Σa_n, Σb_n are so in the usual sense. (See Chandrasekharan and Minakshisundaram [4]). The theorem to be proved is stated as follows.

Theorem 1. 1. Let f(x) be a two component function, p times differentiable on $(0, \pi)$, such that

$$(1.8) \qquad \int_0^t | f^p (x+u)-f^p(x) |^r \, du = o(t^r), \ r \geqslant 1,$$

as t tends to zero, where $x \in (x_0-\delta, x_0+\delta)$, x_0 fixed and $\delta > 0$.

Then

$$(1.8a) \ (i) \ \lim_{\mu \to \infty} \int_0^{\mu} (1-\nu^2/\mu^2)^l \, d_\nu S^{(p)}(x, \nu) = f^{(p)}(x) + o(\mu^{p-l}) \, ;$$

the result holds uniformly for $x_0-\delta \leqslant x \leqslant x_0+\delta$, $\delta > 0$; $S(x, \mu)$ is given by

$$(1.9) \qquad S(x,\mu) = \sum_{k<\mu} C_k \Psi_k (x)$$

ii) The p-times differentiated F-series is summable at x_0 by the Riesz means of order l to $f^{(p)}(x_0)$, $l \geqslant p \geqslant 0$, the result holding uniformly for $x_0-\delta \leq x \leq x_0+\delta$, $\delta > 0$.

iii) In particular, if $f^{(p)}(x) \in L_r(0, \pi)$, $r \geqslant 1$, the the result is valid almost everywhere in $(0, \pi)$.

The second part follows from the first part from definition. The third part is an immediate consequence of the second partand the Minkowsky inequality, since (1.8) holds for almost all x, if $f^{(p)} \in L_r(0, \pi)$, $r \geqslant 1$. (See Zygmund [7], p 237). It is therefore enough to establish the first part of the theorem which we do in the following. We follow Levitan and Sargsyan [3] indicating steps but emphasizing the parts where we considerably differ.

2. **Proof of the theorem.** Let $\{f_n(x)\}$ be a sequence of vectors defined on $(0, \pi)$ satisfying the conditions of validity of the expansion formula (1.4) when $\Psi_n(x)$ is given by (1.7). Then

$$(2.1) \quad f_n(x) = \sum_{k=0}^{\infty} C_k^{(n)} \Psi_k(x), \quad C_k^{(n)} = \int_0^{\pi} (f_n, \Psi_k) \, dt.$$

the series being absolutely and uniformly convergent for $0 \leqslant x \leqslant \pi$.

Let $\{f_n(x)\}$ converge to f (x) in the norm of L $(0, \pi)$.

As in Levitan and Sargsyan ([5], p. 20-22) let $g_\epsilon(x)$ be a scalar function, satisfying

i) $g_\epsilon(x) = g_\epsilon(-x)$, (ii) $g_\epsilon(x) = 0$ for $|x| \geqslant \epsilon$ and (iii) $g_\epsilon(x)$ has a bounded second derivative. If $\phi_\epsilon(\mu$, is the Fourier cosine transform of $g_\epsilon(x)$:

(2.2) $\phi_\epsilon(\mu)=\int_0^\epsilon g_\epsilon(u)\cos\mu u\,du,$

then $\phi(\mu)$ is even and

(2.3) $\phi(\mu)=o(1/\mu^2)$, as μ tends to infinity, by integration by parts.

It easily follows from (1.6) and (1,7) that

(2.4) $\frac{1}{2}[\Psi_n(x+t)+\Psi_n(x-t)]=\Psi_n(x)\cos nt$

Then from (2.1)—(2.4),

(2.5) $\int_0^\epsilon [f_n(x+t)+f_n(x-t)]\,g_\epsilon(t)\,dt=\int_{-\infty}^{\infty}\phi_\epsilon(\mu)\,d_\mu S_n(x,\mu)$

where

$$S_n(x,\mu)=\sum_{k<\mu} C_k^{(n)}\Psi_k(x),\ n=1,2,\ldots$$

The mean convergence consideration permits us to pass on to the limit under the sign of integration on the left hand side of (2.5) and the same operation is permissible on the right hand side of (2.5) in view of (2.3). Therefore from (2.5)

(2.6) $\int_{-\infty}^{\infty}\phi_\epsilon(\mu)\,d_\mu S(x,\mu)=\int_0^\epsilon [f(x+t)+f(x-t)]\,g_\epsilon(t)\,dt$

where S (x, μ) is given by (1.9).

In view of the finiteness of $S(x,\mu)$ and the relation (2.3), it is possible to differentiate the left hand side of (2.6) with respect to x under the sign of integration and the same process is obviously applicable to the right hand side. Hence from (2.6),

(2.7) $\int_{-\infty}^{\infty}\phi_\epsilon(\mu)\,d_\mu S^{(p)}(x,\mu)=\int_0^\epsilon [f^{(p)}(x+t)+f^{(p)}(x-t)]\,g_\epsilon(t)\,dt$

where $x\in(x_0-\frac{1}{2}\delta,\ x_0+\frac{1}{2}\delta)$ and the superscript p indicates the pth derivative of the concerned function.

Put

(2.8) $\alpha(x,\mu)=(\alpha_1,\alpha_2)^T=\int_0^\epsilon [f^{(p)}(x+t)+f^{(p)}(x-t)]\cos\mu t\,dt$

Then applying the Parseval theorem for ordinary Fourier Transform to each component of $\alpha(x,\mu)$ and (2.2), it follows that

$$(2.9)\qquad \int_{-\infty}^{\infty} \phi_{\epsilon}(\mu)\, d_{\mu}[S^{(p)}(x,\mu) - 1/\pi \int_{0}^{\mu} \alpha(x,\nu)\, d\nu] = 0$$

since $\phi_{\epsilon}(\mu)$ and $\alpha(x,\mu)$ are each even functions of μ.

It can be easily verified that

$$\underset{\mu}{\overset{\mu+1}{V}} [S^{(p)}(x,\mu)] = O(\mu^{p}), \text{ as } \mu \text{ tends to infinity,}$$

also $\underset{\mu}{\overset{\mu+1}{V}} \left[\int_{0}^{\mu} \alpha(x,\nu)\, d\nu \right] = o(1)$, as μ tends to infinity.

Hence from (2.9) and the Tauberian theorem A,

$$(2.10)\qquad \int_{0}^{\mu} (1-\nu^2/\mu^2)^{l}\, d_{\nu} [S^{(p)}(x,\nu) - \frac{1}{\pi} \int_{0}^{\nu} \alpha(x,u)\, du] = o(\mu^{p-l})$$

as μ tends to infinity uniformly for $x_0 - \frac{1}{2}\delta \leqslant x \leqslant x_0 + \frac{1}{2}\delta$.

By a change in the order of integration, which is easily justifiable, we have

$$(2.11)\qquad \pi \int_{0}^{\mu} (1-\nu^2/\mu^2)^{l}\, d_{\nu}\, S^{(p)}(x,\nu).$$

$$= \int_{0}^{\epsilon} \phi(t)\, dt \int_{0}^{\mu} (1-\nu^2/\mu^2)^{l} \cos \nu t\, d\nu$$

$$+2f^{(p)}(x) \int_{0}^{\epsilon} dt \int_{0}^{\mu} (1-\nu^2/\mu^2)^{l} \cos \nu t\, d\nu + o(\mu^{p-l})$$

$$= I_1 + I_2 + o(\mu^{p-l}), \text{ as } \mu \text{ tends to infinity,}$$

where $\phi(t) = f^{(p)}(x+t) + f^{(p)}(x-t) - 2f^{(p)}(x)$.

In I_2, (let us evaluate the inner integral by Watson [6] p.48, formula (3)), change the the variable in the intergral and then utilize the Weber integral (Watson [6], P 39).

By fixing ϵ and making μ tend to infinity, it follows that I_2 tends to $f^{(p)}(x)$ as μ tends to infinity.

It follows from (1.8) that

$$(2.12) \quad \Phi(t) = \int_0^t |\phi(u)|^r du < \epsilon_1 t^r$$

where ϵ_1 is a preassigned positive quantity, t being small enough.

Evaluating the inner integral in I_1 and then changing μt to t, we have

$$(2.13) \quad I_1 = 2^{l-\frac{1}{2}} \Gamma(l+\tfrac{1}{2}) \pi^{\frac{1}{2}} \int_0^{\mu\epsilon} \phi(t/\mu) J_{l+\frac{1}{2}}(t)/t^{l+\frac{1}{2}} dt$$

The integral on the right hand side of (2.13) is equal to

$$A = \left[\int_0^1 + \int_1^{\mu} + \int_{\mu}^{\mu\epsilon}\right] (.)\, dt$$

$$= I_{11} + I_{12} + I_{13}, \text{ say.}$$

Then

$$|A|^r \leqslant 3^{r-1}(|I_{11}|^r + |I_{12}|^r + |I_{13}|^r), r \geqslant 1.$$

Using the inequality

$$|J_\nu(z)/z| < 1/2^\nu \Gamma(\nu+1), \nu > -\tfrac{1}{2}, \text{ (Watson [6], p.49)}$$

and the inequality

$$(2.14) \quad |F|^r \leqslant (b-a)^{r-1} \int_a^b |f|^r dt$$

for a vector

$$F = \int_0^b f\, dt, \text{ we have } |I_{11}|^r < \epsilon_1/\mu^{r-1}.$$

Since $|J_\nu(z)| \leqslant B$, for any real value of $z > 1$, where B is a constant, it easily follows that

$$|I_{13}|^r \leqslant B^r/\mu^{r(l-\frac{1}{2})} (\epsilon-1)^{r-1}(\Phi(\epsilon)-\Phi(1)).$$

To estimate I_{12}, we have, by integration by parts,

$$|I_{12}|^r \leqslant H(\mu)\,[\Phi(1)-\mu^{r(l+\frac{1}{2})}\,\Phi(1/\mu)+r(l+\tfrac{1}{2})\int_{1/\mu}^{1} t^{-r(l+\frac{1}{2})-1}\Phi(t)\ dt]$$

$=J_{11}+J_{12}+J_{13}$, say,

where $H(\mu)=B^r/\mu^{r(l-\frac{1}{2})}(1-1/\mu)^{r-l}$ tends to zero, as μ tends to infinity, if $r\geqslant 1$, $l\geqslant\frac{1}{2}$,

Now J_{11} tends to zero, as μ tends to infinity, since $H(\mu)$ does so. Also, by (2.12), $|J_{12}|<\epsilon_1$ as μ tends to infinity.

To estimate J_{13}, we divide the interval of integration $(1/\mu, 1)$ into sub intervals $(1/\mu, \eta)$ and $(\eta, 1)$ and choose η such that for $0\leqslant t\leqslant\mu$, $\Phi(t)<\epsilon_1 t^r$. Then by familiar arguments $|J_{13}|<\epsilon_1$, as μ tends to infinity, uniformly in $x_0-\delta\leqslant x\geqslant x_0+\delta$, $\delta>0$.

Altogether, from (2.13), $I_1=o(1)$, as μ tends to infinity, uniformly in $x_0-\delta\leqslant x\leqslant x_0+\delta$, $\delta>0$.

Hence from (2.11) we obtain (1.8a), valid for $l\geqslant 0$, $p\geqslant 0$.

When $r=1$ in (1,8), (1.8a) follows by an easy adaptation of Hobson ([7], p.567–569) by replacing the function $C_{l+k}(t)$ by the Bessel function $J_{l+\frac{1}{2}}(t)$.

The theorem in thus established.

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REFERENCES

[1] Acharyya, Sudip Kumar — Some eigenvalue problems associated with a matrix differential operator— Calcutta Ph.D. thesis (1983)

[2] Chakravarty, N K. — Some problems in eigenfunction expansions I, Quart J. Math (Oxford) (2) 16 (62) 1965, 135-150.

[3] Chandrasekharan, C. and Minakshisundaram, S.— Typical Means (1952) (Oxford University Press).

[4] Hobson, E. W. — Theory of function of a real variable, Vol. II (Dover) 1926.

[5] Levitan, B. M. and Sargsyan, I. S. — Some problems in the theory of a Sturm-Liouville equation— Russian Math Surveys, Vol 15 (1960) p. 2-95.

[6] Watson, G. N. — Bessel Functions (1944) (Cambridge Univ, Press).

[7] Zygmund, A.— Trigonometrical Series (Dover) (1955).

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Department of Pure Math
Calcutta University

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ON COMPARISONS OF CERTAIN NON-CONTINUOUS MULTIVALUED FUNCTIONS BETWEEN BITOPOLOGICAL SPACES

M. N. Mukherjee and S. Ganguly

Abstract

In this paper certain classes of multivalued functions strictly weaker than continuous one, have been introduced for bitopological spaces, which present extended and generalized versions of their single-valued and multivalued counterparts between topological spaces. They have been characterized and studied specially with regard to their mutual dependence and interrelations.

1. Introduction

For the last quarter of a century, different mathematicians have been taking keen interest in the introduction and study of numerous kinds of mappings in topological spaces, most of which are strictly weaker than the usual continuous or open maps. Such a vast study has not only effectively characterized various concepts of topology but altogether new directions of further research and study have emerged. Some of these maps have been generalized to their multivalued forms too.

The notion of weak continuous map on topological spaces was first introduced and studied by N. Levine [5] followed by its further study by T. Noiri [10. 11, 12] and others. M. K. Singal and A. R. Singal [13] introduced the concept of a very important class of a non-continuous map which they termed almost continuous function. This kind of map was later found to be a natural tool and extremely useful for studying nearly compact spaces, almost regular spaces and for fruitfully characterizing H-closed spaces as the almost continuous images of minimal Hausdorff spaces. Due to its effectiveness and use in application, the concept was subsequently generalized to fuzzy topological situation by Azad [1] and to its multivalued form in a more generalized structure of bitopological

spaces in [9]. Functions between topological spaces under the same terminology viz. almost continuity were also studied by Husain [3] and others [2, 15], but each of those functions is independent of that of Singal and Singal. Investigations of these functions along with their mutual interactions are found in [6, 7, 8]. A certain study of almost continuous (in the sense of Husain) and weakly continuous multifunctions is done by Smithson [14], generalizing some results derived earlier for single-valued case. After the introduction of the theory of bitopological structures by J. C. Kelly [4] in 1963, the last two becades have witnessed a tremendous growth of the theory resulting to a vast literature of papers dealing with numerous concepts of topology in more generalized premises in a very effective manner. Such structures are seen to be naturally inherent in certain situations like quasiuniform, quasi-pseudometric or quasi proximity space and the theory contains the theory of topological spaces in particular. Apart from extension of concepts of topology to a more generalized perspective, the study of bitopological spaces has alreaby shown some real worth in getting newer concepts, more general, more fruitful, specially when the two topologies are very much naturally associated. Though a good number of papers have appeared dealing with some single-valued maps between bitopolo-gical spaces, the multifunctions have recently been touched demanding a substantive theory in this context to be evolved.

With the above motivation in view, our aim in this paper is to introduce and study the multivalued forms of weak continuity of Levine and almost continuity of Husain in bitopological spaces and make a conparative study of these maps along with almost continuous multifunctions studied in [9].

By X and Y we shall always mean the bitopological spaces (X, P_1, P_2) and (X, Q_1, Q_2) where P_1, P_2 (Q_1, Q_2) are two arbitrary topologies on X (respectively Y) and F will denote a multifunction from X to Y. P_i-clA and P_i-intA will respectively stand for closure and interior of a subset A of X with respect to the topology P_i on X, for $i = 1$ or 2. Similarly the notations Q_i-clB and Q_i-intB are defined. We make the convention that in any sentence where the suffixes i & j both appear it is understood that $i, j = 1, 2$ and $i \neq j$.

2. Weakly Continuous Multifunctions Between Bitopological Spaces

Definition 2.1 Let (X, P_1, P_2) aud (Y, Q_1, Q_2) be two bitopotogical spaces and $F : X \to Y$ be a multifunction. Then

(a) the upper inverse $F^+(G)$ and lower inverse $F^-(G)$ of a subset G of Y under F are defined by

$F^+(G) = \{x \in X : F(x) \subset G\}$, $F^-(G) = \{x \in X : F(x) \cap G \neq \emptyset\}$,

(b) The multivalued graph function G_F of F is defined to be the function from (X, P_1, P_2) to $(X \times Y; P_1 \times Q_1, P_2 \times Q_2)$ given by $G_F(x) = \{(x, y) : y \in F(x)\}$ for $x \in X$; by R_i we shall denote the product topology $P_i \times Q_i$ (for i = 1, 2) on the product space $X \times Y$.

Definition 2.2 Let $F : (X, P_1, P_2) \to (Y, Q_1, Q_2)$ be a multifunction.

(a) F is called $Q_i(P_iQ_j)$-upper weakly continuous (u.w.c.-in short) if for each point $x_0 \in X$ and each Q_i-open set V with $F(x_0) \subset V$, there exists a P_i-open neighbourhood (henceforth nbd., in short) U of x_0 such that $F(x) \subset Q_j\text{-cl}V$, for all $x \in U$, i, j = 1, 2 and $i \neq j$.

(b) F is called $Q_i(P_iQ_j)$-lower weakly continuous (l.w c.-in short) if for each point $x_0 \in X$ and each Q_i-open set V with $F(x_0) \cap V \neq \emptyset$, there exists a P_i-open nbd. U of x_0 such that $F(x) \cap Q_j\text{-cl } V \neq \emptyset$, for every $x \in U$ (i,j = 1,2 and $i \neq j$).

(c) F is called pairwise u.w.c. (pairwise l.w.c) if F is $Q_1(P_1Q_2)$-u.w.c. (l.w.c.) and $Q_2(P_2Q_1)$-u.w.c (l. w. c.).

(d) F is called pairwise weakly continuous if F is pairwise u,w.c. as well as pairwise l.w.c.

Theorem 2.3 A multifunction $F : X \to Y$ is $Q_i(P_iQ_j)$-u.w.c. iff its graph function $G_F : X \to (X \times Y, R_1, R_2)$, where $R_i = P_i \times Q_i$ (for i = 1, 2), is $R_i(P_iR_j)$-u.w.c., for i,j = 1,2 and $i \neq j$.

Proof Suppose F is $Q_i(P_iQ_j)$-u.w.c. and $x_0 \in X$ be arbitrary. Let W be a R_i-open set with $G_F(x_0) \subset W$. Then $G_F(x_0) \subset U \times V \subset W$, where $U \in P_i$ and $V \in Q_i$, so that $F(x_0) \subset V \in Q_i$. Since F is $Q_i(P_iQ_j)$-u.w.c , there exists a p_i-open nbd. U' of x_0 with $U' \subset U$ such that $F(U') \subset Q_j\text{-cl}V$ Then $G_F(U') = U' \times F(U') \subset U \times Q_j\text{-cl}V \subset R_j\text{-cl } (U \times V) \subset R_j\text{cl } W$. Hence G_F is $R_i(P_iR_j)$-u.s.c. conversely, let G_F be $R_i(P_iR_j)$-u.s.c. and let $x_0 \in X$ be arbitrary. If $V \in Q_i$ with $F(x_0) \subset V$, then $G(x_0) \subset X \times V \in R_i$ and hence by hypothesis, there exists a P_i-open nbd. U of x_0 such that $G_F(U) \subset R_j\text{-cl } (X \times V) = X \times Q_j\text{-cl } V$. That means $U \times F(U) \subset X \times Q_j\text{-cl } V$ so that $F(U) \subset Q_j\text{-cl } V$ and hence F is $Q_i(P_iQ_j)$-u.w.c.

Corollary 2.4 $F : X \to Y$ is pairwise u.w.c. iff its graph function G_F is so.

Theorem 2.5 A multifunction $F : X \to Y$ is $Q_i(P_iQ_j)$-l.w c. iff its graph function $G_F : X \to (X \times Y, R_1, R_2)$ is $R_i(P_iR_j)$-l.w.c.. for i, j = 1, 2 and $i \neq j$.

Proof. Let F be $Q_i(P_iQ_j)$-l.w.c. and $x_0 \in X$ be arbitrary. If $W \in R_i$ with $G_F(x_0) \cap W \neq \emptyset$, then there exist $U \in P_i$, $V \in Q_i$ such that $U \times V \subset W$ and $F(x_0) \cap V \neq \emptyset$. By hypothesis,

there exists a P_i-open nbd. U of x_0 such that $F(x)\cap Q_j\text{-cl } V \neq \emptyset$, for all $x\in U$. Now, $G_F(x)\cap R_j\text{-cl } (U\times V)=[\{x\}\times F(x)]\cap R_j\text{-cl } (U\times V)=[\{x\}\times F(x)]\cap [P_j\text{-cl}U\times Q_j\text{-cl}V] \neq \emptyset$. for all $x\in U$ so that $G_F(x)\cap R_j\text{-cl } W \neq \emptyset$, for all $x\in U$ and hence G_F is R_i (P_iR_j)-l.w.c. Conversely; let $V\in Q_i$ such that $F(x_0)\cap V \neq \emptyset$. Now $G_F(x_0)\cap(X \times V)\neq\emptyset$, where $X \times V\in R_i$. Since G_F is R_i (P_iR_j) l.w.c. there is P_i-open nbd. U of x_0 such that $G_F(x)\cap R_j\text{-cl } (X\times V) \neq \emptyset$, for all $x\in U$, i.e., $[\{x\}\times F(x)]\cap[P_j\text{-cl } X\times Q_j\text{-cl } V] \neq \emptyset$, so that $F(x)\cap Q_j\text{-cl } V \neq \emptyset$, for all $x\in U$. Hence F is Q_i (P_iQ_j)-l.w.c.

Corollary 2.6 $F : X\to Y$ is pairwise l.w.c. iff its graph function G_F is so.

From Corollaries 2.4 and 2 6 we obtain—

Corollary 2.7 A multifunction $F : X \to Y$ is pairwise weakly continuous iff the multi-valued graph function G_F of F is pairwise weakly continuous.

Theorem 2.8 If a multifunction $F : X\to Y$ is Q_i (P_iQ_j)-u.w.c. then $P_i\text{-cl } [F^-(V)]\subset F^-(Q_j\text{-cl } V)$, for every Q_j-open set V.

Proof. Suppose $x\notin F^-(Q_j\text{-cl } V)$, where $V\in Q_j$ Then $F(x)\subset Y-Q_j\text{-cl}V\in Q_i$. Since F is Q_i (P_iQ_j)-u.w.c.. there exists a P_i-open nbd. U of x such that $F(U)\subset Q_j\text{-cl } (Y-Q_j\text{-cl } V)$. Then $F(U)\cap V=\emptyset$, since V is Q_j-open so that $U\cap F^-(V)=\emptyset$. Hence $x\notin P_i\text{-cl } [F^-(V)]$.

Theorem 2.9 If a multifunction $F : X \to Y$ is $Q_i(P_iQ_j)$-l.w.c., then $P_i\text{cl}[F^+(V)] \subset F^+(Q_j\text{cl } V)$, for every Q_j-open set V.

Proof. Let $x\notin F^+(Q_j\text{-cl}V)$, where $V\in Q_j$. Then $F(x)\not\subset Q_j\text{-cl } V$ so that $F(x)\cap(Y- Q_j\text{-cl}V) \neq \emptyset$. Since F is Q_i (P_iQ_j)-l.w.c., there exists a P_i-open nbd. U of x such that $F(x')\cap Q_j\text{-cl } (Y-Q_j\text{-cl } V) \neq \emptyset$, for all $x'\in U$. Then $F(x')\not\subset V$, for all $x'\in U$, since $V\cap Q_j\text{-cl}(Y-Q_j\text{-cl}V)=\emptyset$. Then $U\cap F^+(V)=\emptyset$, where $x\in U\in P_i$. Hence $x\notin P_i\text{-cl } F^+(V)$.

We know that the continuity of multifunctions between topological spaces is defined by the introduction of two associated concepts viz. lower semi-continuity and upper semi-continuiy. Analogously we define the notion of pairwise continuity of multifunctions between bitopological spaces as follows.

Definition 2.10 A multifunction $F : X \to Y$ is said to be

(a) $Q_i(P_i)$-lower semi-continuous (l.s.c -in short) if for each point x_0 of X and every Q_i-open set V in Y with $F(x_0)\cap V \neq \emptyset$, there is a P_i-open nbd. U of x_0 such that $F(x)\cap V \neq \emptyset$, for all x of U (i=1, 2).

(b) $Q_i(P_i)$-upper semi-continuous (u.s.c -in short) if for each point x_0 of X and every Q_i-open set V in Y with $F(x_0)\subset Y$, there is a P_i-open nbd. U of x_0 such that $F(U)\subset V$ (i=1, 2),

(c) pairwise l.s.c. (u.s.c.) if F is $Q_1(P_1)$-l.s.c. (u.s.c.) as well as $Q_2(P_2)$-l.s.c. (u.s.c.),

(d) pairwise continuous if F is pairwise l.s.c. and pairwise u.s.c.

It is clear that every pairwise l.s.c. (u.s.c., continuous) multifunction is pairwise l.w.c. (respectively u.w.c., weakly continuous). In order to investigate for the converse problem we require the following definitions.

Definition 2.11 [4] A space (X, P_1, P_2) is called P_i (P_j)-regular if for each x in X and each P_i-closed set V with $x \notin V$, there is a P_i-open set U and a P_j-open set W disjoint from U such that $x \in U$ and $V \subset W$, where, as before, $i, j = 1, 2$ and $i \neq j$, X is called pairwise regular iff it is P_1 (P_2)-regular and P_2 (P_1)-regular.

Definition 2.12 A set A of a bitopological space (X, P_1, P_2) is called strictly P_i (P_j) -paracompact iff every cover $\mathcal{Y}$ of A with P_i-open sets has a refinement $\mathcal{B}$ with P_i-open sets, which cover A and $\mathcal{B}$ is P_j, locally finite, i.e., for each point x of X there is a p_j-open nbd. U of x intersecting at most finitely many elements of $\mathcal{B}$. A is called strictly pairwise paracompact if it is strictly P_1 (P_2)-as well as P_2 (P_1)-paracompact.

Theorem 2.13 If a multifunction $F : X \to Y$ is Q_i $(P_i\ Q_j)$-l.w.c. and Y is Q_i (Q_j)-regular, then F is Q_i (P_i)-l.s.c.

Proof : Let $x_0 \in X$ be arbitrary and V be a Q_i-open set with $F(x_0) \cap V \neq \emptyset$. Let $y \in F(x_0) \cap V$. Since Y is Q_i (Q_j)-regular and $y \in V \in Q_i$, there exists $D \in Q_i$ such that $y \in D \subset Q_j\text{-cl } D \subset V$. Now since $D \in Q_i$ and $y \in F(x_0) \cap D$, there exists P_i-open nbd. U of x_0 such that $F(x) \cap Q_j\text{-cl } D \neq \emptyset$, for all $x \in U$. Then $F(x) \cap V \neq \emptyset$, for all $x \in U$ and hence F is Q_i (P_i)-l. s. .c.

Corollary 2.14 For a multifunction F from a bitopological space to a pairwise regular space the concepts of pairwise lower weak continuity and pairwise lower semi-continuity coincide.

Theorem 2.15 Let (Y, Q_1, Q_2) be Q_i (Q_j)—regular and for each $x \in X$, F (x) is strictly Q_i (Q_j)-paracompact, where $F : X \to Y$ is Q_i $(P_i\ Q_j)$- u.w.c. Then F is Q_i (P_i)-u.s.c.

Proof. Being similar to that of Theorem 3.9 of [9] is omitted.

Corollary 2.16 For a multifunction F from a bitopological space X to a pairwise regular space the concepts of pairwise upper weak continuity and pairwise upper semicontinuity coincide, provi 'ed F (x) is strictly pairwise paracompact for each x of X.

3. WEAKLY CONTINUOUS AND S. ALMOST CONTINUOUS MULTIFUNCTIONS

M.K. Singal and A.R. Singal [13] initiated the study of almost continuous single-valued function between topological spaces. The concept was generalized to multivalued case and that too between bitopologicel spaces by us in [9]. It is the purpose of this section to correlate the concept with that of weakly continuous multifunction studied in the last section.

Definition 3.1 [9] For a multifunction $F : (X, P_1, P_2) \rightarrow (Y, Q_1, Q_2)$ we define as follows.

(i) F is Q_i (P_i Q_j)-upper almost continuous in the sense of Singal and Singal (abbreviated as Q_i (P_i Q_j)-S.u.a.c.) if for each x_0 of X and each Q_i-openset V with $F(x_0) \subset V$, there exists a P_i-open nbd. of x_0 such that $F(x) \subset Q_i$-int (Q_j-cl V), for all $x \in U$.

(ii) F is called Q_i (P_i Q_j)-lower almost continuous in the sense of Singal and Singal (abbreviated as Q_i (P_i Q_j)-S.l.a.c) if for each x_0 of X and each Q_i-open set V with $F(x_0) \cap V \neq \emptyset$, there is a P_i-open set U containing x_0 such that $F(x) \cap [$ Q_i-int (Q_j-cl V) $] \neq \emptyset$, for all $x \in U$.

(iii) F is called pairwise S l.a.c. (S.u.a.c.) if F is Q_1 (P_1 Q_2)-S.l.a c. (S.u.a c) as well as Q_2 (P_2 Q_1)-S. l.a.c. (S u.a.c.).

(iv) F is called pairwise S. almost continuous if F is pairwise S.l.a.c and pairwise S.u.a.c..

It is obvious that

Theorem 3.2 If $F : X \rightarrow Y$ is a multifunction, then

(a) F is Q_i (P_i Q_j)-S.u.a.c. (pairwise S.u.a c.)
$\Rightarrow$ F is Q_i (P_i Q_j)-u.w.c. (Pairwise u.w.c.)

(b) F is Q_i (P_i Q_j)-S.l.a.c. (pairwise S.l.a.c.)
$\Rightarrow$ F is Q_i (P_i Q_j)-l.w c. (pairwise l.w.c)

(c) F is pairwise S. almost continuous $\Rightarrow$ F is pairwise weakly continuous.

Definition 3.3 Let $F : X \rightarrow Y$ be a multifunction.

(a) F is called P_i (Q_i)-open, if for each P_i-open set U, F (U) is Q_i-open (i=1 or 2). F is called pairwise open if it is P_1 (Q_1)-open and P_2 (Q_2)-open.

(b) F is called P_i (Q_j)-point open if for each P_i-open set U, F (x) is Q_j-open, for all $x \in U$ (i, j=1, 2 ; $i \neq j$). F is called pairwise point open if it is P_1 (Q_2)-as well as P_2 (Q_1)-point open.

Theorem 3.4 If a multifunction $F : X \to Y$ is Q_i $(P_i\, Q_j)$-u.w.c. and P_i (Q_i)-open, then F is Q_i $(P_i\, Q_j)$-S,u.a.c.

Proof. Let $x_0 \in X$ be taken arbitrarily and let V be a Q_i-open set such that $F(x_0) \subset V$. Then there exists a P_i-open nbd. U of x_0, such that $F(U) \subset Q_j$-cl V. Since F is P_i (Q_i)-open, $F(U) \subset Q_i$-int $(Q_j$cl V) and hence F is Q_i $(P_i\, Q_j)$-S.u.a.c.

Corollary 3.5 For a pairwise open multifunction, the concepts of pairwise S upper almost continuity and pairwise upper weak continuity coincide.

Theorem 3.6 If a multifunction $F : X \to Y$ is Q_i $(P_i\, Q_j)$-l.w.c. and P_i (Q_j)-point open, then F is Q_i $(P_i\, Q_j)$-S.l.a.c.

Proof. Let $x_0 \in X$ be arbitrary and $V \in Q_i$ such that $F(x_0) \cap V \neq \emptyset$. Then there is a P_i-open nbd. U of x_0 such that $F(x) \cap Q_j$-cl $V \neq \emptyset$, for all $x \in U$. Now for each $x \in U$, since $F(x)$ is Q_j-open, we must have $F(x) \cap Q_i$-int $(Q_j$-cl $V) \neq \emptyset$. for all $x \in U$. In fact, if for some $x \in U$, $F(x) \cap Q_i$-int $(Q_j$-cl $V) = \emptyset$, then since $F(x) \cap Q_j$-cl $V \neq \emptyset$ there must exist $y \in F(x)$ such that $y \in Q_j$-cl V but $y \notin Q_i$-int $(Q_j$-cl V). Then $F(x)$ is a Q_j-open nbd. of y and hence $F(x) \cap V \neq \emptyset$, so that $F(x) \cap Q_i$-int $(Q_j$cl V) $\neq \phi$, as $V \subset Q_i$-int $(Q_j$-cl V)-a contradiction.

Hence F is Q_i $(P_i\, Q_j)$-S.l.a.c.

Corollary 3.7 For a pairwise point-open multifunction $F : X \to Y$, the concepts of pairwise S.l.a.c. and pairwise l.w.c. coincide.

The following theorem gives alternative conditions under which pairwise upper and lower weekly continuous muitifunctions may be identical with pairwise S. upper and S. lower almost continuous functions respectively.

Theorem 3.8 Let F be a multifunction from X to a pairwise regular space Y. Then.

(i) F is pairwise l.w,c. iff F is pairwise S.l.a.c.

(ii) F is pairwise u.w.c. iff F is pairwise S u.a.c., provided F (x) is pairwise strictly paracompact for each x of X.

Proof. Follows from Corollaries 2.14 and 2.16, and the fact that every pairwise lower (upper) semi-continuous function is pairwise S.l.a.c. (S.u.a.c.)

4. WEAKLY CONTINUOUS AND H. ALMOST CONTINUOUS MULTIFUNCTIONS

A new class of non-continuous single valued maps under the terminology 'almost continuous functions' was introduced by T. Husain [3]. Husain's almost continuity is seen to be independent of that of Singal and Singal [13]. This section is devoted to the

introduction of an extended form of Husain's almost continuity for multifunctions in a more generalized setting of bitopological spaces. Such a multifunction is characterized and studied briefly and its behaviour with regard to weakly continuous and S. almost continuous multifunctions has been investigated.

Definition 4.1 For a multifunction $F: X \to Y$ we define as follows.

(i) F is Q_i $(P_i\ P_j)$-upper almost continuous in the sense of Husain (Q_i $(P_i\ P_j)$-H.u.a.c. - in short) if for each x_0 of X and each Q_i-open set V with $F(x_0) \subset V$, P_j-cl $F^+(V)$ is a P_i-nbd. of x_0.

(ii) F is Q_i $(P_i\ P_j)$-lower almost continuous in the sense of Husain (Q_i $(P_i\ P_j)$-H l.a.c. - in short) if for each x_0 of X and each Q_i-open set V with $F(x_0) \cap V \neq \emptyset$, P_j-cl $F^-(V)$ is a P_i-nbd. of x_0.

(iii) F is called pairwise H.l a.c. (H.u.a c.) if F is Q_1 $(P_1\ P_2)$-H.l.a.c. (H u.a.c.) as well as Q_2 $(P_2\ P_1)$-H.l.a.c. (H.u.a.c). F is called pairwise H.a.c. if F is pairwise H l.a.c. and pairwise H.u.a.c.

Theorem 4.2 Let $F: X \to Y$ be a multifunction. Then

(a) F is Q_i $(P_i\ P_j)$-H.u.a.c. iff $F^+(V) \subset P_i$-int $[P_j$-cl $F^+(V)]$, for every Q_i- open set V.

(b) F is Q_i $(P_i\ P_j)$-H.l.a.c. iff $F^-(V) \subset P_i$-int $[P_j$-cl $F^-(V)]$. for every Q_i-open set V.

Proof.

(a) Let F be Q_i $(P_i\ P_j)$-H.u.a.c. and $x \in F^+(V)$. then $F(x) \subset V \in Q_i$ Then P_j-cl $F^+(V)$ is a P_i-nbd. of x_0 so that $x \in P_i$-int $(P_j$-cl$F^+(V))$. Conversely, for any x_0 of X and a Q_i-open set V with $F(x_0) \subset V$, we have $x_0 \in F^+(V) \subset P_i$-int $(P_j$-cl $F^+(V))$.

Thus P_j-cl $F^+(V)$ is a P_i-nbd of x_0.

(b) Let F be Q_i $(P_i\ P_j)$-H.l.a.c. and $x \in F^-(V)$. Then $F(x) \cap V \neq \emptyset$. Then there exists P_i-open set U such that $x \in U \subset P_j$-cl $F^-(V)$ and hence $x \in P_i$-int $[P_j$-cl $F^-(V)]$.

Conversely, $x_0 \in X$ and $F(x_0) \cap V \neq \emptyset$, where $V \in Q_i \Rightarrow x_0 \in F^-(V) \subset P_i$ int $[P_j$-cl $F^-(V)] \subset P_j$-cl $F^-(V)$

Theorem 4.3 A multifunction $F: (X, P_1, P_2) \to (Y, Q_1, Q_2)$ is Q_i $(P_i\ P_j)$-H.u a c. (H l.a.c.) if the graph function $G_F: X \to (X \times Y, R_1, R_2)$ is R_i $(P_i\ P_j)$-H.u.a.c (H.l.a.c.), where $R_K = P_K \times Q_K$ (for $K=1, 2$).

Proof. First let G_F be R_i $(P_i\ P_j)$-H.u.a.c. and let $x \in X$ be arbitrary. Suppose $V \in Q_i$ such that $F(x) \subset V$. Now $X \times V \in R_i$ such that $G_F(x) \subset X \times V$. Then P_j-cl $[G_F^+(X \times V)]$ is a P_i-nbd. of x. Now, $G_F^+(X \times V) = \{x \in X : F(x) \subset V\} = F^+(V)$. Thus P_j-cl $(F^+(V))$ is a P_i-nbd. of x and hence F is Q_i $(P_i\ P_j)$-H.u.a.c.

Next, let G_F be R_i $(P_i\ P_j)$-H.l.a.c. and $x \in X$ be arbitrary. If V is any Q_i-open set with $F(x) \cap V \neq \emptyset$, then $G_F(x) \cap (X \times V) \neq \emptyset$, where $X \times V \in R_i$. Hence there exists a P_i-open nbd. U of x such that $x \in U \subset P_j$-cl $[G_F^-(X \times V)]$. But $G_F^-(X \times V) = \{x \in X : F(x) \cap V \neq \emptyset\} = F^-(V)$ and then F is Q_i $(P_i\ P_j)$-H.l.a.c.

Corollary 4.4 A multifunction F is pairwise H.u.a.c. or pairwise H.l.a.c. or pai wise H. almost continuous if its graph function G_F is respectively so.

It can be easily seen that weak continuity and H. almost continuity of a multifunction between bitopological spaces are independent notions. In fact, they are also so even for single-valued case. We shall now derive conditions under which they can be correlated.

Definition 4.5 A multifunction $F : X \to Y$ is called

(i) Q_i $(P_j\ Q_j)$-upper almost open (u.a.o., in short) if $F^+(Q_j\text{-cl}V) \subset (P_j\text{-cl}\,(F^+(G))$. for every $V \in Q_i$,

(ii) Q_i $(P_j\ Q_j)$-lower almost open (l.a.o., in short) if $F^-(Q_j\text{-cl}V) \subset P_j\text{-cl}\,(F^-(V))$, for every $V \in Q_i$,

(iii) pairwise u.a.o. (l.a.o.) if F is Q_1 $(P_2\ Q_2)$-u.a.o. (l.a.o.) and Q_2 $(P_1\ Q_1)$-u.a.o. (l.a.o.).

(iv) pairwise almost open if it is pairwise u.a.o. as well as pairwise l.a.o.

Theorem 4.6 A multifunction $F : X \to Y$, which is Q_i (P_iQ_j)-u.w.c. and Q_i (P_jQ_j)-u.a.o., is Q_i (P_iP_j)-H.u.a.c.

Proof. Let $x \in X$ be arbitrary and $V \in Q_i$ such that $F(x) \subset V$. Since F is Q_i (P_iQ_j)-u.w c, there exists a P_i-open nbd U of x such that $F(U) \subset Q_j$-cl V. It then follows that $U \subset F^+(Q_j\text{-cl}V) \subset P_j\text{-cl}\,(F^+(V))$ (since F is Q_i (P_jQ_j)-u a.o.). Thus P_j-cl $(F^+(V))$ is a P_i-nbd. of x and F is Q_i (P_iP_j)-H.u.a c.

Corollary 4.7. A pairwise u,a.o. multifunction is pairwise H.u,a.c. if it is pairwise u.w.c.

Theorem 4.8. A multifunction $F : X \to Y$ which is Q_i (P_iQ_j)-l.w.c. and Q_i (P_jQ_j)-l.a.o., is Q_i (P_iP_j)-H.l.a c.

Proof. Let $x_0 \in X$ and $V \in Q_i$ such that $F(x_0) \cap V \neq \emptyset$. Since F is Q_i (P_iQ_j)-l.w.c., there exists a P_i-open nbd. U of x_0 such that $F(x) \cap Q_j\text{-cl } V \neq \emptyset$, for all $x \in U$. Thus $U \subset F^-(Q_j\text{-cl } V)$. Again, since F is Q_i (P_jQ_j)-l.a.o., we have $x_0 \in U \subset F^-(Q_j\text{-cl } V) \subset P_j\text{-cl } (F^-(V))$. Thus $P_j\text{-cl } (F^-(V))$ is a P_i-nbd. of x_0 and F is Q_i (P_iP_j)-H.l a.c.

Corollary 4.9- A pairwise l.a.o. multifunction is pairwise H.l.a.c. if it is pairwise l.w.c. From Corollaries 4.7 and 4.9 we immediately have.

Corollary 4.10 A pairwise almost open and pairwise weakly continuous multifunction is pairwise H. almost continuous.

Theorem 4.11 A Q_i (P_iP_j)-H.u.a.c. multifunction $F: X \to Y$ is Q_i (P_iQ_j)-u.w.c. if $P_j\text{-cl } [F^+(V)] \subset F^+(Q_j\text{-cl } V)$, for every Q_i-open set V of Y.

Proof. Let $x \in X$ and $V \in Q_i$ such that $F(x) \subset V$. Since F is Q_i (P_iP_j)-H.u.a.c., $P_j\text{-cl } (F^+(V))$ is a P_i-nbd. of x. Then there is a P_i-open set U such that $x \in U \subset P_j\text{-cl } (F^+(V)) \subset F^+(Q_j\text{-cl } V)$. Thus $F(U) \subset Q_j\text{-cl } V$ and F is Q_i (P_iQ_j)-u.w.c.

Theorem 4.12 A Q_i (P_iP_j)-H.la.c. multifunction $F: X \to Y$ is Q_i (P_iQ_j)-l.w.c. if $P_j\text{-cl } [F^-(V)] \subset F^-(Q_j\text{-cl } V)$, for every Q_i-open set V of Y.

Proof. Let $x_0 \in X$ and $V \in Q_i$ such that $F(x_0) \cap V \neq \emptyset$. Since F is Q_i $(P_i P_j)$-H.l.a.c., there exists a P_i-open nbd. U of x_0 such that $U \subset P_j\text{-cl } (F^-(V))$. Using the given condition we have $x_0 \in U \subset F^-(Q_j\text{-clv})$. Thus $F(x) \cap Q_j\text{-cl } V \neq \emptyset$, for all $x \in U$ and F is Q_i $(P_i Q_j)$-l.w.c.

From Theorems 4.11 and 4.12 we now have

Corollary 4.13 (a) A pairwise H.u.a.c. (H,l a.c.) multifunction $F: X \to Y$ is pairwise u.w.c. (l.w.c.) if $P_j\text{-cl } [F^+(V)] \subset F^+(Q_j\text{-cl } V)$ (respectively $P_j\text{-cl } [F^-(V)] \subset F^-(Q_j\text{-cl } V)$), for every Q_i-open set V of Y, where $i, j = 1$ and 2, $i \neq j$.

(b) A pairwise H. almost continuous multifunction $F: X \to Y$ is pairwise weakly continuous if $P_j\text{-cl } (F^-(V)) \subset F^+(Q_j\text{-cl } V)$ holds for every Q_i-open set V of Y, where $i, j = 1$ and 2, $i \neq j$

The following theorems, obtained as immediate consequences of certain previously deduced results, present connections between S. almost continuity and H. almost continuity.

Theorem 4.14 (a) A multifunction $F: X \to Y$ which is Q_i $(P_i Q_j)$-S.u.a c. (S.l.a.c.) and Q_i $(P_j Q_j)$-u.a.o. (l.a.o.) is Q_i $(P_i P_j)$-H.u.a.c. (H.l.a.c.)

(b) A pairwise u.a,o. (l.a o.) and pairwise S.u.a.c. (S l.a.c.) multifunction is pairwise H.u.a.c. (H.l.a.c.)

(c) A pairwise almost open and paitwise S. almost continuous function is pairwise H. almost continuous.

Proof. (a) Follows from Theorems 3.2, 4.6 and 4.8

(b) Follows from Theorem 3.2 and Corollary 4.7 and 4.9.

(c) Follows from Theorem 3.2 and Corollary 4.10.

Theorem 4.15 (a) A P_i (Q_i) open and Q_i (P_i P_j)-H.u.a.c. multifunction $F ; X \to Y$ is Q_i (P_i Q_j)-S.u.a.c. if P_j-cl $[F^+ (V)] \subset F^+ (Q_j\text{-cl } V)$, for every O_i-open set V of Y.

(b) A pairwise open. pairwise H.u.a.c. multifunction $F : X \to Y$ is pairwise S.u.a.c. provided P_j-cl $[F^+ [V)] \subset F^+ (Q_j\text{-cl } V)$, for every Q_i-open set V of Y, where i, j=1 and 2, $i \neq j$.

Proof. (a) Follows from Theorem 3.4 and 4.11.

(b) A consequence of Corollary 3.5 and 4.13 (a).

Theorem 4.16 (a) A P_i (Q_j)-point open and Q_i (P_i P_j)-H.l.a.c. multifunction $F : X \to Y$ is Q_i (P_i Q_j)-S.l.a.c. if P_j-cl $[F^- (V)] \subset F^- (Q_j\text{-Cl } V)$, for every Q_i-open set V of Y.

(b) A pairwise point open, pairwise H.l.a.c. multifunction $F : X \to Y$ is pairwise S.l.a.c. provided P_j-cl $[F^- (V)] \subset F^- (Q_j\text{-cl } V)$, for every Q_i-open set V of Y, where i, j=1 and 2, $i \neq j$.

Proof. (a) is a direct consequence of Theorem 3.6 and 4.12, whereas (b) follows immediately from Corollary 3.7 and 4.13 (b).

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REFERENCES

1. Azad, K. K. On fuzzy semicontinuity, fuzzy almost continuity and fuzzy Weakly continuity, Math. Analysis and Applications 82, No. 1 (1981), 14-32.
2. Frolik, Z. Remarks concerning the invariance of Baire speces under mappings, Cech, Math. J. 11 (86) (1961), 381-385.
3. Husain, T. Almost continuous mappings, Prace Mat. 10 (17) (1966), 1-7.
4. Kelly, J. C. Bitopological spaces, Proc. London Math. Soc. 13 (3) (1963). 71–89.
5. Levine, N. A decomposition cf continuity in topological spaces, Amer. Math. Monthly 68 (1961), 44-46.
6. Long, P. E. and Carnahan. D. A. Comparing almost continuous functions. Proc. Amet, Math soc. 38 (No. 2) (1973). 413–418.
7. Long, P. E, and Megehee, Jr, E. E. Properties of almost continuous functions, Proc. Amer. Math. Soc. 24 (1970), 175-180.
8. Mathur, A. and Deb, M. A note on almost continuous mappings, Math. Student 41 (No. 2) (1972), 173-184.
9. Mukherjee, M. N. and Ganguly, S. Generalization of almost continuous multifunctions to bitopological spacea, Bull. Cal. Math. Soc. 79 (1987), 274-283.
10. Noiri, T. On weakly continuous mappings, Proc. Amer, Math. Soc, 46 (No. 1) (1974) 120-124.
11. Noiri, T. Bet ween continuity and weak continuity, Boll. Un. Mat. Ital. (4) 9 (1974), 647-654.
12. Noiri. T. Semi-continuity and weak continuity, Czechoslovak Math. Jour. 31 (106) (1981), 314-321.
13. Singal. M. K. and Singal, A. R. Almost continuous mappings, Yokohama Math. Jour. 16 (No. 2) (1968), 63-73.
14. Smithson, R, E. Almost and weak continuity for multifunctions, Bull. Cal. Math. Soc. 70 (1978), 383-390.
15. Stallings, J. Fixed point theorems for connectivity maps, Funda, Math, 47 (1959), 249-263.

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Department of Pure Mathematics,
University of Calcutta,
35, Ballygunge Circular Road,
Calcutta-700 019.

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ON THE MAXIMUM TERMS OF THE DERIVATIVES OF ENTIRE FUNCTIONS IN SEVERAL COMPLEX VARIABLES REPRESENTED BY MULTIPLE DIRICHLET SERIES.

B. C. Chakraborty and Md. Muklesur Rahman

Abstract : A family F of entire functions in several complex variables represented by a multiple Dirichlet series has been considered. Hadamard multiplication and the concept of rank of a maximum term of any function $f_1 \in F$ have been introduced. Partial derivatives of any order of two different functions are considered and a few inequalities involving their maximum terms and ranks have been obtained in R^n, real n-space. After removing the set of discontinuities of the rank from R^n, the special forms and the consequences of the above inequalities are also obtained.

1. **Notations :** We denote complex and real n-space by C^n and R^n respectively and the set of non-negative integers by I, so that I^n will denote the Cartesian product of n copies of I. We indicate the points $(s_1, \ldots\ldots, s_n)$, $(\sigma_1, \ldots\ldots, \sigma_n)$, $(m_1, \ldots\ldots m_n)$ etc. of C^n or R^n by their corresponding unsuffixed symbols s, σ, m respectively and make use of the standard notations of the single variable which are easy to understand from the context.

For s, w $\in$ C^n and $\alpha \in C$ where $s = (s_1, \ldots\ldots, s_n)$, $w = (w_1, \ldots\ldots, w_n)$ we define

i) $s = w$ iff $s_i = w_i$, $i = 1, \ldots\ldots, n$

ii) $s + w = (s_1 + w_1, \ldots\ldots, s_n + w_n)$,

iii) $\alpha s = (\alpha s_1, \ldots\ldots, \alpha s_n)$

iv) $s.w = s_1 w_1 + \ldots + s_n w_n$

v) $|s| = \left\{ |s_1|^2 + \ldots\ldots + |s_n|^2 \right\}^{1/2}$

For $a \in R$, $s \in C^n$,

vi) $s+a = (s_1+a, \ldots\ldots s_n+a)$

Also for $x, y \in R^n$, we say that

vii) $x \leq y$ iff $x_i \leq y_i$, $i = 1, \ldots\ldots, n$

viii) $x < y$ iff $x \leq y$ but $x \neq y$

ix) $x << y$ iff $x_i < y_i$, $i = 1, \ldots\ldots, n$.

The positive hyperoctant R^n_+ in R^n will be

$R^n_+ = \{x : x \in R^n, x_i \geqslant 0, i = 1, \ldots\ldots, n\}$. For $t \in R^n_+$ we set $\| t \| = t_1 + \ldots\ldots + t_n$ and for $m \in I^n$, $m! = m_1! \ldots m_n!$.

Also for $s \in C^n$, $t \in R^n_+$ we shall denote $s_1^{t_1} \ldots s_n^{t_n}$ by s^t, ($s_i^0 = 1$ even if $s_i = 0$). For $k \in R$, $\bar{k}$ will denote the real n-tuple $(k, \ldots\ldots, k)$. For an entire function f with domain C^n, f^k will denote the function $\dfrac{\partial^{\| k \|} f}{\partial s_1^{k_1} \ldots \partial s_n^{k_n}}$ where $k \in I^n$ and $f^{\bar{0}} = f$.

An unorthodox notation: In the definition of a multiple Dirichlet series we take n sequences $\left\{\lambda_{jm_j}\right\}_{m_j = 1}^{\infty}$, $j = 1, \ldots\ldots n$ of exponents.

We shall often require the n-tuple $(\lambda_{1m_1} \ldots, \lambda_{nm_n})$ of those sequences. For brevity we denote this n-tuple by (λ_{nm_n}). Also, for a particular set of values of $m_1, \ldots\ldots m_n$, say $p = (p_1, \ldots\ldots p_n)$, the n-tuple $(\lambda_{1p_1}, \ldots\ldots, \lambda_{np_n})$ will be denoted by (λ_{np}). Thus, $s.(\lambda_{nm_n})$ will mean $s_1 \lambda_{1m_1} + \ldots + s_n \lambda_{nm_n}$.

2. We consider an entire function in C^n represented by the multiple Dirichlet series

$$f_1(s_1, \ldots\ldots s_n) = \sum_{m_1, \ldots\ldots, m_n = 1}^{\infty} a_{m_1 \ldots\ldots m_n} \exp(s_1 \lambda_{1m_1} + \ldots\ldots + s_n \lambda_{nm_n})$$

i. e.

$$(2.1.) \quad f_1(s) = \sum_{m=1}^{\infty} a_m \exp\left\{ s.(\lambda_{nm_n}) \right\}, \text{ where}$$

$s_j=\sigma_j + i\tau_j \epsilon C$ ($j = 1, \ldots\ldots n$), $a_m \epsilon C$ and $\left\{\lambda_j m_j\right\}_{m_j=1}^{\infty}$

are n sequences of exponents satisfying the conditions

(2. 2) $0 < \lambda_{j1} < \lambda_{j2} < \ldots < \lambda_{jk} \to \infty$ as $k \to \infty$, for $j = 1, \ldots\ldots, n$. Throughout we tacitly assume that

(2. 3) $$\lim_{m_j \to \infty} \frac{\log m_j}{\lambda_{jm_j}} = 0, \quad j = 1, \ldots\ldots, n.$$

A. I. Janusauskas [2] had shown that if (2. 3) holds then the domain of convergence of the series (2. 1) coincides with its domain of absolute convergence.

Let F be the family of all entire functions represented by a series of the form (2.1) having the same sequences of exponents $\left\{\lambda_{jm_j}\right\}_{m_j=1}^{\infty}$, ($j=1,\ldots\ldots,n$) and satisfying the condition (2.2).

For $f_1, f_2 \epsilon F$, we define $f = f_1 * f_2$ by

(2. 4) $$f(s) = f_1(s) * f_2(s) = \sum_{m=1}^{\infty} a_m b_m \exp\left\{ s. \left(\lambda_{nm_n}\right)\right\}$$ where

(2 5) $$f_1(s) = \sum_{m=1}^{\infty} a_m \exp\left\{ s. \left(\lambda_{nm_n}\right)\right\}$$ and

(2. 6) $$f_2(s) = \sum_{m=1}^{\infty} b_m \exp\left\{ s. \left(\lambda_{nm_n}\right)\right\}.$$

Throughout this paper f will denote the product function $f_1 * f_2$ as given in (2.4).

Theorem 1. The function f, as defined by (2.4), belongs to F.

Proof : Since (2.5) is entire, the series $\sum_{m=1}^{\infty} | a_m | \exp\left\{\sigma. \left(\lambda_{nm_n}\right)\right\}$ is convergent for all $\sigma \epsilon R^n$. In particular, it is convergent at $\sigma = \bar{0}$, so that $\sum_{m=1}^{\infty} | a_m |$ is convergent. Thus, $\lim_{\| m \| \to \infty} | a_m | = 0$ and hence the n-sequence $\left\{ | a_m | \right\}$ is bounded. Also the series $\sum_{m=1}^{\infty} | b_m | \exp\left\{ \sigma. \left(\lambda_{nm_n}\right)\right\}$ is convergent for all $\sigma \epsilon R^n$

and consequently $\sum_{m=1}^{\infty} | a_m | \; | b_m | \exp \left\{ \sigma . (\lambda_{nm_n}) \right\}$ is convergent for all $\sigma \in R^n$

which implies $\sum_{m=1}^{\infty} a_m b_m \exp \left\{ s . (\lambda_{nm_n}) \right\}$ is absolutely convergent for all $s \in C^n$.

Hence (2.4) represents an entire function and $f \in F$.

3. Corresponding to any $f_1 \in F$ we define the functions: the maximum modulus $M (\sigma, f_1)$, the maximum term $\mu (\sigma, f_1)$ and the indices $\nu_j (\sigma, f_1)$ of the maximum term $\mu (\sigma, f_1)$, ($j = 1, \ldots\ldots n$) on R^n by $M (\sigma, f_1) = \max_{\operatorname{Re} s = \sigma} | f (s) |$,

$$\mu (\sigma, f_1) = \max_{m \in I^n} \left[| a_m | \exp \left\{ \sigma . (\lambda_{nm_n}) \right\} \right.$$

$$\nu_j (\sigma, f_1) = \max \left[m_j : | a_m | \exp \left\{ \sigma . \lambda_{nm_n}) \right\} = \mu (\sigma, f_1) \right], \; j = 1, \ldots\ldots, n.$$

We call $\nu = \nu (\sigma, f_1) = (\nu_1, \ldots\ldots \nu_n)$ as the rank of the maximum term $\mu (\sigma, f_1)$. It is shown in theorem I that the series (2.4) belongs to F. Consequently, for $k \in I^n$,

$$f^k (s) = (f_1 (s) * f_2(s))^k = \sum_{m=1}^{\infty} (\lambda_{nm_n})^k a_m b_m \exp \left\{ s . \lambda_{nm_n}) \right\} \text{ and } \overset{k}{\underset{*}{f}} (s) = \overset{k}{f}_1 (s) * \overset{k}{f}_2 (s)$$

$$= \sum_{m=1}^{\infty} (\lambda_{nm_n})^{2k} a_m b_m \exp \left\{ s . (\lambda_{nm_n}) \right\} \text{ are also elements of F.}$$

Let $M (\sigma, k)$, $M^* (\sigma, k)$ be the maximum modulii of f^k and $\overset{k}{\underset{*}{f}}$ respectively and their respective maximum terms be denoted by $\mu (\sigma, k)$ and $\mu^* (\sigma, k)$. Then,

$$\mu (\sigma, k) = \max_{m \in I^n} \left[(\lambda_{nm_n})^k | a_m b_m | \exp \left\{ \sigma . (\lambda_{nm_n}) \right\} \right],$$

$$\mu^* (\sigma, k) = \max_{m \in I^n} \left[(\lambda_{nm_n})^{2k} | a_m b_m | \exp \left\{ \sigma . (\lambda_{nm_n}) \right\} \right].$$

Also let $\nu_j = \nu_j (\sigma, k)$, and $\nu_j^* = \nu_j^* (\sigma, k)$, $j = 1, \ldots\ldots, n$, be the indices of $\mu (\sigma, k)$ and $\mu^* (\sigma, k)$ respectively and $\nu = \nu (\sigma, k) = (\nu_1, \ldots\ldots, \nu_n)$, $\nu^* = \nu^* (\sigma, k) = (\nu_1^* \ldots\ldots, \nu_n^*)$ be their respective ranks.

Theorem 2. For $\bar{0} \leq \sigma << \zeta$ and $k \in I^n$,

$$M (\sigma, k) \leqslant \frac{k \,!\, M^* (\zeta, \bar{0})}{(\zeta - \sigma)^k}$$

Proof : By Cauchy's integral formula

$$\frac{\partial^{\|k\|} f(s)}{\partial s_1^{k_1} \ldots\ldots \partial s_n^{k_n}} = \frac{k\,!}{(2\pi i)^n} \int_0 \frac{f(w)}{(w-s)^{k+1}}\, dw_1 \ldots\ldots dw_n\,,$$

where $C = C_1 \times .. \times C_n$, $C_1 : | w_i - s_i | = \xi_1 - \sigma_1$, $i = 1, \ldots\ldots, n$.

Hence, $$\left| \frac{\partial^{\|k\|} f(s)}{\partial s_1^{k_1} \ldots\ldots \partial s_n^{k_n}} \right| \leq \frac{k\,!\ M(\zeta, \bar{O})}{(\zeta - \sigma)^k}$$

Since $M(\sigma, \bar{O}) = M^*(\sigma, \bar{O})$ for all $\sigma \in R^n$, the theorem follows.

Theorem 3. For any $\sigma \in R^n$ and $k \in I^n$,

$$(\lambda_{np})^k \leqslant \lambda \frac{\mu^*(\sigma, k)}{\mu(\sigma, k)} \leqslant (\lambda_{n\nu^*})^k,$$

where $p = (p_1, \ldots\ldots, p_n)$ is a position of occurrence of $\mu(\sigma, k)$ and $\nu^*(\sigma, k)$ is th rank of $\mu^*(\sigma, k)$.

Proof : Let $\mu(\sigma, k)$ occur at a position $p = (p_1, \ldots\ldots, p_n)$ and $\mu^*(\sigma, k)$ occur a $p^* = (\overset{*}{p_1}, \ldots\ldots \overset{*}{p_n})$. Then,

$$\mu(\sigma, k) = (\lambda_{np})^k \,| a_p b_p |\ \exp\left\{ \sigma.(\lambda_{np}) \right\}$$

$$\geqslant (\lambda_{np*})^k \,| a_{p*} b_{p*} |\ \exp\left\{ \sigma.(\lambda_{np*}) \right\} = \frac{\mu^*(\sigma, k)}{(\lambda_{np*})^k}$$

Hence, $\dfrac{\mu^*(\sigma, k)}{\mu(\sigma, k)} \leq (\lambda_{np*})^k$, Evidently $p* \leq \nu^*$. Due to (2.2) we have,

$$(3.1)\qquad \frac{\mu^*(\sigma, k)}{\mu(\sigma, k)} \leqslant (\lambda_{np^*})^k.$$

Again, $\mu^*(\sigma, k) = (\lambda_{np*})^{2k} \,| a_{p*} b_p^* |\, \exp\left\{ \sigma.(\lambda_{np*}) \right\}$

$\geqslant (\lambda_{np})^{2k} \,| a_p b_p |\ \exp\{ \sigma. \lambda_{np} \} = (\lambda_{np})^k \mu(\sigma, k)$. Hence,

$$(3.2)\qquad (\lambda_{np})^k \leq \frac{\mu^*(\sigma, k)}{\mu(\sigma, k)}$$

Combining (3.1) and (3.2) the theorem follows.

Theorem 4: For any $k >> \bar{O}$ and $\sigma \in R^n$,

$$\lambda_{1q_1} \cdots\cdots \lambda_{nq_n} \leqslant \frac{\mu(\sigma, k)}{\mu(\sigma, k\text{-}l)} \leqslant \lambda_1^{\nu_1} \cdots\cdots \lambda_n^{\nu_n},$$

where $q = (q_1, \ldots\ldots, q_n)$ is a position of occurrence of $\mu(\sigma, k\text{-}l)$ and $\nu = (\nu_1, \ldots\ldots, \nu_n)$ is the rank of $\mu(\sigma, k)$,

Proof: Let $\mu(\sigma, K)$ occur at $p = (p_1, \ldots\ldots, p_n)$. Then,

$$\mu(\sigma, k\text{-}l) = (\lambda_{nq})^{k-1} \mid a_q b_q \mid \exp\left\{\sigma \cdot (\lambda_{nq})\right\}$$

$$\geqslant (\lambda_{np})^{k-1} \mid a_p b_p \mid \exp\left\{\sigma \cdot (\lambda_{np})\right\} = \frac{\mu(\sigma, k)}{\lambda_{1p_1} \cdots\cdots \lambda_{np_n}}, \quad \text{Hence,}$$

(3. 3) $$\frac{\mu(\sigma, k)}{\mu(\sigma, k\text{-}l)} \leqslant \lambda_{1p_1} \cdots \lambda_{np_n} \leqslant \lambda_1^{\nu_1} \cdots \lambda_n^{\nu_n}.$$

Again $\mu(\sigma, k) \geqslant (\lambda_{nq})^k \mid a_q b_q \mid \exp\left\{\sigma \cdot (\lambda_{nq})\right\}$

$= \lambda_{1q_1} \cdots \lambda_{nq_n} \mu(\sigma, k\text{-}l)$. Hence,

(3. 4) $\lambda_{1q_1} \cdots \lambda_{nq_n} \leqslant \dfrac{\mu(\sigma, k)}{\mu(\sigma, k\text{-}l)}$; combining (3. 3) and (3. 4) the theorem follows.

Theorem 5. For any $k \geqslant \bar{O}$ and $\sigma \in R^n$,

$$\lambda_{1q_1^*} \cdots \lambda_{nq_n^*} \leqslant \left\{\frac{\mu^*(\sigma, k)}{\mu^*(\sigma, k\text{-}l)}\right\}^{\frac{1}{2}} \leqslant \lambda_1^{\nu_1^*} \cdots \lambda_n^{\nu_n^*},$$

where. $q^* = (\overset{*}{q_1}, \ldots\ldots, \overset{*}{q_n})$ is a position of occurrence of $\mu^*(\sigma, k\text{-}l)$ and $\nu^* = (\nu_1^*, \ldots\ldots, \nu_n^*)$ is the rank of $\mu^*(\sigma, k)$.

Proof : The proof is exactly similar to that of theorem 4.

4. For any $f_1 \in F$, let D be the set of all discontinuities of ν in R^n, where $\nu = (\nu_1, \ldots\ldots, \nu_n)$ is the rank of the maximum term $\mu(\sigma, f_1)$. Also let S denote the set of all $\sigma \in R^n$ at which $\mu(\sigma, f_1)$ is attained by more than one term of the series

(4. 1) $$\sum_{m=1}^{\infty} \mid a_m \mid \exp\left\{\sigma \cdot (\lambda_{nm_n})\right\}.$$

R. K. Das [1] had shown that D and S are identical. (A similar result for entire functions represented by multiple power series was proved by J. G. Krishna [3]).

Hence for $\sigma \in R^n\text{-}D$, $\mu(\sigma, f_1)$ is attained by only one term of the series (4. 1) and the position of that term is $\nu = (\nu_1, \ldots, \nu_n)$.

Hence in such a case, the theorems 3,4 and 5 will take the following forms in theorems 6, 7 and 8 respectively.

Theorem 6. For any $\sigma \in R^n$ - D and $k \in I^n$,

$$(4.2) \quad (\lambda_n \nu) \leqslant \frac{\mu^* (\sigma, k)}{\mu (\sigma, k)} \leqslant (\lambda_n \nu^*)$$

where ν and ν^* are the ranks of μ and μ^* respectively and D is the set of all discontinuities of $\nu(\sigma, k)$ in R^n.

Theorem 7. For any $\sigma \in R^n - D_1$ and $k >> \bar{0}$,

$$(4.3) \quad \lambda_1 \nu_1 (\sigma, k\text{-}I) \ldots \lambda_n \nu_n (\sigma, k\text{-}I) \leq \frac{\mu (\sigma, k)}{\mu (\sigma, k\text{-}I)} \leqslant \lambda_1 \nu_1 (\sigma, k) \ldots \lambda_n \nu_n (\sigma, k),$$

where D_1 is the set of all discontinuities of $\nu(\sigma, k\text{-}I)$.

Corollaries : For any $\sigma \in R^n\text{-}D_1$,

i) $\lambda_1 \nu_1 (\sigma, \bar{0}) \ldots \lambda_n \nu_n (\sigma, \bar{0}) \leqslant \lambda_1 \nu_1 (\sigma, \bar{1}) \ldots \lambda_n \nu_n (\sigma, \bar{1}) \ldots) \leqslant \ldots$

ii) $\dfrac{\mu (\sigma, \bar{1})}{\mu (\sigma, \bar{0})} \leqslant \dfrac{\mu (\sigma, \bar{2})}{\mu (\sigma, \bar{1})} \leqslant \ldots\ldots$

iii) Putting $k = \bar{1}, \bar{2}, \ldots\ldots, \bar{p}$ successively in (4.3) and multiplying the resulting p inequalities and using (i) we have,

$$\lambda_1 \nu_1 (\sigma, \bar{0}) \ldots \lambda_n \nu_n (\sigma, \bar{0}) \leqslant \left\{ \frac{\mu (\sigma, \bar{p})}{\mu (\sigma, \bar{0})} \right\}^{\frac{1}{p}} \leqslant \lambda_1 \nu_1 (\sigma, \bar{p}) \cdots\cdots \lambda_n \nu_n (\sigma, \bar{p}) .$$

Theorem 8. For any $\sigma \in R^n - D_1^*$ and $k \geqslant \bar{0}$

$$(4.4) \quad \lambda_1 \nu_1^* (\sigma, k\text{-}I) \cdots \lambda_n \nu_n^* (\sigma, k\text{-}I) \leqslant \left\{ \frac{\mu^* (\sigma, k)}{\mu^* (\sigma, k\text{-}I)} \right\}^{1/2} \leqslant \lambda_1 \nu_1^* (\sigma, k) \cdots \lambda_n \nu_n^* (\sigma, k)$$

where D_1^* is the set of all discontinuities of $\nu^* (\sigma, k\text{-}I)$.

Corollaries : For any $\sigma \in R^n\text{-}D_1^*$

i) $\lambda_1 \nu_1^* (\sigma, \bar{0}) \cdots \lambda_n \nu_n^* (\sigma, \bar{0}) \leqslant \lambda_1 \nu_1^* (\sigma, \bar{1}) \cdots \lambda_n \nu_n^* (\sigma, \bar{1}) \leqslant \ldots$

ii) $\dfrac{\mu^* (\sigma, \bar{1})}{\mu^* (\sigma, \bar{0})} \leqslant \dfrac{\mu^* (\sigma, \bar{2})}{\mu^* (\sigma, \bar{1})} \leqslant \ldots$

iii) $\lambda_1 \nu_1^*(\sigma, \bar{O}) \cdots \lambda_n \nu_n^*(\sigma, \bar{O}) \leqslant \left\{ \dfrac{\mu^*(\sigma, \bar{p})}{\mu^*(\sigma, \bar{O})} \right\}^{\frac{1}{2p}} \leqslant \lambda_1 \nu_1^*(\sigma, \bar{p}) \cdots$

$\lambda_n \nu_n^*(\sigma, \bar{p})$

Theorem 9. For any $\sigma \in R^n - D \cup D_1$ and $k \in I$ $(k > 0)$,

$$\lambda_1 \nu_1(\sigma, \bar{k}\text{-}1) \cdots \lambda_n \nu_n(\sigma, \bar{k}\text{-}1) \leqslant \left\{ \frac{\mu^*(\sigma, \bar{k})}{\mu(\sigma, \bar{k}\text{-}1)} \right\}^{\frac{1}{k+1}} \leqslant \lambda_1 \nu_1^*(\sigma, k)$$

$\ldots \lambda_n \nu_n^*(\sigma, \bar{k})$,

where D is the set of all dicontinuities of $\nu(\sigma, \bar{k})$and D_1 is the set of ali discontinuities of $\nu(\sigma, \bar{k}\text{-}1)$.

Proof : From (4.2), using (4.3), we have

$\mu^*(\sigma, \bar{k}) \leqslant \mu(\sigma, \bar{k}) (\lambda_n \nu^*(\sigma, \bar{k}))^{\bar{k}}$

$\leq \mu(\sigma, \bar{k}\text{-}1) \lambda_1 \nu_1(\sigma, \bar{k}) \cdots \lambda_n \nu^*(\sigma, \bar{k}) (\lambda_n \nu_n^*(\sigma, \bar{k}))$

Hence,

(4.5) $$\frac{\mu^*(\sigma, \bar{k})}{\mu(\sigma, \bar{k}\text{-}1)} \leqslant \lambda_1 \nu_1(\sigma, \bar{k}) \cdots \lambda_n \nu_n(\sigma, \bar{k}) (\lambda_n \nu^*(\sigma, \bar{k}))^{\bar{k}}.$$

Again, from (4.2), $\mu^*(\sigma, \bar{k}) \geqslant \mu(\sigma, \bar{k}) (\lambda_n \nu(\sigma, \bar{k}))^{\bar{k}} \geqslant \mu(\sigma, \bar{k}\text{-}1) \times$

$\lambda_1 \nu_1(\sigma, \bar{k}\text{-}1) \ldots \lambda_n \nu_n(\sigma, \bar{k}\text{-}1) (\lambda_n \nu(\sigma, \bar{k}))^{\bar{k}}.$

Hence,

(4.6) $$\frac{\mu^*(\sigma, \bar{k})}{\mu(\sigma, \bar{k}\text{-}1)} \geqslant \lambda_1 \nu_1(\sigma, \bar{k}\text{-}1) \cdots \lambda_n \nu_n(\sigma, \bar{k}\text{-}1) (\lambda_n \nu(\sigma, \bar{k}))^{\bar{k}}.$$

From (4.5) and (4.6), using (4.2) and the Corollary (i) of theorem 7, the result follows. The second author wishes to express his gratitude to the "Govt. of India" for awarding him a scholarship.

REFERENCES

1. Das R. K. : Study on the growth and the space of entire functions of several complex variables represented by multiple Dirichlet series - Ph. D. thesis, Calcutta University (1985).

2. Janusanskas A. I. : Elementary theorems on the convergence of double Dirichlet series-Soviet Math. Dokl Vol. 18 (1977) No. 3. pp. 610-614.

3. Krishna J. G. : Maximum term of a power series in one and several complex variables - Pacific Jour. Math. Vol 29 No. 3 (1969) pp 609 - 622.

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Dept. of Pure Math
Calcutta University

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A NOTE ON PAIRWISE SEMIOPEN SETS IN A SUBSPACE OF A BITOPOLOGICAL SPACE

M. N. MUKHERJEE

ABSTRACT In this paper, pairwise semiopen and semiclosed sets and their different properties, especially with regard to subspaces of a bitopological space, have been studied.

Norman Levine [2] introduced semiopen and semiclosed sets in a topological space and investigated some of their properties. This paved the way for a very wide research in this direction. The corresponding notions were introduced in the more generalized and richer structure of bitopological spaces by S. Bose [1]. It is the purpose of this paper to make a further study of the concepts of pairwise semiopen, semiclosed sets and pairwise semiclosure and semiinterior of sets of a bitopological space with special reference to subspaces. Such a study turns out to be of immense help and sometimes indispensible when study of certain semitopological concepts e. g semiconnectedness in bitopological space, is carried out in terms of subspace topology. To make the expositions self-contained as far as practicable, we quote a few definitions and results from [1] as follows. By (X, T_1, T_2) or simply by X we shall mean a bitopological space. A subset A of (X, T_1, T_2) will be called (1, 2)-semiopen [1] in X, written as (1,2)-SO(X), if there exists a T_1 open set B such that $B \subset A \subset T_2\text{-cl}B$ (T_2-clB denotes the T_2-closure of B in X). Similarly, sets which are (2, 1)-semiopen in X are defined. A ($\subset X$) is called (i, j)-semiclosed (in short, (i, j)-Scl (X)) if X-A is ij -SO (X) where i, j =1, 2 and $i \neq j$. The set of all subsets that are (i, j)-semiopen (semiclosed) in X will be denoted by (T_i, T_j)-SO (X) (respectively by (T_i, T_j)-SCl (X)), for i, j =1, 2 and $i \neq j$. The subset A of X is called pairwise semiopen or simply p. s. o. (pairwise semi-closed or simply p. s. cl.) in X [1] if A is (1, 2)-SO (X) and (2,1)-SO (X) (respectively, (1, 2)-SCl (X) and (2, 1)-SCl (X)).

It has been shown in [1] that a set in (X, T_1, T_2) may be semiopen in both (X, T_1) and (X, T_2) but is neither $(1, 2)$-SO(X) nor is $(2, 1)$-SO(X). Also a set may be p, s. o. without being either T_1 or T_2 semiopen. Obviously every T_i-open set in (X, T_1, T_2) is ij-SO(X) for i, j $=1, 2$ and $i \neq j$, but the converse is false. Though the union of any collection of sets, each of which is (i, j)-SO(X), is also so, the intersection of even two sets that are (i, j)-SO(X), may not be (i, j)-SO(X) $(i, j = 1, 2$ and $i \neq j)$. This is seen from the following example. :

EXAMPLE 1 Let $X = \{a, b, c, d\}$, $T_1 = \left\{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\right\}$ and $T_2 = \left\{X, \emptyset, \{a\}, \{b, d\}, \{a, c\}, \{a, b, d\}\right\}$. Then (T_1, T_2)—SO$(X)=$ $\left\{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}, \{a, c\}, \{b, c, d\}\right\}$ and (T_2, T_1)—SO$(X)=$ $\left\{X, \emptyset, \{a\}, \{b, d\}, \{a, c\}, \{a, b, d\}, \{a, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\right\}$.

Now, $\{b, c\}$, $\{a, c\}$ are $(1, 2)$—SO(X) but $\{b, c\} \cap \{a, c\} = \{c\}$ is not so, though $\{b, c\} \in T_1$. Again $\{a, c, d\}$ and $\{b, c, d\}$ are $(2, 1)$—SO(X) but their interesection i. e., $\{c, d\}$ is not $(2, 1)$—SO(X).

Theorem 1 In a bitopological space (X, T_1, T_2) if A is $(1,2)$-SO(X) and $B \in T_1 \cap T_2$ then $A \cap B$ is $(1, 2)$-SO(X).

Proof. There is T_1-open set V such that $V \subset A \subset T_2\text{-cl}V$. Since B is T_2-open, we have $T_2\text{-cl } V \cap B \subset T_2\text{-cl }(V \cap B)$.

Thus $V \cap B \subset A \cap B \subset T_2\text{-cl}V \cap B \subset T_2\text{-cl }(V \cap B)$. Since $V \cap B$ is T_1-open, it now follows that $A \cap B$ is $(1, 2)$-SO(X).

Remark 1. It is clear that in the above theorem if A be $(2, 1)$-SO(X), then $A \cap B$ is also so.

Definition 1 (a) Let x be a point of (X, T_1, T_2). A subset N of X is called a $(1, 2)$-semi-neighbourhood of x [1] in X if there is a $(1, 2)$-SO(X) set B (say) such that $x \in B \subset N$. Similarly, $(2, 1)$-semi-neighbourhood of x in X is defined.

Definition 1 (b) Let $A \subset (X, T_1, T_2)$. A point x of X is said to be a $(1, 2)$-semi accumulation point of A in X if every $(1, 2)$-semi-neighbourhood of x intersects A in at least one point other than X. Similar goes the definition of $(2, 1)$-semi accumulation point of A in X.

Definition 1 (c) Let $A \subset (X, T_1, T_2)$. The intersection of all (i, j)-SCl (X) sets, each containing A, is called the (i, j)-semi-closure of A in X [1] and is denoted by $\underline{A}_{T_i(T_j)}$, where i, j=1, 2 and i $\neq$ j.

Therem 2 (Bose [1]) Let $A \subset (X, T_1, T_2)$. (a) A is (i, j)-SCl(X) If and only if $A = \underline{A}_{T_i(T_j)}$,

(b) $x \in \underline{A}_{T_i(T_j)}$ if and only if x is either a point of A or a (i, j)-semi accumulation point of A in X. In (a) and (b), i, j=1, 2 and i $\neq$ j.

From Theorem 2, we have

Theorem 3 A set A in (X, T_1, T_2) is (i, j)-SCl(X) if and only if A contains the set of all (i, j)-semi accumulation points of A in X (i, j=1, 2 and i $\neq$ j),

Theorem 4 Let $Y \subset (X, T_1, T_2)$, Then a subset U of Y is (1, 2)- SO(Y) $\Rightarrow U = V \cap Y$, for some (1, 2)-SO(X) set V (here (1, 2)-SO(Y) means U is (1, 2)-semiopen in the space $(Y, (T_1)_Y, (T_2)_Y)$.

Proof. $U(\subset Y)$ is (1, 2)-SO(Y)

$\Rightarrow$there is a $W \in (T_1)_Y$ such that $W \subset U \subset (T_2)_Y\text{-cl}W$,

But, $W = V_1 \cap Y$, where $V_1 \in T_1$.
Now, $U = (V_1 \cup (U - W)) \cap Y$ and $V_1 \cup (U - W) \subset T_2\text{-cl}V_1$.
In fact, $U - W \subset U = U \cap Y \subset (T_2)_Y\text{-cl}W \cap Y \subset T_2\,\text{cl}W \cap Y$

$= T_2\text{-cl}(V_1 \cap Y) \cap Y \subset T_2\text{-cl}\, V_1 \cap T_2\text{-cl}\, Y \cap Y \subset T_2\text{-cl}\, V_1 \cap Y \subset T_2\text{-cl}\, V_1$,
Thus putting $V_1 \cup (U - W) = V$ we see that $U = V \cap Y$,
where $V_1 \subset V \subset T_2\text{cl}\, V_1$ and $V_1 \in T_1$, i,e., V is (1, 2)-SO(X).

Remark 2 Converse of Theorem 4 is false even if Y be T_1-open: as is seen from the example below.

Example 2. Consider (X, T_1, T_2) of Example 1 and $Y = \{a, c, d\}$.

We have $(T_1)_Y = \Big\{Y, \emptyset, \{a\}, \{c\}, \{a, c\}\Big\}$,

$$(T_2)_Y = \Big\{Y, \emptyset, \{a\}, \{d\}, \{a, c\}. \{a, d\}\Big\}$$

$$\Big((T_1)_Y, (T_2)_Y\Big) - SO(Y) = \Big\{Y, \emptyset, \{a\}, \{c\}, \{a, c\}\Big\}$$

$$((T_2)_Y, (T_1)_Y)-SO(Y) = \{Y, \emptyset, \{a\}, \{d\}, \{a, c\}, \{a, d\}\}$$

Now, $\{b, c, d\} \in (T_1, T_2)-SO(X)$ but $\{b, c, d\} \cap Y = \{c, d\} \notin \left((T_1)_Y, (T_2)_Y\right)-SO(Y)$.

Also, $\{b, c, d\} \in (T_2, T_1)-SO(X)$ but $\{b, c, d\} \cap Y = \{c, d\} \notin$

$\left((T_2)_Y, (T_1)_Y\right)-SO(Y)$.

Again, $Z = \{a, b, d\}$ is T_2—open.

Then, $(T_1)_z = \{Z, \phi, \{a\}, \{b\}, \{a, b\}\}$

$(T_2)_z = \{Z, \phi, \{a\}, \{b, d\}\}$.

$\left((T_1)_z, (T_2)_z\right)-SO(Z) = \{Z, \phi, \{a\}, \{b\}, \{a, b\}, \{b, d\}\}$.

$\left((T_2)_z, (T_1)_z\right)-SO(Z) = \{Z, \phi, \{a\}, \{b, d\}, \{a, d\}\}$.

Now, $\{a, b, c\} \in (T_2, T_1)-SO(X)$, but $\{a, b, c\} \cap Z$

$= \{a, b\} \notin \left((T_2)_z, (T_1)_z\right)-SO(Z)$, though $Z \in T_2$.

Theorem 5 Let $Y \subset (X, T_1, T_2)$. A subset F of Y is $(1, 2)-SCl(Y) \Rightarrow F = V \cap Y$, for some $(1, 2)-SCl(X)$ set V.

Proof. We have $Y - F$ is $(1, 2)-SO(Y)$, Then by Theorem 4, $Y - F = Y \cap O$, where O is $(1, 2)-SO(X)$. Then $F = Y - (Y \cap O) = (X - O) \cap Y$, where $V = (X - O)$ is $(1, 2)-SCl(X)$.

Remark 3 The converse of Theorem 5 is false even if Y is both T_1—closed and T_2—closed. This is shown in the next example

Example 3 Let $(X\ T_1, T_2)$ be same as in Example I, and let $Y = \{b, c, d\}$. Y is T_1—closed as well as T_2—closed.

We have $(T_1)_Y = \{Y, \phi, \{b, c\}\}$, $(T_2)_Y = \{Y, \phi, \{c\}, \{b, d\}\}$,

$\left((T_1)_Y, (T_2)_Y\right)-SO(Y) = \{Y, \phi, \{b, c\}\}$.

$\left((T_2)_Y, (T_1)_Y\right)-SO(Y) = \{Y, \phi, \{c\}, \{b, d\}, \{b, c\}\ \{c, d\}\}$.

Now, $\{b, d\}$ is $(1, 2)-SCl(X)$, but $\{b, d\} \cap Y = \{b, d\}$ is not $(1, 2)-SCl(Y)$, although Y is T_1—closed and T_2—closed.

From Theorem 5, it immediately follows that

Theorem 6 Let $A \subset Y \subset (X, T_1, T_2)$,

(a) If $x\ (\epsilon Y)$ is a (i, j)-semi-accumulation point of A in X then x is also a (i, j)-semi-accumulation point of A in $(Y, (T_i)_Y, (T_j)_Y)$.

(b) $\underset{-}{A}_{T_i(T_j)} \cap Y \subset \underset{-}{A}_{(T_i)_Y((T_j)_Y)}$

In (a) and (b) above, i, j=1, 2 and $i \neq j$.

Remark 4 The reverse inclusion in Theorem 6(b) does not, in general, hold as is seen from the next example.

Example 4 Consider (X, T_1, T_2) and Y of Exampla 2.

Let $A=\{a\} \subset Y$. Then

$\underset{-}{A}_{(T_1)_Y((T_2)_Y)}=\{a, d\}$, $\underset{-}{A}_{(T_2)_Y((T_1)_Y)}=\{a, c\}$.

But $\underset{-}{A}_{T_1(T_2)}=\underset{-}{A}_{T_2(T_1)}=\{a\}$.

Theorem 7 Let $A \subset Y \subset (X, T_1, T_2)$. If $Y \epsilon T_j$, then

$\underset{-}{A}_{T_i(T_j)} \cap Y=\underset{-}{A}_{(T_i)_Y((T_j)_Y)}$, where i, j = 1, 2 and $i \neq j$.

Proof. We prove the theorem by taking i = 1 and j = 2. Similar will be the proof when i = 2 and j = 1.

Let us put $\underset{-}{A}_{(T_1)_Y((T_2)_Y)}=A_1$

By virtue of theorem 6, it is enough to show that

$A_1 \subset \underset{-}{A}_{T_1(T_2)} \cap Y$, where $Y \epsilon T_2$. Let $x \epsilon A_1$ and B be (1, 2)-SO(X) such that $x \epsilon B$. Then there is a T_1-open set 0 such that $0 \subset B \subset T_2\text{-cl } 0$.

Then $O \cap Y \epsilon (T_1)_Y$ and $O \cap Y \subset B \cap Y \subset T_2\text{-cl } O \cap Y \subset (T_2)_Y\text{cl}(O \cap Y)$ (since $Y \epsilon T_2$),Thus $B \cap Y$ is (1,2)-semi neighbourhood of x in Y. Since $x \epsilon A_1$, by Theorem 2 (b) $A \cap (B \cap Y) \neq \emptyset$ and hence $A \cap B \neq \emptyset$, i.e., $x \epsilon \underset{-}{A}_{T_1(T_2)}$ Thus $A_1 \subset \underset{-}{A}_{T_1(T_2)}$ and this completes the proof.

Remark 5 Theorem 7 may not hoid if we replace the hypothesis "$Y \epsilon T_j$" by "$Y \epsilon T_i$" or by "$Y \epsilon (T_j, T_i)$-SO (X)".

This is verified in the following example.

Example : 5 We consider (X, T_1, T_2) and Z of Example 2.

Let $A=\{d\} \subset Z$. Then $\underset{-}{A}_{T_2(T_1)}=\{d\}$ and $\underset{-}{A}_{(T_2)_Z((T_1)_Z)}=\{b,d\}$.

Thus $\underset{-}{A}_{T_2(T_1)} \cap Z \neq \underset{-}{A}_{(T_2)_Z}\big((T_1)_Z\big)$, though $Z \in T_2$.

Also Y of Example 2 is (2,1)—SO (X) and if

A={a} ⊂Y, then as shown In Example 4,

$$\underset{-}{A}_{T_2(T_1)} \cap Y \neq \underset{-}{A}_{(T_2)_Y}\big((T_1)_Y\big).$$

Theorem 8 (Bose [1]) Let $A \subset Y \subset (X, T_1, T_2)$. Then A is (i,j)—SO(X)⇒A is (i,j)—SO(Y), where i,j=1,2 and $i \neq j$.

Remark 6 Converse of Theorem 8 is false as is shown by the example that follows.

Example 6 Consider the bitopological space (X, T_1, T_2) and the subset Y of Example 2. Then {c} (⊂Y) is (1,2)—SO(Y) but is not (1,2)—SO(X) and {d} is (2,1)-SO(Y) but is not (2,1)—SO(X).

Theorem 9 Let $A \subset Y \subset (X, T_1, T_2)$. If $Y \in T_1$, then A is (1,2)—SO (Y) if and only if A is (1,2)—SO(X).

Proof. Let A be (1,2)-SO(Y). Then there exists $(T_1)_Y$- open set O such that $O \subset A \subset (T_2)_Y\text{-cl}\ O$. Since Y Is T_1-open, we have $O \in T_1$ and also $O \subset A \subset (T_2)_Y\text{-cl}\ O \subset T_2\text{-cl}\ O$. Hence A is (1,2)-SO(X).

The other part follows from Theorem 8.

Remark 7 It is evident that the indices 1 and 2 can be interchanged in Theorem 9.

Remark 8 Theorem 9 does not hold, in general, if we replace condition "$Y \in T_1$" by "$Y \in T_2$" or by "$Y \in$ (1,2)-SO(X)". This is evident from the next example.

Example 7 Let us consider the bitopological space (X, T_1, T_2) and the subset Y of X of Example 2. We see that $Y \in$ (2,1)-SO(X) but $\{d\} \in ((T_2)_Y, (T_1)_Y)$-SO(Y), whereas $\{d\} \notin (T_2, T_1)$-SO(X). Again Z of the same Example 2 is T_2-open. Now {b} is (1,2)-SO(Z) but {b} is not (1,2)-SO(X).

Remark 9 Let $A \subset Y \subset (X, T_1, T_2)$. Then A is (1,2)-SCl(Y) does not imply nor is implied by the fact that A is (1,2)-SCl(X). In fact, we have the following.

Example 8 Consider Y and (X, T_1, T_2) of Example 2. Here {a} (⊂Y) is (1,2)-SCl(X) but it is not (1,2)-SC1(Y).

Again, {c,d} (⊂Y) is (1,2)-SC1(Y) but is not (1,2)-SC1(X).

Theorem 10 Let $A \subset Y \subset (X, T_1, T_2)$. If A is (i,j)-SC1(Y) and Y is(i,j)-SC1(X), then A is(i, j)-SC1(X), where i,j=1,2 and $i \neq j$.

Proof. follows easily from Theorem 5 and the fact that intersection of two (i,j)-SC1(X)-sets is also so.

Remark 10 A subset A of Y $\subset$ (X, T_1,T_2) may be (1,2)-SC1(X) but may not be (1,2)-SCl(Y), even if Y is pairwise semiopen, closed in both T_1 and T_2 and pairwise semi closed.

This is seen in the example that follows.

Example 9 Consider Y, and (X, T_1, T_2) of Example 3. Then Y is pairwise semi open, pairwise semi closed and closed in both T_1, T_2. Now {b,c} is (2, 1)-SC1 (X) but it is not (2, 1)-SCl(Y).

Remark 11 It follows from Example 9 that if A is (i, j)-SC1(X) and Y$\subset$X, then A $\cap$ Y may not be (i, j)-SC1(Y) (i, j=1,2 ; i $\neq$ j), even if Y is both T_1 and T_2-closed (and hence pairwise semi closed).

Definition 2 [1] Let A$\subset$ (X, T_1, T_2). The union of all (1, 2)-SO(X) sets, each contained in A, is called the (1, 2)-semi interior of A in X and is denoted by (T_1, T_2)-SInt(A). Similarly (T_2, T_1)-SInt (A) is defined. It is proved in [1] that a set A$\subset$ (X, T_1, T_2) is (i, j)-SO(X) if A=(T_i, T_j)-SInt(A), where i, i=1, 2 and i $\neq$ j.

Theorem 11 Let A$\subset$Y $\subset$ (X, T_1, T_2). Then

(a) (T_1, T_2)-SInt(A) $\subset$ ($(T_1)_Y$, $(T_2)_Y$)-SInt (A), where the reverse inclusion does not hold, in general,

(b) (T_1, T_2)-SInt (A)=($(T_1)_Y$, $(T_2)_Y$)-SInt(A) if Y is T_1-open.

Proof. (a) By virtue of Theorem 8, obviously (T_1, T_2)-SInt(A) $\subset$($(T_1)_Y$, $(T_2)_Y$)-SInt(A).

Now, let us consider (X, T_1, T_2) and Y of Example 2.

Let A={c, d}. Then (T_1, T_2,)-SInt (A)=(T_2, T_1)-SInt (A)=$\emptyset$.
But ($(T_1)_Y$, $(T_2)_Y$)-SInt (A)={c} and ($(T_2)_Y$, $(T_1)_Y$) SInt (A)={d}.

(b) follows from theorem 9.

Remark 12 The converse of Theorem 11 (b) does not hold, in general. For example, if X is the real line and T_1=T_2=the usual topology and Y=[0, 1), then the equality in (b) holds for every A$\subset$ Y but clearly Y is neither T_1 open nor T_2 open,

Theorem 12 In a bitopological space (X, T_1, T_2), T_i=S_i, where S_i=$\{$ T_i-int A : A $\in$ (T_i, T_j)-SO(X)$\}$ (i,j=1, 2 and i $\neq$ j).

Proof. It is simple.

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REFERENCES

[1] Bose S. : Semiopen sets, semi continuity and semi open mappings in bitopological spaces, Bull. Cal. Math. Soc. 73 (1981) 237-243

[2] Levine Norman : Semiopen sets and semi continuity in topological spaces, Amer. Math. Monthly 70 (1963). 36-41

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Depar ment of Pure Mathematics,
University of Calcutta.

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ON NORMAL PSEUDO-IDEALS IN SEMIGROUPS.

T. K. Dutta

1. Introduction.

A semigroup S is called normal if $Sx=xS$ for all elements x of S [7]. A pseudo-ideal A of a semigroup S is called normal if A is a normal subsemigroup of S i. e. if $xA=Ax$ for all $x \in A$. In this paper we have studied some properties of normal pseudo-ideals. The purpose of this paper is to give some properties which characterise normal semigroups and normal regular semigroups in terms of normal pseudo-ideals and bi-ideals.

Let NB (S) denote the set of all normal pseudo-ideals and bi-ideals of a semigroup S. Then NB (S) is a semigroup under the multiplication of subsets and N (S), the set of all normal pseudo-ideals of S is a commutative subsemigroup of NB (S). We have shown that a semig oup S is normal if and only if NB (S) is normal. Lastly we have characterised regula ity of all those semigroups in which all the pseudo-ideals are normal.

2. In [8] Sen has shown that in a group every pseudo-ideal is a normal pseudo-ideal; the result is also true in a commutative semigroup The following example shows that there are also semigroups which are neither groups nor commutative semigroups but contain normal pseudo-ideals.

2. 1 Example. Let $S=G \times J$ where G is a noncommutative group and J is the set of all integers; then S is a semigroup with respect to the multiplication defined componentwise. Let $A=G \times J^{+}$ be a subset of S where J^{+} denotes the set of all nonnegative integers Then A is a pseudo-ideal of S. Also for any element x of S, $xA=Ax$. So A is a normal pseudo-ideal of S.

2 2 Proposition. A normal subsemigroup A of a semigroup S will be a pseudo-ideal if and only if $xAx \subseteq A$ for every $x \in S$.

Proof. Let a normal subsemigroup A of a semigroup S be a pseudo-ideal of S Then $xAx = xxA = x^2A \subseteq A$. On the other hand if for a normal subsemigroup A of a semigroup S, $xAx \subseteq A$ then $x^2A = xxA = xAx \subseteq A$ for every $x \in S$. Similarly $Ax^2 \subseteq A$ for every $x \in S$. So A is a pseudo-ideal of S.

2. 3 Proposition. Every one-sided normal pseudo-ideal of a semigroup S is a pseudo-ideal (two-sided) of S.

Proof. Let A be a normal left pseudo-ideal of a semigroup S and $x \in S$. Then $Ax^2 = x^2A \subseteq A$. So A is also a right pseudo-ideal of S. Similarly we can show that if A is a right pseudo-ideal then A is also a left pseudo-ideal. Hence the proposition.

2. 4 Proposition. Let N (S) denote the set of all normal pseudo-ideals of a semigroup S ; then N (S) is a commutative semigroup under the multiplication of subsets.

Proof. Let $A_1, A_2 \in N(S)$. Let $a_1a_2, b_1b_2 \in A_1A_2$ where $a_1, b_1 \in A_1$ and $a_2, b_2 \in A_2$. Then $a_1a_2b_1b_2 = a_1b_1a_2'b_2 \in A_1A_2$ where $a_2b_1 = b_1a_2'$, $a_2' \in A_2$ and also for any $x \in S$ $x^2 (A_1A_2) = (x^2A_1) A_2 \subseteq A_1 A_2$. So $A_1 A_2$ is a left pseudo-ideal of S. Since A_1, A_2 are both normal, $x (A_1A_2) = (A_1 A_2) x$. Hence $A_1 A_2 \in N$ (S). Lastly let $a_1 \in A_1$ then $a_1 A_2 = A_2 a_1$. So $A_1 A_2 = A_2 A_1$. Evidently $A_1 (A_2 A_3) = (A_1 A_2) A_3$. Consequently N (S) is a commutative semigroup.

2. 5 Proposition. Let NB (S) denote the set of all normal pseudo-ideals and bi-ideals of a semigroup S ; NB (S) is a semigroup under the multiplication of subsets.

Proof. It follows from the above proposition that the product of two normal pseudo-ideals is a normal pseudo-ideal. Also the product of two bi-ideals of S is a bi-ideal [3]. Let $A \in N$ (S) and $B \in B$ (S), the set of all bi-ideals of S. Now (AB) (AB) $=$ A (BAB) $\subseteq$ AB and (AB) S (AB) $=$ AB (SA) B $\subseteq$ AB. So AB $\in$ B (S) $\subseteq$ NB (S). Sincc A is normal AB $=$ BA, So BA $\in$ NB (S). Evidently A (BC) $=$ (AB) C for A, B, C $\in$ NB (S). Hence the proposition.

2. 6 Proposition. B (S) is an ideal of NB (S).

Proof. Let B $\in$ B (S) and X $\in$ NB (S). Then (BX) (BX) $=$ (B X B) X $\subseteq$ BX and also (BX) S (BX) $=$ B (XS) BX $\subseteq$ BX So BX is a bi-ideal of S i, e BX $\in$ B (S). Consequently B (S) is a right ideal of NB (S). Similarly we can show that B (S) is also a left ideal of NB (S). Hence B (S) is an ideal of NB (S).

2. 7 Theorem. A semigroup S is normal if and only if NB(S) is normal.

Proof. Let S be a normal semigroup and X, A $\in$ NB (S). Let a $\in$ A ; then aX $\subseteq$ aS $=$ Sa $\subseteq$ NB (S). a and so A NB (S) $\subseteq$ NB (S) A. Similarly we can prove that the converse

inclusion holds. Thus we obtain that $A\,NB(S) = NB(S)\,A$ for all $A \in NB(S)$ So $NB(S)$ is normal. Conversely let $NB(S)$ be normal. In order to prove that S is normal, let $x \in S$. Then for some $A \in NB(S)$ we have $xS \subseteq (x)_B S = A(x)_B \subseteq S(x)_B \subseteq Sx$ where $(x)_B = \{xSx \cup x \cup x^2\}$ is the bi-ideal generated by x and hence $(x)_B \in B(S) \subseteq NB(S)$. Similarly we can prove that the converse inclusion holds. So S is normal.

2. 8 Theorem. Let S be a normal semigroup ; then the following conditions are equivalent

(1) S is a regular semigroup,

(2) $A \cap B = \bar{B}A$ where A is a left pseudo-ideal and B is a bi-ideal of S,

(3) $A \cap B = A\bar{B}$ where A is a right pseudo-ideal and B is a bi-ideal of S.

Proof. (1) $\Rightarrow$ (2) . Let S be a normal semigroup which is also regular. Let A, B be respectively a left pseudo-ideal and a bi-ideal of S. Then $\bar{B}A \subseteq A$. Let $b^2 a \in \bar{B}A$ where $b \in B$ and $a \in A$. Since S is normal, $bS = Sb$. So $b^2 a = bba \in bbS = bSb \subseteq B$. Consequently $\bar{B}A \subseteq B$. Combining the above two inclusions we get $\bar{B}A \subseteq A \cap B$. Conversely, let $c \in A \cap B$. Since S is regular and normal, $c = cxc = cxcxcxcxc$ (since xc is idempotent, $x \in S$) $= cxx_1c.\,cxx_1c.\,c = (cxx_1c)^2 c \in \bar{B}A$ ($c \in B$ implies that $cxx_1c \in B$). Therefore $A \cap B \subseteq \bar{B}A$. Hence $A \cap B = \bar{B}A$.

(2) $\Rightarrow$ (1). Let $c \in S$. Since S is a left pseudo-ideal we have $S \cap (c) = \overline{(c)}\,S$ i, e $(c) = \overline{(c)}\,S$ where (c) denotes the ideal generated by c and hence (c) is a bi-ideal of S. Since S is normal, $(c) = \{c \cup xc : x \in S\}$ Now $c \in (c) = \overline{(c)}\,S$ implies that $c = c^2 y$ or $(xc)^2 z$ where $x, y, z \in S$. Since S is normal, we can write $c = cxc$ for some $x \in S$. So c and hence S is regular.

Similarly we can show that (1) and (3) are equivalent Hence the theorem.

2. 9 Lemma. ([6] , Corollary II. 4. 12) For an idempotent semigroup S the following conditions are equivalent.

(1) S is normal

(2) S is commutative

2. 10 Lemma. ([1] Theorem 7 6) Following conditions concerning a regular semigroup S are equivalent.

(1) S is normal

(2) es = Se for all idempotents ϵ of S.

2. 11 Lemma. ([4] Theorem 2) For a semigroup S the following conditions are equivalent.

(1) S is a semilattice of groups

(2) S is regular and eS = Se for all idempotents e of S.

2. 12 Lemma, ([2]) For a semigroup S the following conditions are equivalent.

(1) S is regular.

(2) B (S) is regular.

2. 13 Theorem. A normal semigroup S is a semilattice of groups if and only if NB (S) is a semilattice.

Proof : Let S be a normal semigroup which is a semilattice of groups. Then by lemma 2.11 S is regular. Let A ϵ NB (S). If A ϵ N (S) and a ϵ A then a = axa = axaxa (since xa is idempotent, x ϵ S) = ax^2a_1 a ϵ AA ($a_1 \epsilon$ A) So A $\subseteq$ AA. On the other hand AA $\subseteq$ A So A = AA Again, if A ϵ B (S) and a ϵ A then a = axa = axaxa = a ax_1xa (since S is normal, xa = ax_1) ϵ AA. So A $\subseteq$ AA. Also AA $\subseteq$ A. Hence for all A ϵ NB (S), A = AA. So NB (S) is idempotent Since S is normal, by theorem 2,7 NB (S) is normal. Hence by lemma 2.9, NB (S) is commutative. Hence NB (S) is a commutative idempotent semigroup i. e, a semilattice. Conversely, let NB (S) be a semilattice. So NB (S) is an idempotent semigroup and hence regular. Since B (S) is an ideal of NB (S), B (S) is also regular. So it follows from lemma 2.12 that S is regular normal semigroup whence by lemma 2.10 and lemma 2.11 it follows that S is a semilattice of groups.

2.14 Theorem. For a normal semigroup S the following conditions are equivalent.

(1) S is regular,

(2) S is left regular,

(3) S is right regular,

(4) S is completely regular,

(5) $a^n = a\, x^{n-1}$ for all a ϵ S and for every integer n $\geqslant$ 2

(6) NB (S) Is idempotent,

(7) NB (S) is completely regular,

(8) NB (S) is regular,

(9) B (S) is regular.

Proof, It follows from theorem 6.6 of [1] that (1) to (4) are equivalent. Now we assume (4). Let $a \in S$. Then $a=axa$ for some $x \in S$ and $ax=xa$. So $a=a^2x=aax=aa^2xx=a^3x^2$. Continuing this we get $a=a^nx^{n-1}$ for every integer $n\geqslant 2$

(5) $\Rightarrow$ (6). Let $A \in NB$ (S). If $A \in N$ (S) and $a \in A$ then $a=a^3x^2=a^2 \cdot ax^2 \in AA$. So $A \subseteq AA$ Also $AA \subseteq A$. Thus $A=AA$. If $A \in B$ (S) and $a \in A$ then $a=a^3x^2=a\,a\,ax^2=a\,aya \in AA$ (since S is normal, $ax^2=ya$ for some $y \in S$). So $A \subseteq AA$. Also $AA \subseteq A$. Thus in this case also $A=AA$. So NB (S) is idempotent. (6) $\Rightarrow$ (7) $\Rightarrow$ (8) are obvious. Next we assume (8). Since B (S) is an ideal of NB (S), B (S) is regular. Lastly we assume (9). Since B (S) is regular, it follows by lemma 2.12 that S is regular.

A semigroup S is called viable if $ab=ba$ whenever ab and ba are idempotents. A viable semigroup has been studied by M. S. Putcha and J. Weissglass [5]

2.15 Lemma. ([1] Theorem 7.6) For a regular semigroup S the following conditions are equivalent,

(1) S is normal

(2) B (S) is viable

2.16 Lemme. ([5] Theorem 6) If a semigroup S is a semilartice of groups then it is viable.

2.17 Theorem. A regular semigroup S is normal if and only if NB (S) is viable.

Proof Let S be a regular normal semigroup. Then NB (S) is also normal (Theorem 2.7), Also NB (S) is regular (Theeoem 2,14), So NB (S) is regular and normal. Hence NB (S) is a semilattice of groups. So by lemma 2.16, NB (S) is viable. Conversely we assume that NB (S) is viable. Since the property of being viableis hereditary, it follows that B (S) is viable whence by lemma 2.15 it follows that S is normal.

2.18 Theorem. The following conditions concerning a semigroup S are equivalent.

(1) S is a semilattice of groups,

(2) S is regular and normal,

(3) NB (S) is regular and normal,

(4) NB (S) is a semilattice of groups,

(5) NB (S) is regular and viable,

(6) NB (S) is a completely regular semigroup and every bi-ideal of S is two-sided.

(7) B (S) is a completely regular semigroup and every bi-ideal of S is two-sided.

Proof. (1) $\Rightarrow$ (2) follows from the lemma 2.10 and the lemma 2.11. (2) $\Rightarrow$ (3) follows from the proposition 2.7 and the theorem 2.14. (3) $\Rightarrow$ (4) follows from the lemma 2.10 and the lemma 2.11. (4) $\Rightarrow$ (5) follows from the lemma 2.11 and the lemma 2.16. (5) $\Rightarrow$ (6). Since NB (S) is regular and viable it follows readily that NB (S) is a completely regular semigroup. Also since B (S) is an ideal of NB (S) it follows that B (S) is also regular and viable and so every bi-ideal of S is two-sided (Theorem 7.7 of [1]). (6) $\Rightarrow$ (7) follows since B (S), is an ideal of NB (S). Lastly (7) $\Rightarrow$ (1) follows from the theorem 7.7 of [1]. Hence the theorem.

3. In a commutative semigroup or in a group we have noted that every pseudo-ideal is a normal pseudo-ideal. There are also semigroups which are neither commutative semigroups nor groups and in which every pseudo-ideal is normal. In fact the semigroup given in example 2.1 is a semigroup of this type. Now we shall study those semigroups in which every pseudo-ideal is normal. Throughout this article by a semigroup S we shall mean a semigroup in which all the pseudo-ideals are normal.

3.1 Theore. A semigroup S will be regular if and only if $A=\bar{A}A$ where A is a pseudo-ideal of S.

Proof. Let S be a regular semigroup and A be a pseudo-ideal of S. Let $a \in A$, Since S is regular $a=axa$ for some $x \in S$. Now $a=axa=axaxaxaxa$ (since xa is idemotent)$=ax^2a_1\, ax^2a_1\, a \in AA$ (since A is normal, $ax=xa_1$ for some $a_1 \in A$) So $A \subseteq \bar{A}A$, Also $\bar{A}A \subseteq A$, Henca $\bar{A}A=A$, Conversely we assume that $A=\bar{A}A$ for all pseudo-ideals A of S. Let $a \in S$ and (a) be the ideal generated by a. Since every ideal is a pseudo-ideal $(a)=\overline{(a)}\,(a)$ Now $a \in (a)=\overline{(a)}\,(a)$ implies $a=axa$ for seme $x \in S$ (Since S is normal as S is a pseudo-ideal of itself). So a and hence S is regular.

3.2 Theorem. The following conditions concerning a semigroup S are equivalent.

(1) S is regular,

(2) S is left regular,

(3) S is right regular,

(4) S is completely regular,

(5) $a=a^n x^{n-1}$ for every element a ef S and for every integer $n \geqslant 2$.

(6) N(S) is idempotent

(7) N(S) is regnlar

Proof. Since every pseudo-ideal of S is normal S is also normol, so it follows frem the theofem 2.14 that (1) to (5) are equivalent, Now we assume (5). Let $A \in N(S)$ and $a \in A$. Then $a=a^3x^2=a^2.\ ax^2 \in AA$. So $A \subseteq AA$. Also $AA \subseteq A$. Hence $A=AA$. So N(S) is idempotent, (6) => (7) is obvious. Lastly we assume (7). Let $a \in S$ and (a) be the ideal generated by a. Since $(a) \in N(S)$ by our assumption there exists A in N(S) such that (a) = (a) A (a) Now a e (a) = (a) A (a) implies that $a=axa$ for some $x \in S$. So a and hence S is regular.

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REFERENCES

[1] Kuroki, N : On normal semigroup. Czechoslovak Math J. 27 (102) 1977, Praha, P 43-53

[2] Luh, J : A characterisation of regular rings, proc. Japan. Acad. 39 (1964) P 741-742

[3] Lajos, S : On the bi-ideals in semigroups. Proc. Japan. Acad 45 (1969) P 710-712

[4] Lajos S : Characterisation of semilatices of groups. Math Balkanica 3 (1973) P 310-311

[5] Putcha, M. S. and Weissglass. J : A semilattice decomposition into semigroups having at most one fdempotent- Pac. J. Math 39 1971, P 225-228

[6] Petrich, M : Introducfion to semigroups. Bell and Howell company (1973)

[7] Schwarz, S : A theorem on normal semigroups. Czechoslovak Math. J. 10(35), 1960, P 197-200

[∞] Sen, M. K. : On pseudo-ideals of semigroups. Bull. Cal. Math. Soc. 67, P 109-114 (1975)

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Department of Pure Mathematics
University of Calcutta
35. Baliygunge Curcular Road
Calcutta-700 019
INDIA.

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GROUP-THEORETIC ORIGINS OF CERTAIN GENERATING FUNCTIONS FOR MODIFIED HYPERGEOMETRIC POLYNOMIALS—I

Sarama Das

1. INTRODUCTION :

In 1982, we [1] have discussed a unified theory of generating functions for hypergeometric polynomials $F(-n, \beta ; \gamma ; x)$ by giving suitable interpretations to n, β and γ simultaneously. The object of the present paper is to investigate the modified hypergeometric polynomials by giving a suitable interpretation to the index n only and we have observed that some generating functions obtained for $F(-n, \beta ; \gamma ; x)$ by means of triple interpretation are derived with little labour. Moreover, some new generating functions which do not appear in the investigation of the polynomial $F(-n, \beta ; \gamma ; x)$, are obtained. The main results are listed below :

$$(1.1) \quad (1-t)^n F\left(-n, \beta ; \gamma-n ; \frac{x-t}{1-t}\right) = \sum_{k=0}^{n} \frac{(-n)_k (\gamma-\beta-n)_k}{(\gamma-n)_k \, k!} F(-n+k, \beta ; \gamma-n+k ; x) t^k$$

which is due to Das [1, (6.5)] replacing γ by $\gamma+n$ in (1.1),

$$(1.2) \quad (1-y)^{\gamma-n-1} (1-xy)^{-\beta} F\left(-n, \beta ; \gamma-n ; \frac{1-y}{1-xy} . x\right)$$

$$= \sum_{k=0}^{\infty} \frac{(1-\gamma+n)_k}{k!} F(-n-k, \beta ; \gamma-n-k ; x) y^k ,$$

which is again due to Das [1, (6.6)] replacing Υ by $\Upsilon + n$ in (1.2)

$$(1.3) \quad (1-y)^{\gamma-1-n} (1-xy)^{-\beta} (1-y+wy)^n F\left[-n, \beta ; \Upsilon-n ; \left(1+\frac{wxy}{1-xy}\right).\right.$$

$$\cdot \frac{1-y}{1-y+xy}\Big] = \sum_{k=0}^{\infty} \frac{y^k}{k!} \sum_{m=0}^{n} \frac{(-n)_m(\Upsilon-n-\beta)_m}{(\Upsilon-n)_m\, m!}(1-\Upsilon-m+n)_k \cdot$$

$$(wy)^{n-m} \;.\; F(m-n-k, \beta ; \Upsilon-k+m ; x)$$

(1.4) $$(wy-1)^n \left(\frac{1+w-wxy}{w}\right)^{-\beta} \left(\frac{1+w-wy}{w}\right)^{\gamma-1-n} F\Big[-n, \beta ; \gamma-n ; \frac{1-wxy}{1-wy}$$

$$\cdot \frac{1+w-wy}{1+w-wxy}\Big] = \sum_{k=0}^{\infty} \frac{(wy)^{n-k}}{k!} \sum_{m=0}^{k-n} \frac{(1-\gamma+n)_m(-n-m)_k(\gamma-n-m-\beta)_k}{(\gamma-n-m)_k\; m!} y^m.$$

$$.\; F(k-n-m, \beta ; \gamma-m+k-n ; x).$$

(1.5) $$e^{-y}\ {}_1F_1(\beta ; \beta-\gamma+1 ; y-xy)$$

$$= \sum_{n=0}^{\infty} \frac{(1-\gamma)_n(-y)^n}{(\beta-\gamma+1)_n\, n!} F(-n, \beta ; \gamma-n ; x),$$

which do not seem to appear before.

(1.6) $$(1-y)^{\gamma-1}(1-xy)^{-\beta} \exp\left(\frac{-wy}{1-y}\right) {}_1F_1\left[\beta ; \beta-\gamma+1 ; \frac{(1-x)wy}{(1-y)(1-xy)}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(1-\gamma)_n}{(\beta-\gamma+1)_n} F(-n, \beta ; \gamma-n ; x)\ L_n^{\beta-\gamma}(w) y^n,$$

which is a new bilateral generating relation parallel to a result of L. Weisner [2 ; (4.6)].

(1.7) $$x^{-\beta}e^{-t}\ {}_1F_1\left(\beta ; \beta-\gamma+1 ; \frac{x-1}{x} t\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(1-\gamma)_n}{(\beta-\gamma+1)_n\, n!} F(-n, \beta ; \gamma-n ; x)\ (-t)^n ,$$

where $x \neq 0$ and this does not appear before.

2. LINEAR DIFFERENTIAL OPERATORS :

Modified hypergeometric polynomials $F(-n, \beta ; \Upsilon-n ; x)$ satisfy the differential equation

(2.1) $$x(1-x)\frac{d^2y}{dx^2} + [\gamma-n-(\beta-n+1)x]\frac{dv}{dx} + n\beta = 0.$$

Replacing $\frac{d}{dx}$ by $\frac{\partial}{\partial x}$, n by $y\frac{\partial}{\partial y}$ and y by u (x, y), we obtain from (2.1),

(2.2) $x(1-x)\dfrac{\partial^2 u}{\partial x^2}-(1-x)\,y\dfrac{\partial^2 u}{\partial x\partial y}+(\gamma-(\beta+1)\,x)\dfrac{\partial u}{\partial x}+\beta y\dfrac{\partial u}{\partial y}=0,$

of which $u=y^nF(-n,\beta;\gamma-n;x)$ is a solution, since $F(-n,\beta;\gamma-n;x)$ is a solution of (2.1).

As we know that $F(-n,\beta;\gamma-n;x)$ satisfies the following

$$(1-x)\frac{d}{dx}F(-n,\beta;\gamma-n;x)$$

$$=\frac{n(\gamma-\beta-n)}{\gamma-n}F(-n+1,\beta;\gamma-n+1;x)-nF(-n,\beta;\gamma-n;x)$$

and

$$x(1-x)\frac{d}{dx}F(-n,\beta;\gamma-n;x)$$

$$=(\gamma-n-1)F(-n-1,\beta;\gamma-n-1;x)-[\gamma-n-1-(\beta-2n-1)\,x]\,F(-n,\beta;\gamma-n;x)$$

we find two partial differential operators

(2.3) $B=(1-x)y^{-1}\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial y}$

and

(2.4) $C=x(1-x)\dfrac{\partial}{\partial x}-y^2\dfrac{\partial}{\partial y}+(\gamma-1-\beta x)y$

such that

(2.5) $B\,f_n(x,y)=\dfrac{n(\gamma-\beta-n)}{\gamma-n}f_{n-1}(x,y)$

and

(2.6) $C\ f_n(x,y)=(\gamma-n-1)\,f_{n+1}(x,y),$

where $f_n(x,y)=y^n\,F(-n,\beta;\gamma-n;x)$.

3. LIE ALGEBRA :

The set I, A, B, C forms a Lie algebra with the commutator relations

$[A,B]=-B,\quad [A,C]=C$

(3.1) $[B,C]=-2A+\gamma-\beta-1$

Now

$CB=(1-x)\ L+(\gamma-\beta-1)\,A,$

where L denotes the second order partial differential operator used to represent (2.1) in the form $Lu=0$. Thus A, B, C commute with $(1-x)\,L$.

The transformation groups generated by B, C are given by

$$e^{bB} f(x, y) = f\left[\frac{xy+b}{y+b}, \quad y+b\right]$$

(3.2)

$$e^{cC} f(x, y) = (1+cy)^{\gamma-1}(1+cxy)^{-\beta} \cdot f\left[x\,\frac{1+cy}{1+cxy}, \frac{y}{1+cy}\right]$$

Then as $[B, C] \neq 0$, we have

(3.3) $$e^{cC} e^{bB} f(x,y) = (1+cy)^{\gamma-1}(1+cxy)^{-\beta} \cdot f\left[\frac{b+(1+bc)xy}{b+(1+bc)y} \cdot \frac{1+cy}{1+cxy}, \frac{b+(1+bc)y}{1+cy}\right]$$

(3.4) $$e^{bB} e^{cC} f(x, y) = (1+bc+cy)^{\gamma-1}(1+bc+cxy)^{-\beta} \cdot f\left[\frac{(b+xy)(1+bc+cy)}{(b+y)(1+bc+cxy)}, \frac{y+b}{1+bc+cy}\right]$$

4. GENERATING FUNCTIONS :

PART-I : Genereting functions derived from the first order operator conjugate to A-n. From the previous considerations we know that $u(x, y) = y^n F(-n, \beta; \gamma-n; x)$ is annulled by L and A-n, where $A = y\,\dfrac{\partial}{\partial y}$.

To obtain generating functions for $F(-n, \beta; \gamma\text{-}n; x)$ we now transform $u(x, y)$ by means of the operators $e^{cC} e^{bB}$ anb $e^{bB} e^{cC}$. We consider ths following cases :

Case — 1 : $b = 1, c = 0$

Case — 2 : $b = 0, c = -1$

Case — 3 : $bc \neq 0$.

Case — 1 : We have

$$e^{B} y^{n} F(-n, \beta; \gamma-n; x) = (1+y)^{n} F\left(-n, \beta; \gamma-n; \frac{xy+1}{y+1}\right)$$

On the other hand

$$e^{B} y^{n} F(-n, \beta; \gamma-n; x) = \sum_{k=0}^{\infty} \frac{(-n)_k (\gamma-\beta-n)_k (-1)^k y^{n-k}}{(\gamma-n)_k\, k!} F(k-n, \beta; \gamma-n+k; x)$$

Equating these two and writing t in place of $-y^{-1}$, we get (1.1)

$$(4.1)\quad (1-t)^n F\left(-n, \beta ; \gamma-n ; \frac{x-t}{1-t}\right)=\sum_{k=0}^{n} \frac{(-n)_k (\gamma-\beta-n)_k}{(\gamma-n)_k\, k!} F(-n+k, \beta ; \gamma-n+k ; x)t^k$$

which is due to Das [1. (6.5)] replacing γ by $\gamma+n$ in (4.1)

Case—2 : We have

$$e^{-C}\left[y^n F(-n, \beta ; \gamma-n ; x)\right] = y^n (1-y)^{\gamma-n-1} (1-xy)^{-\beta} F\left(-n, \beta ; \gamma-n ; \frac{1-y}{1-xy}\, x\right)$$

Also

$$e^{-C}\left[y^n F(-n, \beta ; \gamma-n ; x)\right]= \sum_{k=0}^{\infty} \frac{(1-\gamma+n)_k}{k!}\, y^{n+k} F\left[-n-k, \beta ; \gamma-n-k ; x\right].$$

Equating these two we get (1.2)

$$(4.2)\quad (1-y)^{\gamma-n-1} (1-xy)^{-\beta} F\left(-n, \beta ; \gamma-n ; \frac{1-y}{1-xy}\, x\right)$$

$$= \sum_{k=0}^{\infty} \frac{(n-\gamma+1)_k}{k!} F(-n-k, \beta ; \gamma-n-k ; x) y^k$$

which is again due to Das [1, (6.6)] replacing γ by $\gamma+n$ in (4.2)

Case—3 : $bc \neq 0$.

We consider $b = w^{-1}$ and $c=-1$. Then from (3.3), we get

$$(4.3)\quad (1-y)^{\gamma-n-1} (1-xy)^{-\beta} (1+(w-1)y)^n F\left[-n, \beta ; \gamma-n ; \frac{(1-y)(1-xy+wxy)}{(1-xy)(1-y+wy)}\right]$$

$$= \sum_{k=0}^{\infty} \frac{y^k}{k!} \sum_{m=0}^{n} \frac{(-n)_m(\gamma-n-\beta)_m(n-\gamma+1-m)_k}{(\gamma-n)_m\, m!\,(-1)^m} (wy)^{n-m} \cdot$$

$$\cdot F(-n-k+m, \beta ; \gamma-n-k+m ; x)$$

and putting $b=-w^{-1}$, $c= -1$. we get from (3.4)

$$(4.4)\quad (wy-1)^n \left(\frac{1+w-wy}{w}\right)^{\gamma-n-1} \left(\frac{1+w-wxy}{w}\right)^{-\beta} F\left[-n, \beta ; \gamma-n ; \frac{(1-wxy)(1+w-wy)}{(1-wy)(1+w-wxy)}\right]$$

$$= \sum_{k=0}^{\infty} \frac{(wy)^{n-k}}{k!} \sum_{m=0}^{k-n} \frac{(1-\gamma+n)_m (-n-m)_k (\gamma-n-m-\beta)_k}{(\gamma-n-m)_k\, m!} \cdot$$

$$\cdot y^m F(-n-m+k, \beta ; \gamma-n-m+k ; x).$$

PART—II : Generating functions derived from operators not conjugate to A$-$n.

Let $S = e^{cC} e^{bB}$. Then for each choice of b and c, $S(A-n)S^{-1}$ represents an operator conjugate to $A-n$.

Let $R = r_1A + r_2B + r_3C + r_4$, for all choices of the coefficients except for $r_1 = r_2 = r_3 = 0$. Then $Lu = 0$ and $Ru = 0$ iff $\Psi(x)\, L(Su) = 0$ and $SRS^{-1}(Su) = 0$.

Now we have

$$e^{aA} B e^{-aA} = e^{-a} B, \quad e^{aA} C e^{-aA} = e^{a} C$$

$$e^{bB} A e^{-bB} = A + bB, \quad e^{cC} A e^{-cC} = A - cC$$

$$\text{(4.5)} \quad e^{bB} C e^{-bB} = -2b\, A - b^2B + C + b(\Upsilon - \beta - 1)$$

$$e^{cC} B e^{-cC} = 2cA + B - c^2C + c(\beta - Y + 1) .$$

Then

$$SAS^{-1} = e^{cC}(A+bB)\, e^{-cC}$$

$$\text{(4,6)} \quad = (1+2bc)A + bB - c(1+bc)C + bc(\beta - Y + 1) .$$

Therefore, for $R = r_1A + r_2B + r_3C + r_4$, $A-n$ is conjugate to R if $r_1^2 + 4r_2r_3 = 1$, so that $A-n$ is not conjugate to the set of operators for which $r_1^2 + 4r_2r_3 \neq 1$.

We choose $r_1^2 + 4r_2r_3 = 0$ for which $A-n$ is not conjugate to $R = r_1A + r_2B + r_3C + r_4$. Then the following cases may arise :

Case—1 : $r_1 = 0,\ r_2 = 1,\ r_3 = 0$

Case—2 : $r_1 = 2,\ r_2 = 1,\ r_3 = -1$

Case—3 : $r_1 = 0\ \ r_2 = 0,\ r_3 = 1$.

Part-II, Case—1 : We find a solution for the system $(B+\eta)\, u = 0$ and $Lu = 0$, where η is a non—zero constant.

Now

$u = e^{-\eta y} f(y-xy)$, a solution of $(B+\eta)\, u = 0$, $B = (1-x)y^{-1}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$, is again a solution of $Lu = 0$, where

$$L = x(1-x)\frac{\partial^2}{\partial x^2} - (1-x)y\frac{\partial^2}{\partial x \partial y} + (\gamma - (\beta+1)x)\frac{\partial}{\partial x} + \beta y\frac{\partial}{\partial y}.$$

Then $f(X)$, $X = y-xy$, satisfy the following ordinary differential equation

$X f''(X) + (\beta-\gamma+1-\eta X) f'(X) - \beta\eta f(X) = 0$

which has a solution (for $\eta = 1$) ${}_1F_1(\beta;\beta-\gamma+1;X)$.

Thus

$u = e^{-y} {}_1F_1(\beta;\beta-\lambda+1;y-xy)$ is a solution for the system $Lu = 0$, $(\beta+1)u = 0$. This function can be expanded in powers of y, say in this form,

$$e^{-y}{}_1F_1(\beta;\beta-\gamma+1;y-xy) = \sum_{n=0}^{\infty} a_n F(-n,\beta;\gamma-n;x)\, y^n$$

Putting $x = 0$ and equating coefficients of y^n from both sides, we get

$$a_n = \frac{(1-\gamma)_n (-1)^n}{(\beta-\gamma+1)_n \; n!}.$$

Thus we have

$$(4.7) \quad e^{-y}{}_1F_1(\beta;\beta-\gamma+1;y-xy) = \sum_{n=0}^{\infty} \frac{(1-\gamma)_n (-y)^n}{(\beta-\gamma+1)_n\, n!} F(-n,\beta;\gamma-n;x),$$

which is believed to be new.

Part-II, Case 2: We are to find a solution for the system

$(2A + B - C + \eta)u = 0$, $Lu = 0$.

We avoid to solve actually, as

$e^{C} B e^{-C} = 2A+B-C+\beta-\gamma+1.$

If u is a solution of $Lu = 0$ and $(B+w)u = 0$, $e^{C}u$ is a solution of $Lu = 0$ and $(2A+B-C+\beta-\gamma+1+w)u = 0$.

Now from Part—II, Case—1,

$u = e^{wy} {}_1F_1(\beta;\beta-\gamma+1;-w(1-x)y)$

is annulled by L and $B-w$.

Then

$$e^{C}u = (1+y)^{\gamma-1}(1+xy)^{-\beta}\exp\left(\frac{wy}{1+y}\right){}_1F_1\left[\beta;\beta-\gamma+1;\frac{-wy(1-x)}{(1+y)(1+xy)}\right]$$

is annulled by L and $2A+B-C+\beta-\gamma+1-w$. This function can be expanded in powers of $-y = t$, say in this from

$$(1-t)^{\gamma-1}(1-xt)^{-\beta}\exp\left(\frac{-wt}{1-t}\right){}_1F_1\left[\beta;\beta-\gamma+1;\frac{(1-x)wt}{(1-t)(1-xt)}\right]$$

$$= \sum_{n=0}^{\infty} a_n F(-n,\beta;\gamma-n;x)\, L_n^{\beta-\gamma}(w)\, t^n.$$

Equating coefficients of t^n from both sides, we get

$$a_n = \frac{(1-\gamma)_n}{(\beta-\gamma+1)_n}.$$

Thus, we obtain a new bilateral generating relation parallel to a result of L. Weisner [2. (4.6)]

$$(4.8)\quad (1-t)^{\gamma-1}(1-xt)^{-\beta} \exp\left(\frac{-wt}{1-t}\right) {}_1F_1\left[\beta ; \beta-\gamma+1 ; \frac{(1-x)wt}{(1-t)(1-xt)}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(1-\gamma)_n}{(\beta-\gamma+1)_n} F(-n, \beta ; \gamma-n ; x)\, L_n^{\beta-\gamma}(w)\, t^n.$$

Part-II, Case-3 : We find a solution for the system $Lu = 0$ and $(C+\eta)u = 0$. From Part-II, Case-1,

$u = e^y\ {}_1F_1(\beta ; \beta-\gamma+1 ; (x-1)y)$

is annulled by L and $B-1$.

As $e^{-B}e^{C}(B-1)e^{-C}e^{B} = -C-1$, we have

$$e^{-B}e^{C}u = x^{-\beta}y^{\gamma-\beta-1} \exp\left(\frac{y-1}{y}\right) {}_1F_1\left(\beta ; \beta-\gamma+1 ; \frac{x-1}{xy}\right)$$

is annulled by L and $C+1$.

This function can be expanded in powers of $y^{-1} = t$. Then, we have

$$(4.9)\quad \sum_{n=0}^{\infty} \frac{(1-\gamma)_n}{(\beta-\gamma+1)_n\, n!} F(-n, \beta ; \gamma-n ; x)\, (-t)^n$$

$$= x^{-\beta}e^{-t}\ {}_1F_1(\beta ; \beta-\gamma+1 ; (1-x^{-1})t), \quad (x \neq 0),$$

which dos not seem to appear before.

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REFERENCES

[1] Das, S. -- Group--theoretic origins of certain generating functions for F (--n, b ; c ; x), Pure Math. Manuscript Vol. I (1982), 15--35.

[2] Weisner, L. -- Group-theoretic origins of certain generating functions, Pacific Math. J. Vol.5 (1955), 1033 - 1039.

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Department of Pure Mathematics
Calcutta University

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AN EXTENSION OF A THEOREM OF FISHER ON COMMON FIXED POINT

S. K. Chatterjea

1. Introduction : In 1977, B. Fisher [2] proved the following theorem :

Theorem 1. In a complete metric space (X, d) if there exist two operators S and T mapping X into itself and satisfying the relation

$$\text{(I.I)}\quad [d(Sx, Ty)]^2 \leqslant b\, d(x, Sx)\, d(y, Ty) + c\, d(x, Ty)\, d(y, Sx)$$

for all $x, y \in X$, where $0 \leqslant b < 1$ and $c \geqslant 0$, then S and T have a common fixed point. Further if $0 \leqslant b, c < 1$, then each of S and T has a unique fixed point and these two points coincide.

It may now be pointed out that the metrics d (x, Sx) and d (y, Ty) occur in the contraction mapping of R. Kannan [3] and the metrics d (x, Ty) and d (y, Tx) occur in the contraction mapping of the present author [1]. Thus the product of two metrics appearing in the right member of (I.I) may be varied in six different ways, viz.

$$d(x, Sx)\, d(y, Ty), \quad d(x, Sx)\, d(x, Ty),$$
$$d(x, Sx)\, d(y, Sx), \quad d(y, Ty)\, d(x, Ty),$$
$$d(y, Ty)\, d(y, Sx), \quad d(x, Ty)\, d(y, Sx).$$

In view of the above fact, it may be of interest to remark that Theorem 1 can be extended as follows :

Theorem 2. In a complete metric space (X, d) if there exist two operators S and T mapping X into itself and satisfying the relation

$$
\begin{aligned}
&[d(Sx, Ty)]^2 \\
(1.2)\quad &\leqslant \alpha_1 d(x, Sx)\, d(y, Ty) + \alpha_2 d(x, Sx)\, d(x, Ty) \\
&+ \alpha_3 d(x, Sx)\, d(y, Sx) + \alpha_4 d(y, Ty)\, d(x, Ty) \\
&+ \alpha_5 d(y, Ty)\, d(y, Sx) + \alpha_6 d(x, Ty)\, d(y, Sx)
\end{aligned}
$$

for all $x, y \in X$, where $\alpha_i \geqslant 0$ and $0 \leq \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 1$, then S and T have a common fixed point. Further if $\alpha_6 < 1$ then S and T have a unique common fixed point.

2. Proof of Theorem 2.

It follows from (1.2) that

$$
\begin{aligned}
(2.1)\quad &[d(Ty, Sx)]^2 \\
&\leq \alpha_1 d(y, Ty)\, d(x, Sx) + \alpha_2 d(y, Ty)\, d(y, Sx) \\
&+ \alpha_3 d(y, Ty)\, d(x, Ty) + \alpha_4 d(x, Sx)\, d(y, Sx) \\
&+ \alpha_5 d(x, Sx)\, d(x, Ty) + \alpha_6 d(y, Sx)\, d(x, Ty).
\end{aligned}
$$

By virtue of the symmetry relation we obtain from (1.2) and (2.1)

$$
\begin{aligned}
(2.2)\quad &[d(Sx, Ty)]^2 \\
&\leq \alpha_1 d(x, Sx)\, d(y, Ty) + \alpha_6 d(x, Ty)\, d(y, Sx) \\
&+ \frac{\alpha_2 + \alpha_5}{2}[d(y, Ty)\, d(y, Sx) + d(x, Sx)\, d(x, Ty)] \\
&+ \frac{\alpha_3 + \alpha_4}{2}[d(y, Ty)\, d(x, Ty) + d(x, Sx)\, d(y, Sx)]
\end{aligned}
$$

Now let $x_0 \in X$ and define

$$
\begin{aligned}
x_{2n+1} &= Sx_{2n}, \quad n = 0, 1, 2, \dots \\
x_{2n} &= Tx_{2n-1}, \quad n = 1, 2, \dots
\end{aligned}
$$

Thus by (2.2) we have

$$
\begin{aligned}
&[d(x_{2n+1}, x_{2n})]^2 \\
&= [d(Sx_{2n}, Tx_{2n-1})]^2 \\
&\leqslant \alpha_1 d(x_{2n}, x_{2n+1})\, d(x_{2n-1}, x_{2n}) \\
&+ \frac{\alpha_2 + \alpha_5}{2} d(x_{2n-1}, x_{2n})\, d(x_{2n-1}, x_{2n+1}) \\
&+ \frac{\alpha_3 + \alpha_4}{2} d(x_{2n}, x_{2n+1})\, d(x_{2n-1}, x_{2n+1}) \\
&\leqslant \frac{\alpha_1}{2}\left[\{d(x_{2n}, x_{2n-1})\}^2 + \{d(x_{2n}, x_{2n+1})\}^2\right] \\
&+ \frac{\alpha_2 + \alpha_5}{2} d(x_{2n}, x_{2n-1})\left[d(x_{2n+1}, x_{2n}) + d(x_{2n}, x_{2n-1})\right] \\
&+ \frac{\alpha_3 + \alpha_4}{2} d(x_{2n}, x_{2n-1})\left[d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n-1})\right]
\end{aligned}
$$

$$\leqslant \frac{\alpha_1}{2}\left[\{d(x_{2n}, x_{2n-1})\}^2 + \{d(x_{2n}, x_{2n+1})\}^2\right]$$

$$+\frac{\alpha_2+\alpha_5}{2}\Big[d(x_{2n}, x_{2n-1})\Big]^2 + \frac{\alpha_3+\alpha_4}{2}\Big[d(x_{2n}, x_{2n+1})\Big]^2$$

$$+\frac{\alpha_2+\alpha_5}{4}\left[\{d(x_{2n}, x_{2n-1})\}^2 + \{d(x_{2n}, x_{2n+1})\}^2\right]$$

$$+\frac{\alpha_3+\alpha_4}{4}\left[\{d(x_{2n}, x_{2n+1})\}^2 + \{d(x_{2n}, x_{2n-1})\}^2\right]$$

i.e. $$\left(1-\frac{\alpha_1}{2}-\frac{\alpha_2+\alpha_5}{4}-\frac{\alpha_3+\alpha_4}{4}-\frac{\alpha_3+\alpha_4}{2}\right)[d(x_{2n+1}, x_{2n})]^2$$

$$\leqslant\left(\frac{\alpha_1}{2}+\frac{\alpha_2+\alpha_5}{2}+\frac{\alpha_2+\alpha_5}{4}+\frac{\alpha_3+\alpha_4}{4}\right)[d(x_{2n}, x_{2n-1})]^2.$$

Thus

(2.3) $d(x_{2n}, x_{2n+1}) \leqslant k d(x_{2n-1}, x_{2n})$,

$$\text{where } k^2=\frac{\frac{\alpha_1}{2}+\frac{\alpha_2+\alpha_5}{2}+\frac{\alpha_2+\alpha_5}{4}+\frac{\alpha_3+\alpha_4}{4}}{1-\frac{\alpha_1}{2}-\frac{\alpha_3+\alpha_4}{2}-\frac{\alpha_3+\alpha_4}{4}-\frac{\alpha_2+\alpha_5}{4}}<1.$$

Similarly we can show

(2.4) $d(x_{2n-1}, x_{2n}) \leqslant k d(x_{2n-2}, x_{2n-1})$.

So $\{x_n\}$ is a Cauchy sequence in X and X being complete, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$.

Now by (1.2) we have

$$\begin{aligned}
&[d(Sz, x_{2n})]^2\\
&=[d(Sz, Tx_{2n-1})]^2\\
&\leqslant \alpha_1 d(z, Sz)\, d(x_{2n-1}, Tx_{2n-1})\\
&+\alpha_2 d(z, Sz)\, d(z, Tx_{2n-1})\\
&+\alpha_3 d(z, Sz)\, d(x_{2n-1}, Sz)\\
&+\alpha_4 d(x_{2n-1}, Tx_{2n-1})\, d(z, Tx_{2n-1})\\
&+\alpha_5 d(x_{2n-1}, Tx_{2n-1})\, d(x_{2n-1}, Sz)\\
&+\alpha_6 d(z, Tx_{2n-1})\, d(x_{2n-1}, Sz)
\end{aligned}$$

Using $n\to\infty$, we thus obtain

$$(1-\alpha_3)[d(Sz, z)]^2 \leqslant 0,$$

which implies $Sz = z$.

Similarly considering $[d(x_{2n+1}, Tz)]^2$ we can show that $Tz = z$

Thns $Sz = z = Tz$.

Finally we prove that z is the unique common fixed point of S and T if $\alpha_6 < 1$.

Let u be any point of X such thet $Su = u$, then we have by (1.2)

$$[d(z, u)]^2 = [d(Su, Tz)]^2$$
$$\leqslant \alpha_6 [d(z, u)]^2,$$

which implies that $z = u$, since $\alpha_6 < 1$,

Similarly we can show that z is the unique fixed point of T.

Lastly if v be any point of X such that $Sv = v = Tv$, then by using (1.2) we can easily prove that $v = z$.

This completes the proof of theorem 2.

References

[1] Chatterjea, S. K—Fixed point theorems, C. R. Acad. Bulgare Sc. 25 (1972), 727—730; also Research Report No 2 (1971) Centre of Adv Studies in Appl. Math., Cal. Univ. p. 12.

[2] Fisher, B.—Common fixed points and constant mapping on metric spaces, Math. Sem. Notes. Kobe Univ. 5 (1977), 319—326.

[3] Kannan, R—Some results on fixed points, Bull. Cal. Math. Soc. 60 (1968), 71—76.

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Dept. of Pure Mathematics
Calcutta University
Calcutta-700019
India

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FIXED POINT THEOREMS IN BANACH SPACES

Kanan Majumdar

In this paper we have proved two fixed point theorems in Banach spaces, which are extensions of the results of K, Goebel and E. Zlotkiewicz [1] and K. Iseki [2]. First we prove the following theorem :—

Theorem 1. Let F be a mapping of a Banach space X into itself satisfying the following conditions :

(a) $F^2=I$ where I is the identity mapping

(b) $$\| F(x)-F(y) \| \leqslant \frac{a \| x-F(y) \| \quad \| y-F(x) \|}{\| x-y \|}$$
$$+b\{ \| x-F(x) \| + \| y-F(y) \| \}$$
$$+c\{ \| x-F(y) \| + \| y-F(x) \| \}$$
$$+d \| x-y \|$$

for every $x, y \in X$ and $x \neq y$, $0 \leqslant a,b,c,d$, $3a+4b+4c+d<2$.

Then F has a fixed point in X. Further the fixed point is unique if $a+2c+d<1$.

Proof. Let x be a point in X. We take $y=\frac{1}{2}(F+I)(x)$, $z=F(y)$ and $u=2y-z$.

Now $\| z-u \| \leqslant \| z-x \| + \| u-x \|$

Again, $\| z-x \| = \| F(y)-F^2(x) \|$

$$\leqslant \frac{a \| y-F^2(x) \| \quad \| F(x)-F(y) \|}{\| y-F(x) \|}$$
$$+b \{ \| y-F(y) \| + \| F(x)-F^2(x) \| \}$$
$$+c\{ \| y-F^2(x) \| + \| F(x)-F(y) \| \}+d \| y-F(x) \|$$
$$=\frac{a \| y-x \| \{ \| F(x)-y \| + \| y-F(y) \| \}}{\| y-F(x) \|}$$
$$+b\{ \| y-F(y) \| + \| F(x)-x \| \}$$

$$+c\{\|y-x\|+\|F(x)-y\|+\|y-F(y)\|\}$$
$$+d\|\tfrac{1}{2}(F+I)(x)-F(x)\|$$
$$\leqslant a\|\tfrac{1}{2}(F+I)(x)-x\|+\frac{a/2\|x-F(x)\|\ \ \|y-F(y)\|}{\|\tfrac{1}{2}(F+I)x-F(x)\|}$$
$$+b\{\|y-F(y)\|+\|x-F(x)\|\}$$
$$+c\|x-F(x)\|+c\|y-F(y)\|+d/2\|x-F(x)\|$$
$$=\frac{a}{2}\|x-F(x)\|+a\|y-F(y)\|+b\|y-F(y)\|$$
$$+b\|x-F(x)\|+c\|x-F(x)\|+c\|y-F(y)\|$$
$$+\frac{d}{2}\|x-F(x)\|$$
$$=\left(\frac{a}{2}+\frac{d}{2}+c+b\right)\|x-F(x)\|+(a+b+c)\|y-F(y)\|$$

Thus

$$\|u-x\|=\|2y-z-x\|=\|(F+I)(x)-F(y)-x\|$$
$$=\|(F+I)(x)-F(y)-x\|$$
$$=\|F(x)-F(y)\|$$
$$\leqslant\frac{a\|x-F(y)\|\ \ \|y-F(x)\|}{\|x-y\|}+b\{\|x-F(x)\|+\|y-F(y)\|\}$$
$$+c\{\|x-F(y)\|+\|y-F(x)\|\}+d\|x-y\|$$
$$\leqslant\frac{a\{\|x-y\|+\|y-F(y)\|\}\|y-F(x)\|}{\|x-y\|}$$
$$+b\{\|x-F(x)\|+\|y-F(y)\|\}$$
$$+c\{\|x-y\|+\|y-F(y)\|\}+\frac{c}{2}\|x-F(x)\|$$
$$+\frac{d}{2}\|x-F(x)\|$$
$$=\frac{a}{2}\|x-F(x)\|+a\|y-F(y)\|+b\|x-F(x)\|$$
$$+b\|y-F(y)\|+\frac{c}{2}\|x-F(x)\|+c\|y-F(y)\|$$
$$+\frac{c}{2}\|x-F(x)\|+\frac{d}{2}\|x-F(x)\|$$
$$=\left(b+c+\frac{a}{2}+\frac{d}{2}\right)\|x-F(x)\|+(a+b+c)\|y-F(y)\|$$

Also $\|z-u\|=\|F(y)-2y+z\|=2\|F(y)-y\|$

$$\therefore \quad 2\,\|F(y)-y\| \leqslant 2(a+b+c)\,\|y-F(y)\| + 2\left(\frac{a}{2}+\frac{d}{2}+b+c\right)\|x-F(x)\|$$

Or, $$\|y-F(y)\| \leqslant (a+b+c)\,\|y-F(y)\| + \left(\frac{a}{2}+\frac{d}{2}+b+c\right)\|x-F(x)\|$$

Or, $$(1-a-b-c)\,\|y-F(y)\| \leqslant \left(\frac{a}{2}+\frac{d}{2}+b+c\right)\|x-F(x)\|$$

Or, $$\|y-F(y)\| \leqslant \frac{\frac{a}{2}+\frac{d}{2}+b+c}{1-a-b-c}\,\|x-F(x)\|$$

$$< \frac{a+d+2b+2c}{2-2a-2b-2c}\,\|x-F(x)\|$$

$$< \alpha\,\|x-F(x)\| \text{ where } \alpha = \frac{a+d+2b+2c}{2-2a-2b-2c} < 1$$

since $3a+4b+4c+d < 2$

Let

$G = \frac{1}{2}(F+I)$, then for any $x \in X$,

$$\|G^2(x)-G(x)\| = \|G(y)-y\|$$

$$= \left\|\frac{1}{2}(F+I)(y)-y\right\|$$

$$= \frac{1}{2}\,\|y-F(y)\| < \frac{\alpha}{2}\,\|x-F(x)\|$$

$\therefore$ $\{G^n(x)\}$ is a Cauchy sequence in X.

As X is complete, $\{G^n(x)\}$ converges to some element x_0 in X, i.e $\lim_{n\to\infty} G^n(x)=x_0$

i.e. $G(x_0)=x_0$

Hence $F(x_0)=x_0$.

i.e. x_0 is a fixed point of X.

Now to show that the fixed point is unique, let us suppose that if possible y_0 be another fixed point of X. Then

$$\|x_0-y_0\| = \|F(x_0)-F(y_0)\|$$

$$\leqslant \frac{a\,\|x_0-F(y_0)\|\;\|y_0-F(x_0)\|}{\|x_0-y_0\|} + b\{\|x_0-F(x_0)\| + \|y_0-F(y_0)\|\}$$

$$+c\{\|y_0-F(x_0)\| + \|x_0-F(y_0)\|\} + d\,\|x_0-y_0\|$$

i.e. $(1-a-2c-d)\,\|x_0-y_0\| \leqslant 0 \Rightarrow x_0=y_0$ since $a+2c+d<1$.

Corollaries :

i) If we put a=0, b=0, c=o, we get result of K. Goebel and E. zlotkiewicz. [i]

ii) If we put a=0, c=0, we get result of K. Iseki. [2]

Next we prove the following theorem :

Theor em 2: Let F_1 and F_2 be two non expansive mappings of a Banach space X into itself and F_1 and F_2 satisfy the following conditions

i) $F_1 F_2 = I = F_2 F_1$ where I is the identity mapping.

and

ii) $$\| F_1(x) - F_2(y) \| \leqslant \frac{a \| x - F_2(y) \| \, \| y - F_1(x) \|}{\| x - y \|}$$
$$+ b\{ \| x - F_1(x) \| + \| y - F_2(y) \| \}$$
$$+ c\{ \| x - F_2(y) \| + \| y - F_1(x) \| \}$$
$$+ d \| x - y \|$$

for every $x \neq y \in X$, $0 \leqslant a, b, c, d$, and $3a + 4b + 4c + d < 2$.

Then F_1 and F_2 have a common fixed pojnt. Further the fixed point is unique if

$$a + 2c + d < 1.$$

Proof. Let $y = \frac{1}{2}(F_2 + I)(x)$, $z = F_1(y)$

$u = 2y - z.$

$$\| z - x \| = \| F_1(y) - F_1 F_2(x) \|$$
$$\leqslant \frac{a \| y - F_1 F_2(x) \| \, \| F_2(x) - F_1(y) \|}{\| y - F_2(x) \|}$$
$$+ b\{ \| y - F_1(y) \| + \| F_2(x) - F_1 F_2(x) \| \}$$
$$+ c\{ \| y - F_1 F_2(x) \| + \| F_2(x) - F_1(y) \| \}$$
$$+ d \| y - F_2(x) \|$$
$$= \frac{a \| \frac{1}{2}(F_2 + I)(x) - F_1 F_2(x) \| \, \| F_2(x) - F_1(y) \|}{\| \frac{1}{2}(F_2 + I)(x) - F_2(x) \|}$$
$$+ b\{ \| \frac{1}{2}(F_2 + I)(x) - F_1(y) \| + \| F_2(x) - F_1 F_2(x) \| \}$$
$$+ c \| \frac{1}{2}(F_2 + I)(x) - F_1 F_2(x) \| + \| F_2(x) - F_1(y) \|$$
$$+ d \| \frac{1}{2}(F_2 + I)(x) - F_2(x) \|$$

$$= \frac{\frac{a}{2} \| F_2(x) - x \| \, \| F_2(x) - F_1(y) \|}{\frac{1}{2} \| F_2(x) - x \|}$$

$$+b\{\| y-F_1(y)\| + \| F_2(x)-x\|\}$$

$$+c\{\| \frac{1}{2}(F_2+I)(x)-x\| + c\| F_2(x)-F_1(y)\|$$

$$+\frac{d}{2}\| x-F_2(x)\|$$

$$\leqslant \frac{a\| x-F_2(x)\| \{\| F_2(x)-y\| + \| y-F_1(y)\|\}}{\| x-F_2(x)\|}$$

$$+b\| y-F_1(y)\| + b\| x-F_2(x)\| + \frac{c}{2}\| x-F_2(x)\|$$

$$+\frac{c}{2}\| x-F_2(x)\| + c\| y-F_1(y)\| + \frac{d}{2}\| x-F_2(x)\|$$

$$=\left(\frac{a}{2}+b+c+\frac{d}{2}\right)\| x-F_2(x)\| + (a+b+c)\| y-F_1(y)\|$$

Thus

$$\| u-x\| = \| 2y-z-x\|$$

$$= \| (F_2+I)(x)-F_1(y)-x\|$$

$$= \| F_2(x)-F_1(y)\|$$

$$\leqslant \frac{a\| x-F_1(y)\| \ \| y-F_2(x)\|}{\| x-y\|}.$$

$$+b\{\| y-F_1(y)\| + \| x-F_2(x)\|\}$$

$$+c\{\| y-F_2(x)\| + \| x-F_1(y)\|\}$$

$$+d\| x-y\|$$

$$\leqslant \frac{a\{\| x-y\| + \| y-F_1(y)\|\} \frac{1}{2}\| x-F_2(x)\|}{\frac{1}{2}\| x-F_2(x)\|}$$

$$+b\{\| y-F_1(y)\| + \| x-F_2(x)\|\}$$

$$+\frac{c}{2}\| x-F_2(x)\| + c\| x-y\|$$

$$+c\| y-F_1(y)\| + \frac{d}{2}\| x-F_2(x)\|$$

$$=\frac{a}{2}\| x-F_2(x)\| + a\| y-F_1(y)\| + b\| y-F_1(y)\|$$

$$+b\| x-F_2(x)\| + \frac{c}{2}\| x-F_2(x)\| + \frac{c}{2}\| x-F_2(x)\|$$

$$+c\| y-F_1(y)\| + \frac{d}{2}\| x-F_2(x)\|$$

$$=\left(\frac{a}{2}+\frac{d}{2}+b+c\right)\|x-F_2(x)\|+(a+b+c)\|y-F_1(y)\|$$

$$\therefore \quad \|z-u\| \leqslant \|z-x\|+\|u-x\|$$

$$\leqslant 2\left(\frac{a}{2}+\frac{d}{2}+b+c\right)\|x-F_2(x)\|$$

$$+2(a+b+c)\|y-F_1(y)\| \quad \ldots\ldots\ldots \quad (1)$$

Also,

$$\|z-u\| = \|F_1(y)-2y+z\|$$

$$= \|F_1(y)-2y+F_1(y)\|$$

$$=2\|y-F_1(y)\| \quad \ldots\ldots\ldots \quad (2)$$

From (1) and (2) we have

$$2\|y-F_1(y)\| \leqslant 2\left(\frac{a}{2}+\frac{d}{2}+b+c\right)\|x-F_2(x)\|$$

$$+2(a+b+c)\|y-F_1(y)\|$$

$$\therefore \quad \|y-F_1(y)\| \leqslant \left(\frac{a}{2}+\frac{d}{2}+b+c\right)\|x-F_2(x)\|$$

$$+(a+b+c)\|y-F_1(y)\|$$

Or, $\|y-F_1(y)\| \leqslant \dfrac{2b+2c+a+d}{2-2(a+b+c)}\|x-F_2(x)\|$

$= \alpha \|x-F_2(x)\|$ where

$\alpha = \dfrac{2b+2c+a+d}{2-2(a+b+c)} < 1$ since $3a+4b+4c+d<2$

Let $G=\frac{1}{2}(F_2+I)$, then for any $x \in X$,

$$\|G^2(x)-G(x)\| \quad \|G(y)-y\|$$

$$= \left\|\frac{1}{2}(F_2+I)y-y\right\|$$

$$= \frac{1}{2}\|y-F_2(y)\|$$

$$= \frac{1}{2}\|F_2F_1(y)-F_2(y)\|$$

$$\leqslant \frac{1}{2}\|F_1(y)-y\|, \text{ since } F_1 \text{ is nonexpansive.}$$

$$\leqslant \frac{\alpha}{2}\|x-F_2(x)\|$$

$\therefore$ $\{G^n(x)\}$ is a Cauchy sequence in X. Also by the completeness, $\{G^n(x)\}$ converges to some element x_0 in X,

i. e. $\lim\limits_{n\to\infty} G^n(x)=x_0$ which implies

$$G(x_0)=x_0$$

Hence $F_2(x_0)=x_0$ i.e. x_0 is a fixed point of F_2.

Again,

$$\| G^2(x)-G(x) \| \leq \frac{\alpha}{2} \| x-F_2(x) \|$$

$$=\frac{\alpha}{2} \| F_2F_1(x)-F_2(x) \|$$

$$\leqslant \frac{\alpha}{2} \| x-F_1(x) \| \quad \text{since } F_2 \text{ is non-expansive.}$$

Therefore we can conclude that

$F_1(x_0)=x_0$ i.e. x_0 is a fixed point of F_1.

Hence $F_1(x_0)=x_0=F_2(x_0)$. Therefore x_0 is a common fixed point of F_1 and F_2.

Proof of the uniqueness of the common fixed point is omitted for the sake of brevity.

REFERENCES

[1] Goebel K and Zlotkiewicz, E—Colloquium Math 23 (1971) 103—106.

[2] Iseki K—Math Balkanica 5 (1975) 143—144

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Dept. of Pure Math.
Calcutta University
Calcutta-700019

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SOME RESULTS ON THE DISTANCE SET OF THE CANTOR TYPE SET C_k^*

D. K. Ganguly and S. Ray

ABSTRACT

In this paper we construct a class of linear sets C_k, which includes the classical Cantor set $C(=C_1)$ and whose construction and properties are very much similar to those of C. Also we have studied some properties of the distance set as well as mid-point set of C_k.

INTRODUCTION :

In 1917, Hugo Steinhaus [9] proved the remarkable property that the distance set of the Cantor set in the unit interval is precisely the interval [0, 1] ; and in 1920, he [10] demonstrated that the distance set of any set with positive Lebesgue measure contains an interval with left end point zero. This result has also been established using alternative methods by S. Ruziewicz [7], T. Šalăt and T. Neubrunn [8], Bose Majumder [1] J. Randolph [6]. In 1947, H. Kestelman [4] considered p-dimensional sets A with the property tnat every sufficiently small vector in Euclidean p-space is the difference of two elements of A i. e. the "directed distances" of A contains a sphere and hence Steinhaus' result is a particular case of Kestelman's result for $p=1$. Utz [11] has also proved the result $D(C)=[0, 1]$ by geometrical way.

In this paper we construct a class of linear sets $\{C_k\}$, which includes the classical Cantor set $C(=C_1)$ and whose construction and properties are very much similar to those of C We have also studied some properties of the distance set of C_k.

DEFINITIONS AND NOTATIONS :

(1) The distance set of $A(\subset R)$ denoted by $D(A)$ is the set

$$D(A)=\{\ |x-y|\ /x\epsilon A,\ y\epsilon A\}.$$

(2) The mid point set of $A(\subset R)$ denoted by $M(A)$ is the set

$$M(A)=\left\{\frac{x+y}{2} \,/ x\epsilon A,\ y\epsilon A\right\}$$

§1. **CONSTRUCTION OF A LINEAR SET** (for a given positive integer k) in the closed interval [0, 1].

We first divide the interval [0, 1] into $(2k+1)$ equal parts and remove the open intervals in the second, fourth, sixth, ..., 2k th positions leaving $(k+1)$ closed intervals each of length $\frac{1}{2k+1}$. We shall call each of these remaining closed intervals class 1 intervals and denote each by A_1. Let $E_1=UA_1$. Each A_1 is again divided into $(2k+1)$ equal parts and the open intervals in the second, fourth, sixth, ..., 2k th positions are again deleted, leaving $(k+1)^2$ closed intervals each of length $\frac{1}{(2k+1)^2}$.

We call each of these remaining closed intervals class 2 closed intervals and each denoted by A_2. Let $E_2=UA_2$. This process is continued inductively. During the n^{th} stage we delete the open intervals at the second, fourth, ..., 2k th position each of length $\frac{1}{(2k+1)^n}$ and denote each by A_n. Let $E_n=UA_n$. Then $E_n(n=1, 2, 3, \ldots)$ form a monotone decreasing sequence of compact sets and thus, have nonempty intersection. Let $C_k=\bigcap_{n=1}^{\infty} E_n$. The set C_k, being the complement of an everywhere dense set of non-overlapping, non-abutting, open intervals is a non-dense perfect set [3]. Now, total length suppressed at the first stage$=\frac{k}{2k+1}$.

Number of closed intervals left at the first stage$=(k+1)$.

Total length suppressed at the second stage$=\frac{k}{(2k+1)^2}(k+1)$

Thus, total length suppressed at the nth stage$=\frac{k(k+1)^{n-1}}{(2k+1)^n}$

Hence total length removed

$$=\frac{k}{(2k+1)}+\frac{k(k+1)}{(2k+1)^2}+\frac{k(k+1)^2}{(2k+1)^3}+\cdots\cdots\cdots+\frac{k\,k+1)^{n-1}}{(2k+1)^n}+\cdots\cdots$$

Hence the Lebesque measure of $C_k=0$.

The set C_k for a given positive integer k may also be described in series notations as the set of all x such that

$$x=\sum_{i=1}^{\infty} \frac{a_i}{(2k+1)^i}, \quad a_i=(0, 2, 4, \ldots\ldots, 2k) \text{ for all } i.$$

It is easy to see that the set C_k is symmetric i. e. if $x \epsilon C_k$ then $1-x \epsilon C_k$.

D. Ganguly [2] proved that $D(C_k)=[0, 1]$. We shall present an alternative proof of this result which seems to be shorter.

THEOREM : 1. 1. $D(C_k)=[0, 1]$.

PROOF : Let $K=\{x-y/x \epsilon C_k, y \epsilon C_k\}$.

To prove this result it is sufficient to show that $K=[-1, 1]$.

Let $x=\sum_{i=1}^{\infty} \frac{a_i}{(2k+1)^i}$ and $y=\sum_{i=1}^{\infty} \frac{b_i}{(2k+1)^i}$

where $a_i, b_i=\{0, 2, 4, \ldots\ldots, 2k\}$ for each i.

Since $1=\sum_{i=1}^{\infty} \frac{2k}{(2k+1)^i}$, hence

$$x-y+1=\sum_{i=1}^{\infty} \frac{a_i-b_i+2k}{(2k+1)^i}=\sum_{i=1}^{\infty} \frac{2c_i}{(2k+1)^i}$$

where $2c_i=a_i-b_i+2k$, $c_i=\{0, 1, 2, \ldots \ldots, 2k\}$.

Thus $\frac{x-y+1}{2}=\sum_{i=1}^{\infty} \frac{c_i}{(2k+1)^i}$ is any point in $[0, 1]$.

Thus every $p \epsilon [0, 2]$ may be expressed as $p=x-y+1$, where $x, y \epsilon C_k$.

Thus $[0, 2]=K+1$ and hence $K=[-1, 1]$.

Theorem 1. 2 : If d is any point in $[0, 1]$ then there exists a pair of points from C_k whose mid point is d i.e. $M(C_k)=[0, 1]$.

Proof : Let $0 \leqslant d \leqslant 1$. Then $1-d \epsilon [0, 1]$. By theorem 1. 1 there exist x and y in C_k such that $y-x=1-d$.

Hence $(1-y)+x=d$.

As C_k is symmetric hence $1-y \epsilon C_k$. Thus given any $d \epsilon [0, 1]$ there exist x and y in C_k whose sum is d.

If $0 \leq 2d \leqslant 1$, then there exist x and y in C_k such that $x+y=2d$ i.e. $d=\frac{x+y}{2}$.

If $1 \leqslant 2d \leqslant 2$, then $0 \leq 2-2d \leqslant 1$ and hence there exists a pair of points $x, y \epsilon C_k$ such that $x+y=2-2d$.

Then $(1-x)+(1-y)=2d$

or $x'+y'=2d$ where $x'=1-x\epsilon C_k$

and $y'=1-y\epsilon C_k$.

Therefore, in any case, for every $d\epsilon[0,1]$ there is a pair of points from C_k whose middle point is d.

We shall now generalize the result 1. 2. Before going to prove the generalized result we shall now state a result due to Ganguly [2].

Result. Given any real number d and m such that $0 \leqslant d \leqslant 1$ and $\frac{1}{2k+1} \leq |m| \leq 2k+1$, there exists at least one pair of points $(x, y)\epsilon C_k x C_k$ s.t $y=mx+d$.

Theorem 1. 3. Given two positive real numbers μ and ν satisfying $\frac{1}{2k+1} \leq \frac{\mu}{\nu} \leq 1$, each point d in $0 \leq d \leq 1$, divides a segment $[x, y] \subseteq [0, 1]$ in the ratio $\nu : \mu$ whose end points x and y are the points of C_k.

Proof : Let d be any point in $0 < d \leq \frac{\nu}{\mu+\nu}$; we now choose d' such that

$$d' = \left[\frac{\mu+\nu}{\nu}\right] d.$$

Hence $d=\frac{\nu d'}{\mu+\nu}$.

Since $0 < d \leqslant \frac{\nu}{\mu+\nu}$, we have $0 < d' \leqslant 1$. We now choose $m=-\frac{\mu}{\nu}$ in above type of result on C_k.

Therefore $\frac{1}{2k+1} \leq |m| \leq 1$ ($<2k+1$) and thus $y=(-\mu/\nu)x+d'$,

where $x\epsilon C_k$ and $y\epsilon C_k$.

Hence $\frac{\nu y+\mu x}{\nu}=d'$

or, $\frac{\nu y+\mu x}{\nu}=\frac{\mu+\nu}{\nu}d$

or, $d=\frac{\nu y+\mu x}{\mu+\nu}$.

Taking $\frac{\nu}{\mu+\nu} < d < 1$, we get

$$1-\frac{\nu}{\mu-\nu} > 1-d > 0,\ 0 < 1-d < \frac{\mu}{\mu+\nu}\left(< \frac{\nu}{\mu+\nu}\right).$$

Hence, by previous argument, we get

$$x \epsilon C_k, \ y \epsilon C_k \text{ and } 1-d=\frac{\nu y+\mu x}{\mu+\nu}$$

or, $\mu+\nu-d(\mu+\nu)=\nu y+\mu x$

or, $(\mu+\nu)d=\mu(1-x)+\nu(1-y)=\mu x'+\nu y'$

where $x'=(1-x)\epsilon C_k$

and $y'=(1-y)\epsilon C_k$.

Thus $d=\dfrac{\mu x'+\nu y'}{\mu+\nu}$

where $\dfrac{\nu}{\mu+\nu} < d < 1$.

Hence the result follows.

§ 2. We are now interested to determine for a given $d \epsilon [0, 1]$ how many pairs of points x and y belonging to C_k are there such that $d=y-x$.

Let $T=C_k X C_k$ and let l denote the line $y=x+d$, $0 \leqslant d \leqslant 1$. Since $D(C_k)=[0, 1]$ the line $y=x+d$ must intersect T at least in one point.

Definition : Given $0 \leq d \leq 1$, we define $\triangle_d$ to be the set

$$\{(x, y) \mid x \epsilon C_k, \ y \epsilon C_k, \ y-x=d\}$$

Note that whenever $y-x=d$, then $|y-x|=d$ but also $|x-y|=d$, and the pair $(y, x) \bar{\epsilon} \triangle_d$. $\overline{\overline{\triangle}}_d$ describes precisely the number of distinct pairs of Cantor type points with the property that they are d unit apart.

Theorem 2.1 : For all but a countable number of d in C_k, $\overline{\overline{\triangle}}_d=c$.

Proof : Let $d \epsilon C_k$ such that d is not an end point of deleted intervals in the construction of C_k. Then we can express d as

$$d=\sum_{i=1}^{\infty} \frac{a_i}{(2k+1)^i}$$

where $a_i=(0, 2, 4, \ldots, 2k)$ for all i, and each of the values $0, 2, 4, \ldots, 2k$ occurs infinitely many times.

Let x be a number expressed in the scale of $(2k+1)$ such that

$$x=\sum_{i=1}^{\infty} \frac{x_i}{(2k+1)^i},$$

where

$$x_i=0 \text{ if } a_i=(2, 4, \ldots, 2k)$$

and

$x_i=(2, 4, \ldots, 2k)$ if $a_i=0$.

Then the number of distinct x's for a given d, attainable in this manner is the number of sequences 0's, 2's, ... 2k's i. e. c (cardinal number of the continum). According to the construction of x, we can say $x \epsilon C_k$. If we let $y=x+d$, then since x and d never have the digit 2, 4, ..., 2k in the same position, $y \epsilon C_k$. Hence $\overset{=}{\triangle}_d=c$.

Theorem 2.2 : For a dense set of $d \,\bar{\epsilon}\, C_k$, $\overset{=}{\triangle}_d=c$.

Proof : It can be easily proved that the numbers having a terminating expansion of the scale $(2k+1)$ form a dense set in the complement of C_k in [0, 1]. Let d be one such number. There exists at least one pair of points (x, y) in Cantor type set C_k such that $y-x=d$. As d terminates, x and y must have identical digits from some index m onwards when they are expressed in the scale of $(2k+1)$.

$$\text{Then } x=\sum_{i=1}^{\infty} \frac{a_i}{(2k+1)^i} \quad \text{and } y=\sum_{i=1}^{\infty} \frac{b_i}{(2k+1)^i}$$

where $a_i, b_i=(0, 2, 4, \ldots, 2k)$ with the condition $a_i=b_i$ for $i>m$.

We can choose a_i and b_i in $(k+1)$ ways so that $a_i=b_i$. Therefore the number of pairs of points (x, y) satisfying $x \epsilon C_k$ and $y \epsilon C_k$ and $y-x=d$ is $(k+1)^{\aleph_0}=c$ [5]. Hence $\overset{=}{\triangle}_d=c$ for a dense set of $d \,\bar{\epsilon}\, C_k$.

We shall now state the following theorem which is proved by Ganguly to determine the number of pairs of points of C_k associated with almost every $d \epsilon [-1, 1]$ such that $d=y-x$.

Theorem 2.3 : For almost all $d \,\epsilon\, [0, 1]$, $\overset{=}{\triangle}_d=c$.

We now elaborate the theorem by

Theorem 2.4 : For every $d \epsilon [0, 1]$, the set $\triangle_d$ is either finite or perfect set.

The following lemma is needed to establish the theorem.

LEMMA : For every d in [0, 1], $\triangle_d$ is a closed subset of the unit square.

Proof : Let $A=\{(x, y)/x \epsilon C_k,\ y \epsilon C_k \text{ and } y>x\}$. Let $f: A \to [0, 1]$ be a function defined by $f(x, y)=y-x$. Then obviously f is continuous. Let d be any point in the unit interval [0, 1]. Then $f^{-1}\{d\}=\triangle_d$ and hence $\triangle_d$ is a closed set.

Proof of the theorem : By the lemma $\triangle_d$ is closed. We have to prove that $\triangle_d$ is dense-in-itself. It has been proved by Gangaly [2] if there are an infinite number of

pairs of points $x \in C_k$, $y \in C_k$ such that $y-x=d$ for a given $d \in [0, 1]$ then each pair, when expressed In the form

$$x=\sum_{i=1}^{\infty} \frac{2\alpha_i}{(2k+1)^i}, \quad y=\sum_{i=1}^{\infty} \frac{2\beta_i}{(2k+1)^i}, \quad \alpha_i, \beta_i = \begin{Bmatrix} 0 \\ 1 \\ 2 \\ \cdot \\ \cdot \\ \cdot \\ k \end{Bmatrix} \quad \ldots(1)$$

has the property that $\alpha_i=\beta_i$ for infinitely many i.

Suppose $\triangle_d$ is an infinite set for a given $d \in [0, 1]$. Let (x, y) be any element of $\triangle_d$, where x and y are expressed in the form of (1).

Let $\epsilon (>0)$ be chosen prevlously. Then we choose a positive integer N such that $\frac{2}{(2k+1)^N} < \sqrt{\frac{\epsilon}{2}}$ and $\alpha_N = \beta_N$. If $\alpha_N = \beta_N = 0$, then $x + \frac{2}{(2k+1)^N}$ and $y + \frac{2}{(2k+1)^N}$ are the points of C_k. If $\alpha_N = \beta_N = (1,2,3,\ldots, k)$.

then

$$x - \frac{2}{(2k+1)^N} \text{ and } y - \frac{2}{(2k+1)^N} \text{ are the points of } C_k.$$

Hence $\left| \left(y \pm \frac{2}{(2k+1)^N}\right) - \left(x \pm \frac{2}{(2k+1)^N}\right) \right| = |y-x| = d$

Therefore $\left(x + \frac{2}{(2k+1)^N}, y + \frac{2}{(2k+1)^N}\right)$

or $\left(x - \frac{2}{(2k+1)^N}, y - \frac{2}{(2k+1)^N}\right)$ is an element of $\triangle_d$. Also

$$\left[\left(y \pm \frac{2}{(2k+1)^N}\right) - y\right]^2 + \left[\left(x \pm \frac{2}{(2k+1)^N}\right) - x\right]^2$$

$$= 2 \left(\frac{2}{(2k+1)^N}\right)^2 < \epsilon.$$

Thus (x, y) is a limit point of $\triangle_d$.

Hence the theorem.

REFERENCES

[1] Bose Majumder, N. C.: 'On the distance set of the Cantor middle third set' Bull. Cal. Math. Soc. 51 (1959) pp 93—102.

[2] Ganguly D. K.: Study of some properties of functions and sets connected with cantorset and similer type Ph.D. Thesis, 1979, Calcutta University.

[3] Hobson, E. W.: "Theory of Function of a real variable" Vol. I Dover Publication Inc. N.Y. p 117

[4] Kestelman, H.: "The convergent sequence belonging to a set' J. Lond. Math. Soc. 22 (1947), pp 130—136.

[5] Kamke, E: "Theory of sets' Dover Publication Inc. N.Y. p 49.

[6] Randolph, J. F.: "Distance between points of the Cantor set" Amer. Math. Monthly, 47 (1940), pp. 549—551.

[7] Ruziewicz, S.: "Contribution a Le'tude des ensembles de distances de points' Fund. Math 7 (1925), pp. 141—143.

[8] Šalát, T. and Neubrunn, T.: "Distance sets, Ratio sets and certain transformations of sets of Real number" Cas. pro. pes. mat. 94 (1969) pp. 381—393.

[9] Steinhaus, H.: "Nowa Vlanosě Monogoscĭ G. Cantor Wecktor, 1917, pp. 105—107.

[10] Steinhaus, H.: "Sur les distances des points dans les ensembles de measure positive' Fund. Math. (1920), pp. 93—104.

[11] Utz, W.R.: "The distance set for the Cantor discontinuum' Amer. Math. Monthly 58 (1951) pp. 407—408.

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Department of Pure Mathematics.
University of Calcutta,
35, Ballygunge Circular Road,
Calcutta-700 019
INDIA

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ON COMPATIBLE TOPOLOGIES OF A GROUP AND THOSE OF ITS LATTICE OF SUBGROUPS.

A. Das Gupta

Abstract :

A topology t of a group G which makes G a topological group will be called a compatible topology of G. Likewise, a topology t' of a lattice L will be called compatible if the upper semilattice operation of L is continuous under t. It has been shown in this paper that a compatible topology of a group G induces a compatible topology of the lattice L (G) of subgroups of G. Conversely, under certain condition a compatible topology of L (G induces a compatible topology of G.

Introduction :

Topological groups as also topological lattices have been studied by many prominent Mathematicians. The aim of this paper is to study them from different point of view. As a matter of fact, an attempt has been made to connect the two studies and thereafter to study the topological groups by the lattice of compatible topologies, as introduced in this paper, of those groups.

A topology t of a group G which makes G a topological group, will be called a compatible topology of G. Likewise, a topology t' of a lattice L will be called compatible, if the upper semilattice operation of L is continuous under t'.

Our aim is to study the topological groups by studying the corresponding lattices of the compatible topologies of the groups concerned.

In this paper we have considered the problem as to whether a compatible topology of a group G induces a compatible topology in L (G) and conversely and to this question we have a positive answer.

In fact, we have shown that a compatible topology of a group G induces a compatible topology of the lattices L (G) of subgroups of G.

Conversely, under certain condition, a compatible topology of L (G) induces a compatible topology of G.

1. Definition :

A set G of elements is called a generalised topological group if

(1) G is an abstract group

(2) G is a topological space.

(3) The group operations in G are continuous in the topological space G.

If the topology in G is a t_1 space i.e each point set is closed then it is called a topological group.

The set L of elements is called a topological lattice if

(1) L is a lattice

(2) L is a topological space

(3) The lattice operations are continuous in the topological space L.

We first prove the following theorem :

Theorem 1. Let G be an abstract group and L (G) be the lattice of all subgroups of G. If G be a generalised topological group, then the topology of G induces a topology in L (G) for which L (G) is a topological upper semilattice.

Proof : Let G be a generalised topological group. Let Σ be the complete system of neighbourhoods of the topological space G.

Let $U \epsilon \Sigma$. We shall denote by U^* the smallest sugroup of G generated by U.

Correspondingly, for $V \epsilon \Sigma$ one gets V^*. Let $W^* = U^* \bar{\cup} V^*$ = smallest subgroup of G containing U^* and V^*.

We say that W^* is the union of U^* and V^* and we shall denote this union by $\bar{\cup}$, to distinguish it from the set union.

We consider the set $\Sigma^* = \{\bar{\cup}\, U^*,\ \forall U \epsilon \Sigma\}$

we shall show that this Σ^* becomes the complete system of neighbourhocds of a topology in L (G) for which L (G) becomes a topological upper semilattice.

Let $H \epsilon L(G)$.

Then H must have a system of generators, say, s.

Hence there exists an open set U | $S \subset U$ and from U we get U^*. So, $H \epsilon U^*$

Next, Σ^* satislies t e following conditions :—

1) $L(G) = \bigcup U^*, \forall U^* \epsilon \Sigma^*$.

It is obvious, as for every $H \epsilon L(G), \exists U^* \epsilon \Sigma^*$ | $H \epsilon U^*$ and so, $L(G) = \bigcup U^*, \forall U^* \epsilon \Sigma^*$.

(2) For any two sets U^* and $V^* \epsilon \Sigma^*$ which contain the subgroup $H \epsilon L(G)$, there exists a $W^* \epsilon \Sigma^*$ | $H \epsilon W^* \subseteq^* U^* \cap^* V^*$.

It is obvious that U^* and V_* are open sets of G and as $H \epsilon U^* \cap V^* \Rightarrow \exists$ open set W containing a system of generators S of H | $S \subseteq W$ and $W \subseteq V^* \cap V^* \Rightarrow H \epsilon W^* \subseteq U^* \cap V^*$.

HenceΣ^* is a complete system of neighbourhood of the topological space L (G).

The upper lattice operation is continuous i.e.

(A) If $H_1 \bar{\cup} H_2 \epsilon U_* \Rightarrow \exists H_1 \epsilon U_1{}^*$ and $H_2 \epsilon U_2{}^*$ | $U_1{}^* \bar{\cup} U_2{}^* \subseteq U^*$ where H_1 and H_2 are any two subgroups of G.

Let $H_1 \bar{\cup} H_2 \epsilon U^*$

Let S_1 and S_2 are two system of generators of H_1 and H_2 respectively.

Hence we can find two open sets $S_1 \subseteq U_1$ and $S_2 \subseteq U_2$ $U_1 \subset U^*$ and $U_2 \subset U^*$.

From U_1 and U_2 we get $U_1{}^*$ and $U_2{}^*$ and $H_1 \epsilon U_{1*}$, $H_2 \epsilon U_2{}^*$ | $U_1{}^* \bar{\cup} U_2{}^* \subseteq U^*$.

Hence L (G) is a topological upper semilattice.

We note that Σ^* satisfies the following condition :

(B) Let $a \epsilon G$ and $U^* \epsilon \Sigma^*$, then we can find a $V^* \epsilon Z^*$ | $aV^*a^{-1} \subset U^*$.

As, U^* is an open set in G and identity $e \epsilon U^*$ it follows that there exists a neighbourhood V of e such that for any element 'a'ϵG, $aVa^{-1} \subset U^* \Rightarrow aV^*a^{-1} \subset U^*$.

2. **Theorem 2 :** If L (G) be a topological upper semilattice, the complete system of neighbourhoods of which satisfies the condition (B), then the topology of L (G) induces a topology in G, for which G becomes a generalised topological group.

Proof ; Let Σ^* be the complete system of neighbourhood of L (G) satisfying condition (B).

Let $U^* \epsilon \Sigma^*$

Let U_i be the set of all subgroups of G contained in U^*.

Let U' be the set of all elements of $\bigcup u_i$

Let $\Sigma' = \{U', \forall U^* \epsilon \Sigma^*\}$

It can be easily shown that Σ' satisfies the following conditions :

(1) Identity is a common element to all the sets.

(2) The intersection of any two sets belonging to Σ' contains a third set of the system Σ'.

(3) For any set U' of the system Σ' there exists a set $V' \epsilon \Sigma'$ such that $V'V'^{-1} \subset U'$

(4) For any set U' of the system Σ' and an element $a \epsilon U'$ there exists a set $V' \epsilon \Sigma' \mid V'a \subset U'$

(5) If $U' \epsilon \Sigma'$ and $a \epsilon G$, there exists a set $V' \epsilon \Sigma'$ such that $a^{-1}V'a \subset U'$.

(1), (2), (4) and (5) are obvious.

For (3), let $U' \epsilon \Sigma'$. We get a $U^* \epsilon \Sigma^*$, Let $H \epsilon U^*$

Now, $H.H = H \tilde{\cup} H = H$.

As this lattice operation is continuous, for every neighbourhood U^* of H, there exists a neighbourhood V^* of H such that $V^*.V^* = V^* \tilde{\cup} V^* \subseteq U^*$. Thus $V'V'^{-1} \subseteq U'$

Thus Σ' satisfies all the conditions stated above.

Hence Σ' is a complete system of neighbourhood of the identity and G is a generalised topological group.

3. A topology of the group G for which the group operations are continuous will be called compatible, similarly, the topologies of L (G) for which the upper semilattice operation is continuous will be called compatible in the lattice.

Thus, from a compatible topology t in G we get a compatible topology t^*, say, in L (G) and from the compatible topology t^* in L (G), we get a compatible topology, say, t' in G.

Preposition 1 : $t' \leqslant t$.

It is obvious.

Proposition 2 : (a) $t_1 \leqslant t_2 \Rightarrow t_1^* \leqslant t_2^*$

(b) $t_1^* \leqslant t_2^* \Rightarrow t_1' \leqslant t_2'$.

Proof : (a) Let $t_1 \leqslant t_2$.

Let Σ_1 and Σ_2 are the complete system of neighbourhoods of the topologies t_1 and t_2 respectively and let Σ_1^* and Σ_2^* be the complete system of neighbourhoods of t_1^* and t_2^* obtained from Σ_1 and Σ_2 respectively.

Let $a \epsilon G$ and let $H = (a)^*$ be the cyclic subgroup generated by a.

Then $(a)^* \epsilon L$ (G). Let $a \epsilon U_1 \epsilon t_1$. From U_1 we get U_1^* and so, $H \epsilon U_1^*$.

As, $t_1 \leqslant t_2$, therefore, $\exists$ $U_2 \mid a \epsilon U_2 \subseteq U_1 \Rightarrow$

$H = (a)^* \epsilon U_2^* \subseteq U_1^* \Rightarrow t_1^* \leqslant t_2^*$.

(b) It can be easily proved.

Theorem 3 : The set T of all compatible topologies of G, is a complete laftice.

Proof : Now, the weakest topology $J=\{G, \phi\}$ where ϕ is the empty set, is a compatible topology.

Let $t_1, t_2 \epsilon T$.

But $t_1 \cap t_2$ we shall mean the largest compatible topology of G contained in t_1 and t_2.

Let $\tau \subseteq T$ be a non-empty set of topologies in T.

Then $\bigcap_{ti \,\epsilon\, \tau} ti \,\epsilon\, T$, as $J \epsilon T$.

As the discete topology is the laagest compatible topology in G, it follows that T is a complete lettice.

Theorem 4 : The set L (T) of all compatible topologies of L (G) is a complete lattice.

Proof : Now, the weakest topology $\bar{J} = \{L\ (G), \phi\} \,\epsilon\, L\ (T)$.

Also the discrete topology belongs to L (T) Thus if t_1 and $t_2 \,\epsilon\, L\ (T)$, then by $t_1 \cap t_2$ we shall mean the largest compatible topology contained in t_1 and t_2. Then as in the above case, L (T) is a complete lattice.

4. The construction of the eystem Σ^* of neighbourhoods of the compatible topoloy t^* of L (G), from a compatible topology t of G, given by the system of neighboutrhoobs Σ, as in theorem 1, is unique.

Hence one can define a mapping $f: T \rightarrow L\ (T)$ as $f\ (t) = t^*, \forall\, t \,\epsilon\, T$. This mapping f is isotone, as has been seen in Prop 2.

Also, the compatible topology t' of G, obtained from a compatible topology t^* of L (G), as in Theorem 2, is unique. Hence we cao define a mapping $f': L\ (T) \rightarrow T$ by $f'\ (t^*) = t', \forall\, t^* \,\epsilon\, L\ (T)$.

This f' is isotone, by Prop. 2.

We define a relation ρ in T such that $t_1\ \rho\ t_2$, iff $f\ (t_1) = f\ (t_2)$ holds in L (T).

ρ is an equivalence relation.

We shall show that ρ is a congruence for the lower semilattice operation.

Let $t_1\ \rho\ t_2$ and $\bar{t}_1\ \rho\ \bar{t}_2$ holds.

Then $f\ (t_1) = f\ (t_2)$ and $f\ (\bar{t}_1) = f\ (\bar{t}_2)$.

As, $t_1 \cap \bar{t}_1 \leqslant t_1, \bar{t}_1$, $f\ (t_1 \cap \bar{t}_1) \leqslant f\ (t_1), f\ (\bar{t}_1)$ by Prop. 2.

i. e. $f\ (\bar{t}_1 \cap t_1) \leqslant f\ (t_1) \cap f\ (\bar{t}_1)$ by Theorem 4. ... (1)

So, $f'\ (f\ (t_1 \cap \bar{t}_1)\ ) \leqslant f'\ (f\ (t_1) \cap f\ (\bar{t}_1)\ )$

Hence, $f'(f(t_1) \cap f(\bar{t}_1)) \leqslant f'(f(t_1)), f'(f'(\bar{t}_1)) \leqslant t_1, \bar{t}_1$

So, $f'(f(t_1) \cap f(\bar{t}_1)) \leqslant t_1 \cap \bar{t}_1$

Hence, $f(t_1) \cap f(\bar{t}_1) \leqslant f(t_1 \cap \bar{t}_1)$ by Prop. 2 ... (2)

From (1) and (2) it follows that

$f(t_1 \cap \bar{t}_1) = f(t_1) \cap f(t_1)$ and similarly, it can be shown that $f(t_2 \cap \bar{t}_2) =$
$f(t_2) \cap f(\bar{t}_2)$.

Hence it follows that $f\, t_2 \cap \bar{t}_2) = f(t_2) \cap f(\bar{t}_2 = f(t_1 \cap \bar{t}_1)$

Hence, ρ is a congruence relation for the lower-semilattice. Thus we have :

Theorem 5 : The lower semilattice T/ρ is monomorphic to L (T).

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REFERENCES

[1] Birkhoff, G. : Lattice Theory. Amer, Math. Soc. Coll. Publ. 1968.

[2] Chatterjee, B.C : Abstract Algebra Das Gupta & Co. (Pvt. Ltd.).

[3] Huassiau, T. : Introduction to Topological Groups. W. B. Saunders Company, Philadelphia and London 1966.

[4] Pontrajagin, L. : Topological Groups, Gorden, 2nd Edition, Princeton University Press 1966.

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Department of Mathematice,
Bengal Engineering College,
Shibpur, Howrah.

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EMBEDDING OF A REGULAR RING IN A REGULAR RING WITH IDENTITY.

Jayasree Ghosh

1. Regular rings were first introduced by J. V. Neumann. In his definition of regular rings, the presence of identity was always assumed and the whole theory was developed with that assumption Subsequently many prominent mathematicians avoided this assumption of identity. So the question naturally arises under what condition a regular ring can be imbedded in a regular ring with identity. In this paper, an attempt has been made to attack this problem.

2. By a regular ring R we mean an associative ring in which $axa=a$ is solvable for all a in R.

Proposition 1. For any regular ring R, aR is a principal right ideal generated by $a\epsilon R$ and $aR=eR$, where e is an idempotent in R.

Proof : Let $a\epsilon R$, then $\exists x\epsilon R/axa=a$. Put $xa=f$, then $f^2=f$ and $af=a$. Thus $a=af\epsilon aR$.

Thus aR is a principal right ideal generated by a.

Let $z\epsilon aR$, Put $ax=e$, then $ea=a$, $e^2=e$.

Thus $z\epsilon aR \Rightarrow z=ar$, for some $r\epsilon R$.

$\Rightarrow z=ear\epsilon eR$.

Thus $aR \subseteq eR$. ....(1)

Now for any $p \epsilon R$,

$ep\epsilon eR \Rightarrow ep=axp=aq\epsilon aR$.

$eR \subseteq aR$ ....(2)

Hence $aR=eR$ by (1) and (2).

Proposition 2. For any two principal right ideals eR and fR of a regular ring R, we have $eR+fR=(e+g)R$, where g is an idempotent with $gR=(f-ef)R$.

Proof. As $g=g.g=(f-ef)\alpha$, then $eg=0$, which shows $e=(e+g)-(e+g)g$. Put $e+g=a$, then $a\epsilon R \Rightarrow \exists x\epsilon R$ such that $axa=a$. Let $xa=k$, then $k^2=k$ and $ak=a$.

Thus $e=ak-ag=a(k-g)=(e+g)(k-g)\epsilon(e+g)R$.

But $f-ef=g(f-ef)$ as by proposition 1, $(f-ef)\epsilon gR$.

Therefore $f=ef+g(f-ef)+e(f-ef)$.

$=ef+(e+g)(f-ef)\epsilon(e+g)R$.

Hence $eR+fR \subseteq (e+g)R$. ....(1)

On the othe hand $e+g=e+(f-ef)\alpha=e(e-f\alpha)+f\alpha\epsilon eR+fR$.

and $(e+g)R\subseteq eR+fR$ ....(2)

From (1) and (2) we get $(e+g)R=eR+fR$.

Proposition 3. For any two principal right ideals eR and fR of a regular ring R, we have $eR\cap fR=(f-fg)R$, where g is an idempotent with $Rg=R(f-ef)$.

Froof : Indeed $f-fg=f(f-g)\epsilon fR$

$f-fg=(f-ef)+(ef-fg)=(f-ef)g+(ef-fg)$

[as by proposit on 1, $(f-ef)\epsilon Rg$.]

$=e(f-fg)\epsilon eR$.

Then $(f-fg)R\epsilon fR\cap eR$. ....(1)

On the otherhand, let $x\epsilon\ eR\cap fR$.

Then $x=ex=fx$ and $g=\beta(f-ef)$.

Thus $x=x-f\beta(x-x)=x-f\beta(f-ef)x$

$=x-fgx=(f-fg)x$. Hence $eR\cap fR\subseteq(f-fg)R$ ......(2)

From (1) and (2) we have $eR\cap fR=(f-fg)R$.

Proposition 4. Let e be a given idempotent in a regular ring R. Then the set of all idempotents $f\epsilon R$ such that $eR=fR$ is exactly the set : $\{e+(ey-eye)\ ;\ y\epsilon R\}$.

Proof : First we prove that $eR=fR$ if and only if $e=fe$; $f=ef$. Now these two equations themselves imply that f is idempotent, $f^2=f.f.=f(ef)=(fe)f=ef=f$.

Hence the equations alone characterize the elements f.

Let us define x by the relation $f=e+x$. Then the relation $e=fe$, $f=ef$ means $e=(e+x)e$; $e+x=e(e+x)$ i.e. $xe=0$, $ex=x$. The latter two equations clearly hold if x is of the form $x=ey-eye$ and conversely imply $x=ey-eye$ with $y=x$. Hence our elements f are given by $f=e+(ey-eye)$, $\forall y\epsilon R$.

Proposition 5. If R is a regular ring, then an idempotent $e\epsilon R$ is central if and only if $ey-eye=0$ for every $y\epsilon R$.

Proof : If e is central, then $ey-eye=ey-ey=0$, $\forall$ $y\epsilon R$ Conversely let $ey-eye$ $=0\forall y\epsilon R$ ....(1)

Now $ye-eye \in R$ Thus by regularity of R, $\exists a \in R$
such that $(ye-eye)\,a.\,(ye-eye)=(ye-eye)$.
Hence $ye-eye=(yea-eyea)(ye-eye)$
$=(yeaye-eyeaye-yeaeye+eyeaeye)$
$=y(ea-eae)\,ye-ey(ea-eae)ye$
$=0$, (from 1).
Thus $ye=eye=ey$, $\forall y \in R$ Thus e is central.

From propositions (4) and (5) we get a principal right ideal eR is uniquely generated iff e is central.

Thus by propositions (1), (2) & (3), for a regular ring R, the set {eR} of all principal right ideals forms a lattice $\{\mathcal{R}\,(R), \cup, \cap\}$ with respect to usual set inclusion relation.

This lattice need not be complete. If for a collection of elements $\{e_\lambda R\}$ of $\mathcal{R}$ (R) we can find a unique element e, for which eR is the lub of $\{e_\lambda R\}$ and in that case, by propositions (4) and (5) e is central in R, we define lub $\{e_\lambda\}=e$.

Proposition 6 If C (R) is the centre of a regular ring R, then C (R) is also a regular ring,

Proof : Let $a \in C(R)$ Then $a \in R \Rightarrow axa=a$ for some $x \in R$.
Define $y=ax^2$ then $aya=a.ax^2a=(axa)xa=axa=a$.
Also $u \in R \Rightarrow yu=ax^2u=x^2ua=x^2uaxa=x^2a^2ux$.
$=xu(axa)x=xa^2ux^2=aux^2=uax^2=uy \Rightarrow y \in C\ (R)$
Thus C (R) is a regular ring.

Definition : By a complete direct sum $\overset{C}{\Sigma}\oplus R_\alpha$ of the rings R_α we mean the set of all infinite rows $\{r_\alpha\}$ where $r_\alpha \in R_\alpha$

If we define equality, addition and multiplication componentwise, then $\overset{C}{\Sigma}\oplus R_\alpha$ is a ring.

Proposition 7. If $\{R_\alpha\}$ be any collection of regular rings, then the complete direct sum $\overset{C}{\Sigma}\oplus R_\alpha$ of all regular rings R_α is also a regular ring.

Proof : Let $\{r_\alpha\}$ be any element of $\overset{C}{\Sigma}\oplus R_\alpha$ where $r_\alpha \in R_\alpha$ then each R_α being regular,

$\exists x_\alpha \in R_\alpha$ such that $r_\alpha x_\alpha r_\alpha = r_\alpha$

Let us collect all $x_\alpha \in R_\alpha$ and form the infinite row $\{x_\alpha\}$.

Then $\{x_\alpha\} \in \overset{C}{\Sigma}\oplus R_\alpha$ and $\{r_\alpha\}\{x_\alpha\}\{r_\alpha\}=\{r_\alpha\}$

Hence $\overset{C}{\Sigma}\oplus R_\alpha$ is a regular ring.

3. We now proceed to prove our main theorem viz, the theorem of embeddability of a regular ring into a regular ring with identity.

Theorem 1. Let R be a regular ring with C (R) its centre such that annihilator $(C(R))=(0)$ in R. Then R can be embedded in a regular ring R' with 1, where $1=\text{lub}\,\{e_\lambda\}$ and $\{e_\lambda\}$ denoting the set of all central idempotents in R.

Proof : Let R be any regular ring, C(R) is its centre.

Let $e \in C(R)$, then eR is a regular ring with e as the identity.

Let $R'=\overset{C}{\Sigma}\oplus(e_\nu R)$, $e_\nu \in C(R)$

By proposition 7, R' is a regular ring with identity

Now $(\ldots\ldots, e, \ldots, f, \ldots\ldots, g\ldots)$ is the identity of R,

where $\{\ldots e, \ldots f, \ldots g, \ldots\}$ denotes the total collection of central idempotents of R.

We define ϕ ; $R \to R'$ as follows :

$a\phi=(\ldots\ldots, ea, \ldots\ldots, fa, \ldots\ldots ga, \ldots\ldots)$

Then $(a+b)\phi=(\ldots\ldots, e(a+b), \ldots\ldots f(a+b), \ldots\ldots, g(a+b), \ldots\ldots)$

$=(\ldots\ldots, ea, \ldots\ldots, fa, \ldots\ldots, ga, \ldots\ldots)+(\ldots\ldots, eb, \ldots\ldots, fb, \ldots\ldots, gb, \ldots\ldots)$

$=a\phi+b\phi$

$(ab)\phi=(\ldots\ldots, e(ab), \ldots\ldots, f(ab), \ldots\ldots, g(ab), \ldots\ldots)$

$=(\ldots\ldots, (ea)(eb), \ldots\ldots, (fa)(fb), \ldots\ldots, (ga)(gb), \ldots\ldots)$

$=(\ldots\ldots, ea, \ldots\ldots, fa, \ldots\ldots, ga, \ldots\ldots)(\ldots\ldots, eb, \ldots\ldots, fb, \ldots\ldots, gb, \ldots\ldots)$

$=(a\phi)(b\phi)$

Now to prove injectivity,

let $a\phi=0$. Now $a \in R \Rightarrow \exists x \in R / axa=a$.

Put $h=xa$ then $h^2=h$ and $h\phi=(x\phi)(a\phi)=0$.

Therefore $(\ldots\ldots, he, \ldots\ldots, hf, \ldots\ldots, hg, \ldots\ldots)=0$

[$e, f, g \ldots\ldots \in C(R)$, $he=eh$ etc]

$\Rightarrow he_\nu=0, \ \forall e_\nu \in C(R)$

We have to show $h=0$

Let $p \in C(R)$, $\exists$ idempotent $l \in C(R) / pl=p$.

Now $hl=0 \Rightarrow hp=0$,

thus $h \in \text{Ann}(C(R)) \Rightarrow h=0$ by hypothesis.

Thus R is embedded in R'.

Therefore from now on an element of R can be considered as an element of R'.

Thus $e \in C(R)$ will take the form $e=(\ldots, e, \ldots, fe, \ldots, ge, \ldots)$ in R'

And $eR'=(\ldots, e, \ldots, fe, \ldots, ge, \ldots)R'$

Let $\{e_\nu\}$ be the set of all idempotents in $C(R)$.

Let $\cup\{e_\nu R'\}=I$ then I is a both sided ideal in R'

We will show $I=R'$

To show this we will use induction to prove that :

$(\ldots, e, \ldots, f, \ldots, g, \ldots) \in I$

Let $\mathcal{A}$ denote the set of all possible non-null rows obtained from the identity [i. e. the row $(\ldots, e, \ldots, f, \ldots, g, \ldots)$] replacing some or none components by zero.

Note that $\mathcal{A}$ is a subset of R'.

Also $(\ldots, e, \ldots, f, \ldots, g, \ldots) \in \mathcal{A}$

Let $A, B \in \mathcal{A}$

We define $A \leqslant B$, iff $AB=A$.

Then $A \leqslant A$, as $AA=A$.

Let $A \leqslant B$, $B \leqslant A$, then $AB=A$ and $BA=B$.

But $AB=BA$, therefore $A=B$.

Let $A \leqslant B$, $B \leqslant C$, then $AB=A$, $BC=B$,

thus $AC=(AB)C=A(BC)=AB=A$,

Hence $\leqslant$ is a relation of partial order.

Note that the minimal elements of $\mathcal{A}$ are of the form $(\ldots, 0, \ldots, 0, \ldots, e, \ldots)$ etc

i. e. the rows obtained from $(\ldots, e, \ldots, f, \ldots, g, \ldots)$

by replacing all but only one component by zero.

We define a property P on A as follows :

$A\in \mathcal{A}$ is said to have P iff $A\in I$

Now the minimals of $\mathcal{A}$ have property P.

Indeed $(\ldots, e\ldots, ef, \ldots, eg, \ldots)\in I$

and $(\ldots, 0, e, \ldots, 0, \ldots, 0, \ldots)\in R'$

I being both sided ideal, the product of these two elements viz.

$(\ldots, 0, \ldots, e, \ldots, 0, \ldots)_\epsilon I$

Suppose the property P holds for every $X\leqslant A$

We will show that property holds for A. Let B be any element of $\mathcal{A}$ but $B\underset{\neq}{<}A$,

then $B\epsilon I$ by hypothesis.

Put $C=A-B[A, B\epsilon R' \Rightarrow A-B$ is defined]

Evidently $C\epsilon A$,

thus $CA=(A-B)A=A-BA=A-B=C$,

thus $C\leqslant A$. But $C\neq A$. For otherwise $C=A\Rightarrow B=0$, contradiction as B is a non-null row.

Hence $C\epsilon I$. That is, $A=B+C\epsilon I$.

Thus by induction hypothesis every element of $\mathcal{A}$ has property P.

So $(\ldots, e, \ldots f, \ldots, g, \ldots)\epsilon I$

Hence $I=R'$

If possible $\bigcup \{e_\nu R'\}=gR'$

$e_\nu \in C(R)$

Now $R'=gR' \Rightarrow 1=g.x \Rightarrow g=g.x=1$

Hence lub $\{e_\nu\}=1$

This completes the proof.

4. Let us now study the situation, when a regular ring is embedded in a regular ring with identity. To do this, we start with the following proposition.

Proposition 8. Let C denote the set of all central idempotents in a regular ring R, then $C^*. C^*=C^*$

Proof : Let $ef\epsilon C^*. C^*$, then $(ef)x=ef(x)=e(xf)$

$=x(ef)\Rightarrow ef\epsilon$ cent R.

Also $(ef)^2=(ef)\ (ef)=e^2f^2=ef$

Thus $ef \in C^*$. Conversely let $e \in C^*$, then $e=e.e \in C^*$. $C^* \Rightarrow C^*.C^*=C^*$

Theorem 2. If a regular ring R is imbedded in a-regular ring R′ with identity, then we can construct a regular ring D with identity having the properties.

i) C (R) can be imbedded in D.

ii) Annihilator (C (R)) = (0) in D.

Proof : Suppose R is imbedded in R′ with identity and C (R) denote its centre. We define a binary relation on R′ as follows $a \sim b$ iff $ae_\lambda = be_\lambda$, $\forall e_\lambda \in C(R)$

Then i) $a \sim a$

ii) $a \sim b \Rightarrow b \sim a$

ii) Let $a \sim b$, $b \sim c$ hold

Then $ae_\lambda = be_\lambda$, $\forall e_\lambda \in C(R)$

$be_\nu = ce_\nu$, $\forall e_\nu \in C(R)$

Thus $ae_\lambda e_\nu = ce_\lambda e_\nu$, $\forall e_\lambda, e_\nu \in C(R)$

Now by proposition 8, $a\,e_p = ce_p$, $\forall e_p \in C(R)$

Thus $a \sim c$.

Hence $\sim$ defines an equivalence relation on R′ and partitions elements of R′ into disjoint classes.

Let (a) denote the class corresponding to a.

Let $D=\{(a)\}$

We define in D, +, as follows.

$(a)+(b)=(a+b)$

$(a)\ (b)=(ab)$

Operations are well defined.

Indeed $(b)=(c) \Rightarrow be_\lambda = ce_\lambda$, $\forall e_\lambda \in C(R)$.

Also $(a+b)\,e_\lambda = ae_\lambda + be_\lambda = ae_\lambda + ce_\lambda = (a+c)\,e_\lambda$, $\forall e_\lambda \in C(R)$

$\Rightarrow (a+b)=(a+c) \Rightarrow (a)+(b)=(a)+(c)$

Similarly $(ab)\,e_\lambda = (ae_\lambda)\,(be_\lambda) = (ae_\lambda)\,(ce_\lambda) = (ac)\,e_\lambda\ \forall e_\lambda \in C(R)$

$\Rightarrow (a)\,(b)=(a)\,(c)$.

Let $(a) \in D$

Then $a \in R$, and by regularity, $\exists x \in R / axa = a$.

Therefore $(axa) = (a)$. Hence $(a)(x).(a) = (a)$.

Thus D is regular.

(1) is the identity of D. Note that (1) contains those elements $x \in R'$ such that $xe_\lambda = 1e_\lambda = e_\lambda$, $\forall e_\lambda \in C(R)$

Thus D is a regular ring with identity.

i) Let $a \in C(R)$.

We define ϕ : $C(R) \to D$

by $a\phi = (a)$, $\forall a \in C(R)$

Now $(a+b)\phi = (a+b) = (a)+(b) = a\phi + b\phi$

$(ab)\phi = (ab) = (a)\ (b) = (a\phi)(b\phi)$

Let $a\phi = (0)$, so $(a) = (0)$, $ae_\lambda = 0$, $\forall e_\lambda \in C(R)$

Now $C(R)$ being regular, $\exists e \in C(R)/ae = a$,

thus $aee_\lambda = 0$, $\forall e_\lambda \in C(R)$

In particular $ae = 0$, as $e \in C(R)$.

$\Rightarrow a = 0$, thus ϕ is injective and $C(R)$ in imbedded in D.

ii) Let $(g) \in \operatorname{Ann}(C(R))$

$\Rightarrow (g)(e_\lambda) = (0)$, $\forall e_\lambda \in C(R)$.

$\Rightarrow (ge_\lambda) = (0) \Rightarrow ge_\lambda e_\nu = 0$, $\forall e_\lambda, e_\nu \in C(R)$

$\Rightarrow ge_p = 0$, $\forall e_p \in C(R)$, by propositions.

$\Rightarrow (g) = (0)$

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REFERENCES

1. J. V. Neumann. Continuous Geometry University Press 1960.

2 M. C Waddel. Properties of regular ring—Duke Math Journal J.19 (1952) 623-627.

3 A. G. Kurosch. Lectures in General Algebra. London Pergamon Press 1965.

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Department of Pure Mathematics,
Calcutta University
35, Ballygunge Circular Road,
Calcutta-700 019.

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ON SEMILINEAR TENSOR PRODUCT

A. K. Maity

ABSTRACT : The object of this paper is to introduce the concept of semilinear tensor product of normed linear spaces over fields with real valued valuations and to consider norm on such tensor product spaces.

1.1. INTRODUCTION :

F.F. Bonsall and J. Duncun [2] have introduced norms in tensor product of normed linear spaces over the field of real numbers. We have, in this paper, introduced semilinear tensor product of normed linear spaces over fields having real valued valuations, using semilinear transformation [4] and finally we have discussed norm (weak) on such tensor product spaces.

1.2. Normed linear space over a field with real valuation.

DEFINITION 1.2.1. :

A linear space X over a field F, with a real valued valuation p is said to be a normed linear space over F, if there exists a map $X \to R$ (denoted by $\| \ \|$) s. t.

i) $\| x \| \geqslant 0 \quad \| x \| = 0$ iff $x=0$, $x \epsilon X$.

ii) $\| x+y \| \leqslant \| x \| + \| y \|$, $y \epsilon X$

iii) $\| \alpha x \| = p(\alpha) \| x \|$, $(\alpha \epsilon F)$

Remarks : 1) Under the above definition F is a normed linear space over itself where $\| \alpha \| = p(\alpha)$.

2) It is known that [1], if K be a field with real valued valuation, then the Cauchy sequences over K form a commutative ring Λ containing the identity element, and the null sequences form a maximal ideal N of Λ. Hence the quotient ring Λ/N is a field: Now let $\{a\}$ be the sequence, every element of which is $a \epsilon K$, then $\{a\}$ is obviously a Cauchy sequence, it is called a constant sequence, the corresponding class $(\{a\})$ is called the

principal class. It can be shown that the principal classes belonging to Λ/N. form a subfield isomorphic to k. Hence there exists an extension Ω of K isomorphic to Λ/N, Ω is called the derived field of K. Therefore, *identifying the elements of K with the corresponding principal classes in* Λ/N, K may be imbedded in Λ/N and Ω may then be considered to be identical with Λ/N. It can be verified that the derived field Ω of K, with real valued valuation ϕ, is a completion of K in the sense that

i) Ω has a real valuation Ψ which is an extension of ϕ, where $\Psi(A) = \underset{n\to\infty}{\text{Lt.}}\ \phi(a_n)$,

for any $\{a_i\} \in A$,

A denoting the residue class in Λ/N.

ii) Ω is complete w.r. to Ψ.

iii) K is dense in Ω.

Following remark (2), we have the following definition.

1.2.2. A map $f : X \to F$, X being an arbitrary set and F being an arbitrary field with a real valued valuation p is said to be bounded on X, if $\| [\{f(x)\}] \| \leqslant M$, M being an arbitrary real number and $[\{f(x)\}] \in \Omega$, Ω being the derived field of F.

i. e. if $p'[\{f(x)\}] \leqslant M$, $\{f(x)\}$ being a Cauchy Sequence over the set $(f(x)) \subset F$, $[\{f(x)\}] \in \Lambda/N$, where Λ is the set of all Cauchy Sequences over $(f(x))$, N is the set of all null (Cauchy) sequences over $(f(x))$ and p' is the extension of the real valued valuation p of F.

1.2.3. A linear map $f : X \to F$, X being a normed linear space over an arbitrary field of scalars F with a real valued valuation p is said to be bounded on X, if $\| [\{f(x)\}] \| \leqslant M \| x \|$, $\forall\, x \in X$, M being real.

Since in the definition of Ω, each element $f(x) \in F$ has been identified with the corresponding principal class of $f(x)$, norm of $f(x)$ will be given by $\| f(x) \| = p\, f(x)$, where p is the restriction of p' to the set $(f(x))$. Hence it follows from the above definition that f is continuous on X iff f is bounded on X.

From now on we shall mean by X(F), a normed linear space over an arbitrary field of scalars F, with a real valued valuation p.

PROPOSITION 2. 1.

The set BL (X, F) of all bounded (continuous) linear maps from X(F) of F is a Banach space under pointwise addition and scalar multiplication and norm defined by

$\| f \| = \sup \| [\{f(x)\}] \|, \quad \| x \| \leqslant 1.$

$f \in BL(X, F)$. $[\{f(x)\}] \in \Omega$.

PROOF : BL (X, F) is obviously a linear space over F under pointwise addition and scalar multiplication. To show that it is a normed linear space, we note that,

i) $\| f \| \geq 0$, $\| f \| = 0$ iff $f = 0$ [By definition]

ii) $\| f+g \| \leq \| f \| + \| g \|$ [By definition]

iii) $\| \alpha f \| = p(\alpha) \| f \|$, for

$$\| \alpha f \| = \sup_{\| x \| \leqslant 1} p' [\{(\alpha f) x\}] = \sup_{\| x \| \leqslant 1} p' [\{\alpha(f(x)\}] \text{ by definition}$$

$$= \sup_{\| x \| \leqslant 1} p'[\{\alpha\} \{f(x)\}], \{\alpha\} \text{ being the constant Cauchy sequence } \{\alpha, \alpha, \ldots\ldots, \alpha\}.$$

$$= \sup_{\| x \| \leqslant 1} p' [\{\alpha\}] \sup_{\| x \| \leqslant 1} p' [\{f(x)\}]$$

$$= p(\alpha) \| f \|$$

Hence BL (X, F) is a normed linear space over F.

To show that BL (X, F) is complete.

Let {fn} be a Cauchy Sequence in BL (X, F), then

$\| f_m - f_n \| < \epsilon$, for $m, n \geqslant N$ (intger). Hence for some fixed

$$x \in X, \sup_{\| x \| \leqslant 1} \| [\{(f_m - f_n) x\}] \| = \sup_{\| x \| \leqslant 1} \| [\{f_m (x) - f_n (x)\}] \|$$

$$= \sup_{\| x \| \leqslant 1} \| [\{f_m (x)\}] - [\{f_n (x)] \| < \epsilon, \text{ for } m, n \geqslant N$$

Thus the classes $[\{f_n(x)\}]$ form Cauchy sequences in Ω, therefore, Ω being complete, the Cauchy sequence of $[\{f_n (x)\}]$ converges in Ω.

Hence $[\{f_n (x)\}]$ tends to $[\{f(x)\}] \in \Omega$, as $n \to \infty$.

Since $x \in X$ is arbitrary, this defines a mapping $f : X \to F$. It remains to show that f is linear, f is bounded and that $f_n \to f$ as $n \to \infty$.

But these are routine verifications by considering Ω as a complete normed linear space. Therefore, BL (X, F) is a Banach space over F.

PROPOSITION : 2.2. We may similarly prove that the set B (X, F) of all bounded maps $f : X \to F$, X being an arbitrary set and F being an arbitrary field with a real valued valuation is a Banach space over F under pointwise addition and scalar multiplication and norm defined by $\| f \| = \sup_{x \in X} \| [\{f(x)\}] \|$

DEFINITION : 2.4. Let X (F_1) and Y (F_2) be normed linear spaces over arbitrary fields F_1 and F_2, which are respectively isomorphic to an arbitrary field F, with a real valuation, under isomorphisms ξ_1 and ξ_2; then a map $f : X \times Y \to F$ is called a bounded bi-semilinear map, if.

$\| [\{f(x, y)\}] \| \leqslant \| M \| \; \| x \| \; \| y \|$,

$x \epsilon X$, $y \epsilon Y$, $[\{f(x, y)\}] \epsilon \Omega$, Ω being derived field of F.

[A mapping $\Psi : X \times Y \to F$ is called bi-semilinear if

$\Psi\ r_1 x_1 + s_1 x_2, y) = \xi_1 r_1\ \Psi\ (x_1, y) + \xi_1 s_1\ \Psi\ (x_2, y)$

$\Psi\ (x, r_2 y_1 + s_2 y_2) = \xi_2 r_2\ \Psi\ (x, y_1) + \xi_2 s_2\ \Psi\ (x, y_2)$:

$x, x_1, x_2 \epsilon X$; $y, y_1, y_2\ \epsilon Y$; $s_1, r_1 \epsilon F_1$; $r_2, s_2 \epsilon F_2$.

Now proceeding as in Prop. 2. 1, we may prove

PROPOSITION : 2 3. The set BL (X, Y ; F) of all bounded bi-semilinear maps $f : X \times Y \to F$ is a Banach space over F under pointwise addition and scalar multiplication and norm defined by

$\| f \| = \sup \| [\{f(x, y)\}] \|$, $\| x \| \leqslant 1$, $\| y \| \leqslant 1$.

3.0. SEMILINEAR TENSOR PRODUCT :

3. 1. Let X (F_1) and Y (F_2) be linear spaces over arbitrary fields F_1 and F_2 and let Ψ be a bi-semilinear map Ψ: X $(F_1) \times$ Y $(F_2) \to$ Z (F), where Z is a linear space over the field F, F_1 and F_2 being isomorphic to F under isomorphisms ξ_1 and ξ_2 respectively. The couple (Z, Ψ) is called the semilinear tensor product of X and Y if (Z, Ψ) possesses universal factorization property [4] in the sense that for every bi-semilinear map $f : X \times Y \to S$ (F), there exists exists a unique linear map $g : Z \to S$ such that $f = g \circ \Psi$.

3.2. SEMILINEAR TENSOR PRODUCT OF NORMED LINEAR SPACES :

Let X (F_1) and Y (F_2) be normed linear spaces over F_1 and F_2 ; ξ_1, ξ_2 be isomorphisms of F_1 and F_2 to F, which possesses a real valued valuation p. Let X' (F_1) and Y' (F_2) be linear dual spaces of X and Y respectively, i.e. $f : X \to F_1$ and $g : Y \to F_2$ where $f \epsilon X' (F_1)$ and $g \epsilon Y' (F_2)$ and B L^s (X' , Y' ; F) denote the space of bounded bi-semilinear maps Ψ : $X' \times Y' \to F$. Let $(X \otimes Y)_F$ denote an element of BL^s (X' , Y' ; F) such that

$(X \otimes Y)_F\ (f, g) = \xi_1 f(x)\ \xi_2\ g\ (y)$; $f \epsilon X'$, $g \epsilon Y'$.

Then the semilineer tensor product $(X \otimes Y)_F$ is defined to be the linear span of $(x \otimes y)_F$ in BL^s (X' , Y' ; F) for $\tau : X \times Y \to (X \otimes Y)_F$ being a bi-semilinear map. it can be show as below that $(X \otimes Y ; \tau)$ has universal factorization proyerty.

LEMMA 1. Given $u \epsilon (X \otimes Y)_F$, there exist linearly independent sets $\{x_i\}$, $\{y_i\}$ such that

$$u = \sum_1^n (x_i \otimes y_i)_F$$

PROOF : If possible, let $y_n = \sum_{1}^{n-1} d_i y_i$, $d_i \in F_2$

$$\text{Then } u = \sum_{1}^{n-1} (x_i \otimes y_i)_F + \sum_{1}^{n-1} (x_n \otimes d_i y_i)_F$$

$$= \quad ,, \quad + \sum_{1}^{n-1} \xi_2 (d_i) (x_n \otimes y_i)_F$$

$$= \quad ,, \quad + \sum_{1}^{n-1} \xi_1 (c_i) (x_n \otimes y_i)_F ,$$

$$[\xi_1 (c_i) = \xi_2 (d_i), c_i \in F_1]$$

$$= \sum_{1}^{n-1} (x_i \otimes y_i)_F + \sum_{1}^{n-1} (c_i x_n \otimes y_i)_F$$

$$= \sum_{1}^{n-1} ((x_i + c_i x_n) \otimes y_i)_F$$

which is a contradiction as n is minimal.

Hence y_i and therefore x_i's are linearly independent.

LEMMA : 2. Let $\sum_{1}^{n} (x_i \otimes y_i)_F = 0$, where x_i s are linearly independent. Then

$$y_i = 0, \ i = 1, 2. \ldots\ldots n.$$

PROOF : We have, for $f \in X' (F_1)$ and $g \in Y' (F_2)$,

$$\left(\sum_{1}^{n} (x_i \otimes y_i)_F\right) (f, g) = 0 \in F$$

$$\therefore \sum_{1}^{n} (\xi_1 f (x_i)\, \xi_2\, g (y_i)) = 0$$

$$\text{i. e. } \xi_1 f \left(\sum_{1}^{n} \left(\xi_1^{-1} \{\xi_2\, g (y_i)\} \right) x_i\right) = 0$$

$$\text{i. e. } \sum_{1}^{n} (\xi_1^{-1} \{\xi_2 (g (y_i))\} \quad x_i = 0 \in X \text{ [f being arbitrary]}$$

i.e $\xi_1^{-1} \{\xi_2 (g (y_i))\} = 0 \in F_1$ [x_i 's are L. I]

i.e $g(y_i) = 0 \in F_2$, i.e $y_i = 0$ [g being arbitrary]

LEMMA : 3. If $\{x_i\}$; $i=1, 2, \ldots\ldots, m$ and $\{y_j\}$; $j=1, 2, \cdots, n$ are linearly independent subsets of X and Y respectively, then $\{x_i \otimes y_j ; i=1, 2, \ldots \ldots, m ; j=1, 2, \ldots\ldots, n\}$ is a linearly independent subset of $(X \otimes Y)_F$.

Proof is immediate from Lemma 1 and Lemma 2.

Now to prove that $((X \otimes Y)_F \; ; \; \tau)$ has Universal factorization property, we consider $\phi : X \times Y \to Z\,(F)$, any bi-semilinear map and show that there exists a unique linear map $\sigma : ((X \otimes Y)_F) \to Z\,(F)$ such that $\sigma((x \otimes y)_F) = \phi(x, y)$; $x \in X$, $y \in Y$.

Considering an element of $(X \otimes Y)_F$ as $\left(\sum_{r=1}^{n} (x_r \otimes y_r)\right)_F$, it is enough to show that $\left(\sum_{r=1}^{n} (x_r \otimes y_r)\right)_F = 0$ implies $\sum_{r=1}^{n} \phi(x_r, y_r) = 0$, if we claim that σ is defined as $\sigma\left(\sum_{r=1}^{n} x_r \otimes y_r\right) = \sum_{r=1}^{n} \phi(x_r, y_r)$. The proof follows directly from Lemma 3.

3.3. Norms on semilinear tensor product spaces :

DEFINITION : 3.3.1 : Let $X\,(F_1)$ and $Y\,(F_2)$ be given nermed linear spaces over arbitrary fields F_1 and F_2 which are respectively isomorphic to another field F having a real valued valuation. Then weak norm on $u = \sum_i (x_i \otimes y_i)_F$ is defined by

$$\omega(u) = \sup \| [\{\sum_i \xi_1 f(x_i)\, \xi_2\, g(y_i)\}] \| \;,\; \|f\| \leqslant 1,\; \|g\| \leqslant 1,$$
$$f \in X',\; g \in Y'$$

Obviously $\omega((x \otimes y)_F) = \|x\| \; \|y\|$ [Considering X and Y as linear dual spaces of X′ and Y′ respectively.]

Since $BL^s(X', Y' ; F)$ is a Banach space, therefore $(X \otimes Y)_F$ is closed in $BL^s(X', Y' ; F) \Longleftrightarrow (X \otimes Y)_F$ is complete in $BL^s(X', Y' ; F)$ under ω.

DEFINITION 3.3.2.

The weak semi-linear tensor product of X and Y is defined as the closure of $(X \otimes Y)_F$ in $BL^s(X', Y' ; F)$ [i. e., the completition of $(x \otimes y)_F$ in $BL^s(X', y' ; F)$ and it is denoted by $(X \otimes_\omega Y)_F$.

PROPOSITION 3.1. Let X and Y be two non-empty arbitrary sets and F_1 and F_2 be two arbitrary field of scalars isomorphic to an associative field F (having a real valued valuation) under isomorphisms ξ_1 and ξ_2 respectively. Then there exists a linear isometric isomorphism T′ of $((B(X, F_1) \otimes_\omega B(Y, F_2))_F \to B(X \times Y, F)$ such that

$(T(f \otimes g))' = \xi_1 f(x) \xi_2 g(y)$; $x \in X$, $y \in Y$, $f \in B(X, F_1)$, $g \in B(Y, F_2)$ where T is the restriction of T′ to $B(X, F_1) \otimes B(Y, F_2)$.

Proof : We first define a map $\Psi : B(X, F_1) \otimes B(Y, F_2) \to B(X \times Y ; F)$ such that $(\Psi(f, g))(x, y) = \xi_1 f(x) \xi_2 g(y)$; Then Ψ is bi-semilinear as shown below :

$$[\Psi(f+h, g)](x,y) = \xi_1(f+h)(x)\xi_2 g(y)$$
$$= \xi_1[f(x)+h(x)]\xi_2 g(y) \text{ [By definition of } f+g]$$
$$= [\xi_1 f(x) + \xi_1 h(x)] \xi_2 g(y) \ [\xi_1 \text{ being an isomorphism}]$$
$$= \xi_1 f(x) \xi_2 g(y) + \xi_1 h(x) \xi_2 g(y) \text{ [By distributive property of F]}$$
$$= \Psi(f,g)(x, y) + \Psi(h,g)(x,y)$$
$$= [\Psi(f,g) + \Psi(h,g)](x, y) \text{ [By definition of the sum in } B(X \times Y ; F)$$

Hence $\Psi[(f+h), g] = \Psi(f, g) + \Psi(h, g)$

Similarly $\Psi(f, g+k) = \Psi(f, g) + \Psi(f, k)$

Also $\Psi(\alpha f, g) = \xi_1(\alpha) \Psi(f, g)$; $\alpha \in F_1$, for

$$[\Psi(\alpha f, g)](x, y) = \xi_1(\alpha f)(x) \xi_2 g(y)$$
$$= \xi_1[\alpha f(x)] \xi_2 g(y) \text{ [By definition of } \alpha f \text{ in } B(X, F_1)]$$
$$= [\xi_1(\alpha) \xi_1(f(x))] \xi_2 g(y) \ [\xi_1 \text{ being an isomorphism}]$$
$$= \xi_1(\alpha) \psi(f, g), \text{ by associative property of F.}$$

Similarly $\psi(f, \beta g) = \xi_2(\beta) \psi(f, g)$; $\beta \in F_2$.

Hence there exists a unique linear mapping $T : (B(X, F_1) \otimes B(Y, F_2))F \to B(X \times Y; F$;

By definition of the tensor product, T is an isomorphism. To show that T is isometric, we consider

$u = \sum_i (f_i \otimes g_i) \in (B(X F_1) \otimes B(Y, F_2))F$; $f_i \in B(X, F_1)$, $g_i \in B(Y, F_2)$ and prove that $\| T(u) \| = \omega(u)$; infact, denoting norm of the derived field Ω of F by ($\|\ '$)

$$\| T(u) \| = \| T(\sum_i (f_i \otimes g_i)) \|$$
$$= \sup_{X \times Y} \| [\{T(\sum_i (f_i \otimes g_i))(x, y)\}] \|'$$
$$= \sup_{X \times Y} \| [\{\sum_i \xi_1 f_i(x) \ \xi_2 g_i(y)\}] \|'$$
$$= \sup_Y \| \sum_i \xi_2 g_i(y) f_i \|$$
$$= \sup_{\|\mu\| \leqslant 1} \sup_Y \| [\{\sum_i \xi_2 g_i(y) \xi_1 \mu(f_i)\}] \|', \ \mu \in (B(X, F_1))'$$
i. e, $\mu : B(X, F_1) \to F_1$

$$= \sup_{\|\mu\| \leqslant 1} \| \sum_i \xi_1 \mu (f_i) g_i \|$$

$$= \sup_{\|\mu\| \leqslant 1} \sup_{\|\nu\| \leqslant 1} \| [\{\sum_i \xi_1 \mu (f_i) \xi_2 \nu (g_i)\}] \|', \ \nu \in B(Y, F_2))' \quad \text{i. e. } \nu : B(Y, F_2) \to F_2$$

$$= \omega (\sum_i f_i \otimes g_i) = \omega (u)$$

Thus T is on isometry of $B(X, F_1) \otimes B(Y, F_2) \to B(X \times Y; F)$; also $B(X \times Y; F)$ is a Banach space, hence there exists an extension $T' : B(X, F_1) \otimes_{\omega} B(Y, F_2) \to B(X \times Y, F)$.

This proves the proposition.

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REFERENCES

1) Bachman G—Introducfion to p-adic numbers and valuation theory—Academic press, 1964.

2) Bonsall F. F, snd Duncun. J—Complete Normrd Algebras, Springer Verlag, Burlin Heidelburg, New York (1973)

3) Greub W. H.—Multilinear Algebra, Springer Verlag, New York Enc. (1967)

4) Gruenburg K. W. and Wein, A. G.—Linear Geometry, D, Van Norstrad Company, Inc. Princeton, New Jersey, 1967.

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Department of Pure Mathematics,
Calcutta University

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ON A GENERALIZATION OF HERMITE POLYNOMIALS-I

S. K. Chatterjea

Some years ago H. W. Gould and A. T. Hopper [2] introduced a generalization of the usual Hermite polynomials by making the definition

(1) $H_n^r(x,a,p) = (-1)^n x^{-a} e^{px^r} D^n\left(x^a e^{-px^r}\right)$, $D \equiv d/dx$,

which bears a close relationship with a generalization of the generalized Laguerre polynomials due to the present author [1], viz.

(2) $H_{kn}^{(\alpha)}(x, p) = (-1)^n n!\, T_{kn}^{(\alpha-n)}(x, p)$,

where

(3) $T_{kn}^{(\alpha)}(x,p) = \dfrac{1}{n!} x^{-\alpha} e^{px^k} D^n \left(x^{\alpha+n} e^{-px^k}\right)$

and

(4) $H_{kn}^{(\alpha)}(x,p) = x^n H_n^k(x, \alpha, p)$,

the reason of multiplying $H_n^k(x, \alpha, p)$ by x^n lies in the fact that $H_n^k(x, \alpha, p)$ is not a polynomial, while $H_{kn}^{(\alpha)}(x, p)$ is a polynomial of degre kn, provided k is a natural mumber.

Now from [2, p. 53], we notice that

(5) $D^k H_n^r(x, a, p) = (-1)^k \sum_{j=0}^{k} \binom{k}{j} H_{k-j}^r(x,-a,-p)\, H_{n+j}^r(x, a, p)$.

Then operating e^{-tD} on $H_n^r(x,a,p)$, we obtain

$$(6)\quad H_n^r(x-t,a,p)=\sum_{m=0}^{\infty}\frac{t^m}{m!}\sum_{j=0}^{m}\binom{m}{j}H_{m-j}^r(x,-a,-p)\,H_{n+j}^r(x,a,p)$$

$$=\sum_{m=0}^{\infty}\frac{t^m}{m!}H_m^r(x,-a,-p)\sum_{j=0}^{\infty}\frac{t^j}{j!}H_{n+j}^r(x,a,p).$$

The relation (6) can be easily verified by means of the followring two generating relations of Gould—Hopper :

$$(7)\quad \sum_{n=0}^{\infty}\frac{t^n}{n!}H_n^r(x,a,p)=x^{-a}(x-t)^a\,e^{p(x^r-(x-t)^r)}$$

$$(8)\quad \sum_{n=0}^{\infty}\frac{t^n}{n!}H_{n+m}^r(x,a,p)=x^{-a}(x-t)^a\,e^{p(x^r-(x-t)^r)}H_m^r(x-t,a,p)$$

The relation (6), in terms of our polynomials $T_{kn}^{(\alpha)}(x,p)$, reveals that

$$(9)\quad T_{rn}^{(a-n)}(x-t,p)$$

$$=\sum_{m=0}^{\infty}(-x\,t)^m\,T_{rm}^{(-a-m)}(x,-p)\sum_{j=0}^{\infty}\binom{n+j}{j}(-xt)^j\,T_{r(n+j)}^{(a-n-j)}(x,p)$$

Next from [2, p. 53] we observe that

$$(10)\quad \mathcal{D}^n=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}H_{n-k}^r(x,a,p)\,D^k,$$

where $\mathcal{D}\equiv D-pr\,x^{r-1}+a/x$.

Then operating $e^{-t\mathcal{D}}$ on $H_n^r(x,a,p)$ we obtain

$$(11)\quad x^{-a}(x-t)^a\,e^{p(x^r-(x-t)^r)}H_n^r(x-t,a,p)$$

$$=\sum_{m=0}^{\infty}\frac{t^m}{m!}\sum_{k=0}^{m}(-1)^k\binom{m}{k}H_{m-k}^r(x,a,p)\,D^kH_n^r(x,a,p),$$

by means of the formula of Gould—Hopper :

$$(12)\quad e^{-t\mathcal{D}}f(x)=x^{-a}(x-t)^a\,e^{p(x^r-(x-t)^r)}f(x-t).$$

The relation (11) when combined with (5), yields the relation

(13) $$x^{-a}(x-t)^{a}\, e^{p(x^r-(x-t)^r)}\, \overset{r}{H}_n(x-t,a,p)$$
$$= \sum_{m=0}^{\infty} \frac{t^m}{m!} \sum_{k=0}^{m} \binom{m}{k} \overset{r}{H}_{m-k}(x,a,p) \sum_{j=0}^{k} \binom{k}{j} \overset{r}{H}_{k-j}(x,-a,-p)\, \overset{r}{H}_{n+j}(x,a,p),$$

which can be easily verified by means of (7) and (8).

Comparing (8) and (13) we get

(14) $$\overset{r}{H}_{n+m}(x,a,p)$$
$$= \sum_{j=0}^{m} \binom{m}{j} \overset{r}{H}_{n+j}(x,a,p) \sum_{k=j}^{m} \binom{m-j}{k-j} \overset{r}{H}_{m-k}(x,a,p)\, \overset{r}{H}_{k-}(x,-a,-p)\;.$$

It may be of much interest to compare (8) and (11). Indeed, then we have

(15) $$\overset{r}{H}_{n+m}(x,a,p)$$
$$= \sum_{k=0}^{m} (-1)^{k} \binom{m}{k} \overset{r}{H}_{m-k}(x,a,p)\, D^k \overset{r}{H}_n(x,a,p).$$

Now if we let $r=2$, $a=0$ and $p=1$ in (15) we find the formula for the classical Hermite polynomials

(16) $$H_{n+m}(x) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} H_{m-k}(x)\, D^k H_n(x).$$

Since $D^k H_n(x) = \dfrac{2^k n!\, H_{n-k}(x)}{(n-k)!}$, we obtain from (16)

(17) $$H_{n+m}(x) = \sum_{k=0}^{\min(m,n)} (-2)^k \binom{m}{k}\binom{n}{k} k!\, H_{m-k}(x)\, H_{n-k}(x),$$

which is the well-known formula of N. Nielsen.

Next suppose that

(18) $$G(x,t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!} H_n^r(x,a,p).$$

Then operating $e^{-t\mathcal{D}}$ on $G(x, t\, z)$ we obtain by means of (12) and the following formula of Gould-Hopper

(19) $$\mathcal{D}^m \overset{r}{H}_n(x,a,p) = (-1)^m \overset{r}{H}_{n+m}(x,a,p):$$

the bilateral generating relation

$$(20)\quad x^{-a}(x-t)^{a}\, e^{p(x^r-(x-t)^r)}\, G(x-t,\, tz)$$

$$=\sum_{m=0}^{\infty}\frac{(-t)^m}{m!}\sum_{n=0}^{\infty}\frac{a_n\,(tz)^n}{n!}\,\mathcal{D}^m H_n^{r}(x,a,p)$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{t^{m+n}}{m!\,n!}\,a_n z^n\, H_{n+m}^{r}(x,a,p)$$

$$=\sum_{m=0}^{\infty}\frac{t^m}{m!}H_m^{r}(x,a,p)\sum_{n=0}^{m}\binom{m}{n}a_n z^n.$$

The above bilateral generating relation can be easily generalized in the form of the follwing theorem :

Theorem. If there exists a generating relation of the form

$$(21)\quad G(x,t,u)=\sum_{n=0}^{\infty}\frac{a_n\,t^n}{n!}\,H_{n+m}^{r}(x,a,p)\,q_n(u),$$

where $q_n(u)$ is any polynomial or function in u, then the following more general generating relation holds

$$(22)\quad x^{-a}(x-t)^{a}\, e^{p\{x^r-(x-t)^r\}}\, G(x-t,\, tz,\, u)$$

$$=\sum_{n=0}^{\infty}\frac{t^n}{n!}\,H_{n+m}^{r}(x,a,p)\sum_{k=0}^{n}\binom{n}{k}a_k z^k q_k(u)$$

Proof.

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\,H_{n+m}^{r}(x,a,p)\sum_{k=0}^{n}\binom{n}{k}a_k z^k q_k(u)$$

$$=\sum_{k=0}^{\infty}\frac{(tz)^k}{k!}a_k q_k(u)\sum_{n=0}^{\infty}\frac{t^n}{n!}H_{n+m+k}^{r}(x,a,p)$$

$$=x^{-a}(x-t)^{a}\,e^{p(x^r-(x-t)^r)}\sum_{k=0}^{\infty}\frac{(tz)^k}{k!}a_k H_{m+k}^{r}(x-t,a,p)\,q_k(u)\quad\text{[By (8)]}$$

$$=x^{-a}(x-t)^{a}\,e^{p(x^r-(x-t)^r)}\,G(x-t,\,tz,\,u)\quad\text{[By (21)]}.$$

Again returning to the action of the operator $e^{-t\mathcal{D}}$ on G (x, tz) defined by (18) we have by means of (10) and (5)

$$(23)\quad x^{-a}(x-t)^{a}\, e^{p(x^r-(x-t)^r)}\, G(x-t,\, tz)$$

$$= \sum_{n=0}^{\infty} \frac{(tz)^n}{n\,!} a_n \sum_{m=0}^{\infty} \frac{t^m}{m\,!} \sum_{k=0}^{m} \binom{m}{k} \overset{r}{H}_{m-k}(x,a,p).$$

$$\cdot \sum_{j=0}^{k} \binom{k}{j} \overset{r}{H}_{k-j}(x,-a,-p) \quad \overset{r}{H}_{n+j}(x,a,p),$$

which may be compared with (20) and which can be easily verified by means of (7), (8) and (18).

Lastly considering the action of the operator e^{tD} on G (x, tz) defined by (18), we have by means of (5) and (8)

$$(24) \quad x^a (x+t)^{-a} e^{-p(x^r-(x+t)^r)} G(x+t, tz)$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n\,!} \sum_{p=o}^{n} (-1)^{n-p} \binom{n}{p} a_p z^p \overset{r}{H}_{n-p}(x,-a,-p) \overset{r}{H}_p(x+t, a, p),$$

which can be easily verified by means of (7) and (18). we should like to examine one special case of (24). If we let $a_n=1$ for all n and z=1, we obtain from (24) the result

$$(25) \quad \left(\frac{x}{x+t}\right)^{2a} e^{-2p\,(x^r-(x+t)^r)}$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n\,!} \sum_{p=0}^{n} (-1)^{n-p} \binom{n}{p} \overset{r}{H}_{n-p}(x,-a,-p) \overset{r}{H}_p(x+t, a, p)$$

REFERENCES

[1] Chatterjea, S. K.—On a generalization of Laguerre polynomials, Rend. Sem. Mat, Univ, Padova 34 (1964), 180—190,

[2] Gould, H.W. & Hopper, A.T.—Operational formulas connected with two generalizations of Hermite polynomials, Duke Math. J. 29 (1962), 51—64.

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Department of Pure Mathematics
Calcutta University

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ON A GENERALIZATION OF HERMITE POLYNOMIAL-II

S. K. Chatterjea

Some years ago H.W. Gould and A. T. Hopper [2] introduced a second generalization of the usual Hermite polynomials by making the definition

(1) $g_n^r(x, h) = e^{hD^r} x^n,\ D \equiv d/dx,$

a particular case of which was studied by L. R. Bragg [1]

In [1, p. 58] we notice that

(2) $D^j g_n^r(x, h) = j!\binom{n}{j} g_{n-j}^r(x, h).$

Now operating e^{-tD} on $g_n^r(x, h)$ we obtain

$$g_n^r(x-t, h) = \sum_{m=0}^{\infty} \frac{(-t)^m}{m!} D^m g_n^r(x, h)$$

(3) $$= \sum_{m=0}^{n} \binom{n}{m} (-t)^m g_{n-m}^r(x, h),$$

which can be compared with the result (6·19) of Gould-Hopper and which has an interesting special case when $h=-1$ and $r=2$, viz.

(4) $$H_n\left(\frac{x-t}{2}\right) = \sum_{m=0}^{n} \binom{n}{m} (-t)^m H_{n-m}(x/2),$$

which can well be compared with the following result [3, p. 255]

(5) $$H_n(x+y) = \sum_{m=0}^{n} \binom{n}{m} H_m(x)(2y)^{n-m}.$$

Next let

(6) $$G(x, t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!} g_n^r(x, h).$$

Then operating e^{-tD} on $G(x, t)$ we get after some calculation

(7) $$G(x-t, t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!} \sum_{m=0}^{n} \binom{n}{m} (-t)^m g_{n-m}^r(x, h),$$

which, when compared with (3), inplies that it can be verified easily by means of (3) and (6).

Next we consider the generating series

$$\sum_{m=0}^{\infty} \frac{t^m}{m!} g_{n+m}^r(x, h).$$

Here we have

$$\sum_{m=0}^{\infty} \frac{t^m}{m!} g_{n+m}^r(x, h)$$

$$= \sum_{m=0}^{\infty} \frac{t^m}{m!} e^{hD^r} x^{n+m}$$

$$= e^{hD^r}(x^n e^{tx}).$$

Now from [2, p. 59] we know that

(8) $$e^{D_x^r}(x^n e^{tx}) = D_t^n (e^{ht^r} e^{tx}).$$

Thus we obtain

(9) $$\sum_{m=0}^{\infty} \frac{t^m}{m!} g_{n+m}^r(x, h) = D_t^n (e^{ht^r} e^{tx}).$$

When $n=0$, we get as special case

(10) $$\sum_{m=0}^{\infty} \frac{t^m}{m!} g_m^r(x, h) = e^{tx+ht^r},$$

which is mentioned in the work of Gould-Hopper. Again when $h = -1$ and $r=2$, we obtain from (9) the interesting special case of the usual Hermite polynomials, viz.

(11) $$\sum_{m=0}^{\infty} \frac{t^m}{m!} H_{n+m}(x/2) = D_t^n (e^{tx-t^2}).$$

In other words,

$$\sum_{m=0}^{\infty} \frac{t^m}{m!} H_{n+m}(x) = D_t^n \left(e^{2tx-t^2}\right)$$

$$= e^{x^2} \cdot D_t^n \, e^{-(x-t)^2}$$

$$= e^{x^2} \cdot (-1)^n D_\omega^n \, e^{-\omega^2} \quad [\omega = x-t]$$

$$= e^{x^2} \cdot e^{-\omega^2} H_n(\omega)$$

$$= e^{2xt-t^2} H_n(x-t),$$

which is the well-known form of the generating functon for Hermite polynomials.

Let us now consider the action of e^{tD} on $g_n^r(x, h)$, where $D \equiv x + hr\, D^{r-1}$.
we have

$$e^{tD} g_n^r(x, h)$$

$$= \sum_{m=0}^{\infty} \frac{t^m}{m!} D^m g_n^r(x, h).$$

Now we know from [2, p. 59]

(12) $D^m g_n^r(x, h) = g_{n+m}^r(x, h).$

Thus we obtain

(13A) $e^{tD} g_n^r(x, h) = \sum_{m=0}^{\infty} \frac{t^m}{m!} g_{n+m}^r(x, h)$

(13B) $e^{tD} g_n^r(x, h) = D_t^n \left(e^{tx} e^{ht^r}\right)$

(13C) $e^{tD} g_n^r(x, h) = e^{hD_x^r} (x^n e^{tx}).$

When $h = -1$ and $r = 2$, we have the following interesting special cases of the usual Hermite polynomials.

(14A) $e^{t(2x-D)} H_n(x) = \sum_{m=0}^{\infty} \frac{t^m}{m!} H_{n+m}(x) = e^{2xt-t^2} H_n(x-t)$

(14B) $e^{t(2x-D)} H_n(x) = D_t^n (2tx - t^2)$

(14C) $e^{t(2x-D)} H_n(x) = e^{-D_x^2/4} (x^n e^{tx})$,

of which (14A) is a special case of the result (5.4) of Gould-Hopper.

REFERENCES

[1] Bragg. L.R.—Products of certain generalised Hermite polynomials, associated relations, Bol. Un Mat. Ital. 3 (1968), 347—355.

[2] Gould, H.W. and Hopper, A.T.—Operational formulas connected with two generalizations of Hermite polynomials, Duke Math. J. 29 (1962), 51—64.

[3] Magnus, W. Oberhettinger, F. and Soni, R.P.—Formulas and theorems for the special functions of Mathematical physics, Springer-Verlag (1966).

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Calcutta University

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